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ON STARLIKENESS AND CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS

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Let N be the class of normalised regular functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \quad |z| < 1 .$$

For $0 \leqq \lambda < 1, \gamma \geqq 1$, let $f(z), g(z) \in N$ be such that

$$|f(z)/[\lambda f(z) + (1-\lambda)g(z)] - \gamma| < \gamma , \quad |z| < 1 .$$

We establish the radius of starlikeness of $f(z)$ under the assumption $\mathrm{Re}\{g(z)/z\} > 0$, or $\mathrm{Re}\{g(z)/z\} > 1/2$, or $\mathrm{Re}\{zg'(z)/g(z)\} > \alpha, 0 \leqq \alpha < 1$, or $\mathrm{Re}\{1 + zg''(z)/g'(z)\} > 0$ for $|z| < 1$. The analysis may be extended to the problem of finding the radius of convexity for certain subclasses of N.

1. Introduction and notation. Let S, S^*, S^c denote the subclasses of N which are univalent, univalent starlike, univalent convex in $|z| < 1$ respectively.

A necessary and sufficient condition for $f(z) \in N$ to be univalent starlike in $|z| < r$ is

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 , \quad |z| < r .$$

A necessary and sufficient condition for $f(z) \in N$ to be univalent convex in $|z| < r$ is

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 , \quad |z| < r .$$

A function $f(z)$ belongs to $S^*(\beta)$, i.e., is starlike of order β, $0 \leqq \beta < 1$, if it satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta , \quad |z| < 1 .$$

A function $f(z)$ belongs to $S^c(\beta)$, i.e., is convex of order β, $0 \leqq \beta < 1$, if it satisfies the condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta , \quad |z| < 1 .$$

Let $\mathscr{P}_\alpha$ denote the class of regular functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k , \quad |z| < 1 ,$$

satisfying the inequality $\operatorname{Re}\{p(z)\} > \alpha$ for $|z| < 1, 0 \leqq \alpha < 1$ and $\mathscr{Q}_\gamma$ the class of functions $q(z)$ with expansion of the above form but satisfying the inequality $|q(z) - \gamma| < \gamma$ for $|z| < 1, \gamma \geqq 1$. We note that both $\mathscr{P}_0$ and $\mathscr{Q}_\infty$ reduce to the class $\mathscr{P}$ of functions with positive real part.

Let N_n, $n \geqq 1$, denote the subclass of N consisting of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$. Then $N_1 = N$.

Shah [8] considered the problem of determining the radius of starlikeness of $f(z) \in N_n$ for the following cases:

(a) $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathscr{P}$ with $g(z) \in N_n$ and $g(z)/z \in \mathscr{P}$, or $g(z)/z \in \mathscr{P}_{1/2}$ (with $n = 1$), or $g(z) \in S^*(\alpha)$;

(b) $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathscr{Q}_1$ with $g(z) \in N_n$ and $g(z)/z \in \mathscr{P}$, or $g(z) \in S^*(\alpha)$.

The conditions were shown to be sharp only when $\lambda = 0$. In this paper, we solve the problem for the subclasses of N mentioned at the beginning, subject to certain restrictions on the values of λ. Letting $\gamma \to \infty$ we obtain the radii of starlikeness of $f(z)$ satisfying $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathscr{P}$. All the bounds obtained are best possible. Furthermore, the same technique may be used to establish the radius of convexity of $f(z) \in N$ satisfying $f'(z)/[\lambda f'(z) + (1-\lambda)g'(z)] \in \mathscr{Q}_\gamma$, where $g(z)$ belongs to various subclasses of N. The results proved here generalize those of MacGregor [3, 4, 5] and Ratti [6, 7].

It should be remarked that parallel results for subclasses of N_n, $n > 1$, may be derived in an analogous manner. The manipulations involved are, however, more complicated.

The lemmas required for the proofs of our theorems are given in §2. Section 3 contains theorems giving the conditions for starlikeness. We outline the conditions for convexity in §4.

2. Some lemmas.

Let $\mathscr{B}$ denote the class of functions $w(z)$ regular in $|z| < 1$ and satisfying $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$.

