ON SPACES WHICH HAVE A CLOSURE-PRESERVING COVER BY FINITE SETS

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The purpose of the present paper is to give a characterization of spaces which have a closure-preserving cover by finite sets, in terms of hereditary metacompactness and generalized σ-discreteness. Several results with respect to spaces which have a closure-preserving cover by finite sets can be deduced easily from our characterization. Moreover, we can conclude more easily from it whether or not a given space has such a cover.

1. **Introduction.** H. Tamano [10] and R. Telgársky [11] raised independently the question of whether or not a space which has a closure-preserving cover by compact sets must be paracompact. H. B. Potoczny [7] showed that even if a space has a closure-preserving cover by finite sets, it is not necessarily paracompact. Thus, the study of spaces which have a closure-preserving cover by finite sets has been taken up by several authors in [9], [11], [12], [13] and [14]. In particular, R. Telgársky [12] gave a necessary condition and a sufficient condition for a given space to have such a cover. But each of these conditions is not necessary and sufficient as we shall show by examples below. In the present paper, we shall first find a necessary and sufficient condition for a topological T_1-space to have a closure-preserving cover by finite sets. It will be shown that the necessary condition and the sufficient condition of R. Telgársky can be deduced easily from our necessary and sufficient condition. Secondly, we shall prove that it is undecidable under set theoretic assumptions whether or not a first countable normal space with a closure-preserving cover by finite sets is metrizable.

2. **Preliminaries and main theorem.** Throughout this paper, N denotes the set of all natural numbers and all spaces are assumed to be T_1-spaces.

A space X is said to be *weakly σ-discrete*, if X is the countable union of discrete subsets $\{X_n : n \in N\}$ such that $\bigcup_{i=1}^{n} X_i$ is closed in X for each $n \in N$. A space X is said to be *σ-discrete* ([12], [13]), if X is the countable union of discrete closed subsets. A space X is said to be *hereditarily metacompact*, if each (open) subset of X is metacompact.

Our main theorem is as follows.

THEOREM 1. *The following are equivalent for a space X.*
(a) *X has a closure-preserving cover by finite sets.*
(b) *X is hereditarily metacompact and weakly σ-discrete.*

REMARK. It is easy to see from Theorem 1 that the spaces which are given by examples in [6], [7] and [14] have closure-preserving covers by finite sets.

3. Proof of Theorem 1.

LEMMA 2. *If a space X has an open cover $\mathfrak{U} = \{U(x): x \in X\}$ such that $x \in U(x)$ for each $x \in X$ and $U(x) \subset U(y)$ whenever $x \in U(y)$, then $\mathfrak{F} = \{F_x: x \in X\}$ is a closure-preserving cover of X, where $F_x = \{y \in X: x \in U(y)\}$ for each $x \in X$.*

Proof. Since $x \in F_x$ for each $x \in X$, $\mathfrak{F}$ is a cover of X. Let $Y \subset X$. Let $x \in \overline{\bigcup_{y \in Y} F_y}$. Then there is a point