LEMMA 2.1 [9]. *If $w(z) \in \mathscr{B}$, then for $|z| < 1$,*

$$|zw'(z) - w(z)| \leqq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

Proof. Write $w(z) = z\phi(z)$, where $\phi(z)$ is regular in $|z| < 1$ and $|\phi(z)| \leqq 1$. The assertion now follows from the well-known result due to Caratheodory

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

LEMMA 2.2. *Let $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$, where $w(z) \in \mathscr{B}$,*

$\beta \geqq 0$. *Then, fo* $|z| = r < \min(1, 1/\beta)$,

$$\operatorname{Re}\left\{-\beta w_1(z) + \frac{1}{w_1(z)}\right\} + \frac{r^2|1+\beta w_1(z)|^2 - |1 - w_1(z)|^2}{(1-r^2)|w_1(z)|} \leqq \frac{1-\beta+(3\beta+1)r+\beta(\beta+3)r^2+\beta(\beta-1)r^3}{(1-r^2)(1+\beta r)}.$$

Proof. By Schwarz's lemma, $|w(z)| \leqq r$ on $|z| = r < 1$. The transformation $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$ maps the disc $|w(z)| \leqq r$, $r < \min(1, 1/\beta)$, onto the disc $|w_1(z) - a| \leqq d$, where

$$a = \frac{1-\beta r^2}{1-\beta^2 r^2}, \quad d = \frac{(1+\beta)r}{1-\beta^2 r^2}.$$

Clearly,

$$0 < a - d = \frac{1+r}{1+\beta r} < a + d = \frac{1+r}{1-\beta r}.$$

Put $w_1(z) = a + u + iv$, $R = |a + u + iv|$; then

$$\begin{aligned} S(u, v) &= \operatorname{Re}\left\{-\beta w_1(z) + \frac{1}{w_1(z)}\right\} + \frac{r^2|1+\beta w_1(z)|^2 - |1-w_1(z)|^2}{(1-r^2)|w_1(z)|} \\ &= -\beta(a+u) + \frac{a+u}{R^2} + \frac{1-\beta^2 r^2}{1-r^2} \cdot \frac{d^2-u^2-v^2}{R}. \end{aligned} \tag{2.1}$$

Now,

$$\frac{\partial S}{\partial v} = -\frac{v}{R^4}\left\{2(a+u) + \frac{1-\beta^2 r^2}{1-r^2}[(d^2-u^2-v^2)R + 2R^3]\right\}.$$

The terms inside the curly brackets are always positive for $r < \min(1, 1/\beta)$. Hence the maximum of $S(u, v)$ in the disc $|w_1(z) - a| \leqq d$ is attained when $v = 0$ and $u \in [-d, d]$. Setting $v = 0$ in (2.1) we obtain

$$S(u, 0) = \frac{2(1-\beta^2 r^2)a}{1-r^2} - \frac{(1+\beta)(1-\beta r^2)}{1-r^2}(a+u). \tag{2.2}$$

Since $dS(u, 0)/du < 0$ for $r < \min(1, 1/\beta)$, the maximum of $S(u, 0)$ occurs at the end point $u = -d$ and the result follows.

LEMMA 2.3. *If* $w(z) \in \mathscr{B}$, $\beta \geqq 0$, *then for* $|z| = r < \min(1, 1/\beta)$,

$$\operatorname{Re}\left\{\frac{zw'(z)}{[1-w(z)][1+\beta w(z)]}\right\} \leqq \frac{r}{(1-r)(1+\beta r)}. \tag{2.3}$$

Proof. From Lemma 2.1, we have

$$\operatorname{Re}\left\{\frac{zw'(z)}{(1-w(z))(1+\beta w(z))}\right\} \leqq \operatorname{Re}\left\{\frac{w(z)}{(1-w(z))(1+\beta w(z))}\right\} + \frac{r^2-|w(z)|^2}{(1-r^2)|1-w(z)|\,|1+\beta w(z)|}.$$

Put $w_1(z) = [1-w(z)]/[1+\beta w(z)]$, then the above inequality becomes

$$\operatorname{Re}\left\{\frac{zw'(z)}{(1-w(z))(1+\beta w(z))}\right\} \leqq \frac{1}{(1+\beta)^2}\left[\beta - 1 + \operatorname{Re}\left\{-\beta w_1(z) + \frac{1}{w_1(z)}\right\} + \frac{r^2|1+\beta w_1(z)|^2 - |1-w_1(z)|^2}{(1-r^2)|w_1(z)|}\right].$$

An application of Lemma 2.2 to the right hand side will give the result which is easily seen to be sharp for $w(z) = z$ at $z = r$.

The following lemma is a consequence of [2, Theorem 3].

LEMMA 2.4. *If $p(z) \in \mathscr{P}$, then on $|z| = r$,*

$$\operatorname{Re}\left\{\frac{zp'(z)}{1+p(z)}\right\} \geqq \begin{cases} -\dfrac{r}{1+r}, & \text{for } r < \dfrac{1}{3} \\ \dfrac{r^2 + 2^{3/2}(1-r^2)^{1/2} - 3}{1-r^2}, & \text{for } \dfrac{1}{3} \leqq r < 1. \end{cases} \tag{2.4}$$

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqq -\frac{2r}{1-r^2}. \tag{2.5}$$

3. Radii of starlikeness.

THEOREM 3.1. *Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathscr{Q}_\gamma$, where $g(z) \in N$ and $g(z)/z \in \mathscr{P}$, $0 \leqq \lambda < (1+\sqrt{3}+1/2\gamma)/(2+\sqrt{3})$. Then the radius of starlikeness σ_1 of $f(z)$ is given by the only positive root in $(0, 1)$ of the equation*

$$\beta r^3 + (2+3\beta)r^2 + 3r - 1 = 0,$$

where $\beta = [(1+\lambda)\gamma - 1]/(1-\lambda)\gamma$.

Proof. Put $\psi(z) = 1 - f(z)/\gamma[\lambda f(z) + (1-\lambda)g(z)]$. Then $|\psi(z)| < 1$ for $|z| < 1$ and $\psi(0) = 1 - 1/\gamma = A$. Let $w(z) = [\psi(z) - A]/[1 - A\psi(z)]$. It is clear that $w(z) \in \mathscr{B}$ and $\psi(z) = [w(z) + A]/[1 + Aw(z)]$ from which we deduce

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{1+A}{1-\lambda}\cdot\frac{zw'(z)}{(1-w(z))(1+\beta w(z))}, \tag{3.1}$$

$\beta = (A+\lambda)/(1-\lambda)$, provided $1 - \lambda(1-w(z))/(1+Aw(z)) \neq 0$. Since $|w(z)| \leqq r$ for $|z| = r$ by Schwarz's lemma, it follows that

$$1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$$

if, in particular, $|z| < 1/\beta$.

Now, as $g(z)/z \in \mathscr{P}$, write $g(z)/z = p(z)$, some $p(z) \in \mathscr{P}$. Then $zg'(z)/g(z) = 1 + zp'(z)/p(z)$. An application of (2.5) gives

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqq \frac{1 - 2r - r^2}{1 - r^2}, \quad |z| = r < 1 . \tag{3.2}$$

This result together with (3.1) and (2.3) yield

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1 - 3r - (2 + 3\beta)r^2 - \beta r^3}{(1 - r)(1 + \beta r)} .$$

For the cubic polynomial

$$F(r) = \beta r^3 + (2 + 3\beta)r^2 + 3r - 1 ,$$

$F(0) < 0$, $F(1) = 4 + 4\beta > 0$, $F(1/\beta) = (3 + 6\beta - \beta^2)/\beta^2$. Thus the equation $F(r) = 0$ has exactly one root in $(0, 1)$ which is in the range $(0, 1/\beta)$ if $\beta < 3 + 2\sqrt{3}$, i.e., if $\lambda < (1 + \sqrt{3} + 1/2\gamma)/(2 + \sqrt{3})$.

REMARK 3.1. The theorem is sharp for

$$f(z) = \frac{1 - z}{1 + \beta z} \cdot \frac{z(1 - z)}{(1 + z)} .$$

When $\lambda = 0$, $f(z)$ is starlike in $|z| < \sqrt{5} - 2$ if $\gamma \to \infty$ and in $|z| < (\sqrt{17} - 3)/4$ if $\gamma = 1$ as previously shown by Ratti [6, Theorems 1 and 4].

THEOREM 3.2. *Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathscr{Q}_\gamma$, where $g(z) \in N$ and $g(z)/z \in \mathscr{P}_{1/2}$. Then the radius of starlikeness of $f(z)$ is*

$$\sigma_2 = \begin{cases} r_1 , & \text{for } 0 \leqq \lambda \leqq 1/2\gamma , \\ r_2 = [2^{1/2}(1 + \beta)^{1/2} - 1]/(1 + 2\beta) , & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 \\ & \quad + 1/\gamma)/(\sqrt{5} + 3) , \end{cases}$$

where $\beta = [(1 + \lambda)\gamma - 1]/(1 - \lambda)\gamma$ and r_1 is the smallest positive root in $(0, 1)$ of the equation

$$(1 + 2\beta + 9\beta^2)r^4 + 2(1 + 12\beta + 3\beta^2)r^3 + (13 + 10\beta + \beta^2)r^2 + 4(1 - \beta)r - 4 = 0 .$$

Proof. Since $g(z)/z \in \mathscr{P}_{1/2}$, there exists $p(z) \in \mathscr{P}$ so that $g(z)/z = 1/2 + p(z)/2$. Hence

$$\text{(3.3)} \qquad \frac{zg'(z)}{g(z)} = 1 + \frac{zp'(z)}{1 + p(z)} .$$

Applying (2.4) to this equation gives, on $|z| = r$,

$$\text{(3.4)} \qquad \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqq \begin{cases} 1/(1 + r) , & \text{for } 0 < r < 1/3 \\ 2[2^{1/2}(1 - r^2)^{1/2} - 1]/(1 - r^2) , & \text{for } 1/3 \leqq r < 1 . \end{cases}$$

This result together with (3.1) and (2.3) yield, for $|z| = r < 1/3$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1 - 2r - (1 + 2\beta)r^2}{(1 - r)(1 + \beta r)} = G(r)$$

and for $1/3 \leqq r < 1$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq -\frac{(1 + \beta)r}{(1 - r)(1 + \beta r)} + \frac{2[2^{1/2}(1 - r^2)^{1/2} - 1]}{1 - r^2} ,$$

which yields the equation giving the condition of starlikeness of $f(z)$ to be

$$(1 + 2\beta + 9\beta^2)r^4 + 2(1 + 12\beta + 3\beta^2)r^3 + (13 + 10\beta + \beta^2)r^2 + 4(1 - \beta)r - 4 = 0 .$$

The only root in (0, 1) of the numerator of $G(r)$ is r_2 which is less than 1/3 if $\beta > 1$, i.e., if $\lambda > 1/2\gamma$, and is the range $(0, 1/\beta)$ if $\beta < \sqrt{5} + 2$, i.e., if $\lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$. Thus $f(z)$ is starlike in $|z| < r_2$ if $1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$. Now, for $0 \leqq \lambda \leqq 1/2\gamma$, $\beta < 1$, and r_1 is in the interval $(0, 1/\beta)$ and the theorem is proved.

REMARK 3.2. The results are sharp. The extremal functions are

$$f(z) = \begin{cases} \dfrac{1 - z}{1 + \beta z} \cdot \dfrac{z}{2}\left[1 + \dfrac{1}{2}\left(\dfrac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} + \dfrac{1 + ze^{i\theta}}{1 - ze^{i\theta}}\right)\right\} , & \text{for } 0 \leqq \lambda \leqq 1/2\gamma \\ \dfrac{1 - z}{1 + \beta z} \cdot \dfrac{z}{1 + z} , & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)(\sqrt{5} + 3) , \end{cases}$$

where θ satisfies the equation

$$H(r_1)(1 + r_1^2) + r_1^2 - [3H(r_1) + 1/2 + r_1^2(H(r_1) + 1/2)]r_1 \cos\theta + 2H(r_1)r_1^2 \cos^2\theta = 0$$

with

$$H(r_1) = [r_1^2 + 2^{3/2}(1 - r_1^2)^{1/2} - 3]/2(1 - r_1^2) .$$

When $\lambda = 0$, the cases $\gamma \to \infty$ and $\gamma = 1$ give Theorems 2 and 5 of [6].

REMARK 3.3. For $g(z) \in S^c$, the result [10]

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqq \frac{1}{1+r}, \quad |z| = r < 1$$

together with (3.1) and (2.3) give the radius of starlikeness of $f(z) \in N$ with $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{Q}_\gamma$ to be $[2^{1/2}(1+\beta)^{1/2} - 1]/(1+2\beta)$ for $0 \leqq \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$, $\beta = [(1+\lambda)\gamma - 1]/(1-\lambda)\gamma$. The bound is attained for the function

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{1+z}.$$

When $\lambda = 0$, the cases $\gamma \to \infty$ and $\gamma = 1$ become Theorem 4 of [4] and Theorem 4 of [5] respectively.

THEOREM 3.3. *Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{Q}_\gamma$, where $g(z) \in S^*(\alpha)$, $0 \leqq \lambda < \lambda_0$, some $\lambda_0 < 1$. Then the radius of starlikeness σ_3 of $f(z)$ is given by the smallest positive root in* (0, 1) *of the equation*

$$\beta(2\alpha - 1)r^3 + (3\beta + 2\alpha - 2\alpha\beta)r^2 + (3 - 2\alpha)r - 1 = 0,$$

where $\beta = [(1+\lambda)\gamma - 1]/(1-\lambda)\gamma$.

Proof. Since $g(z) \in S^*(\alpha)$, we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqq \frac{1 + (2\alpha - 1)r}{1+r}, \quad |z| = r < 1.$$

Applying this result and (2.3) to (3.1) gives the required equation from which σ_3 may be obtained. λ_0 is determined by the condition $\sigma_3 < 1/\beta$.

REMARK 3.4. The theorem is sharp for

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{(1+z)^{2-2\alpha}}.$$

When $\lambda = 0$, the cases $\gamma \to \infty$ and $\gamma = 1$ correspond to Theorems 3 and 6 of [6].

4. Radii of convexity. In this section, we briefly look at the problem of determining the radius of convexity of $f(z) \in N$ with $f'(z)/[\lambda f'(z) + (1-\lambda)g'(z)] \in \mathcal{Q}_\gamma$, where $g(z)$ belongs to various subclasses of N. For such $f(z)$, we can deduce in a similar manner as in Theorem 3.1 that

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}=\operatorname{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\}-\frac{1+A}{1-\lambda}\cdot\frac{zw'(z)}{(1-w(z))(1+\beta w(z))}, \tag{4.1}$$

provided $1-\lambda(1-w(z))/(1+Aw(z))\neq 0$, $w(z)\in\mathscr{B}$, $A=1-1/\gamma$, $\beta=(A+\lambda)/(1-\lambda)$. With some restriction on λ, we may apply (2.3) and the known bounds for $\operatorname{Re}\{1+zg''(z)/g'(z)\}$ to (4.1) to get the equations from which the radii of convexity of $f(z)$ may be obtained. We consider the following six cases.

(i) $g'(z)\in\mathscr{P}$. The radius of convexity of $f(z)$ is equal to σ_1 as given by Theorem 3.1.

(ii) $g'(z)\in\mathscr{P}_{1/2}$. The radius of convexity of $f(z)$ is equal to σ_2 as given by Theorem 3.2.

(iii) $g(z)\in S^c(\alpha)$. The radius of convexity of $f(z)$ is equal to σ_3 as given by Theorem 3.3.

(iv) $g(z)\in S$.

The result [1, p. 166]

$$\operatorname{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\}\geqq\frac{1-4r+r^2}{1-r^2}, \quad |z|=r<1,$$

together with (2.3) and (4.1) yield the radius of convexity of $f(z)$ to be the smallest positive root (less than 1) of the equation

$$\beta r^3-5\beta r^2-5r+1=0,$$

with $0\leqq\lambda<(2-\sqrt{6}+1/2\gamma)/(3-\sqrt{6})$.

(v) $g(z)\in S^*$. The radius of convexity of $f(z)$ is the same as that of part (iv).

(vi) $g(z)\in S^*(1/2)$. Theorem 4.1 of [9] with $\beta=1/2$ gives

$$\operatorname{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\}\geqq\frac{1-r}{1+r}, \quad |z|=r<1/2.$$

This result together with (2.3) and (4.1) yield the radius of convexity of $f(z)$ to be the smallest positive root ρ of the equation

$$\beta r^3-3\beta r^2-3r+1=0,$$

with $0\leqq\lambda<(1+\sqrt{2}+1/2\gamma)/(2+\sqrt{2})$.

All these results are best possible and generalise those obtained by Ratti [7, Theorems 1-6].

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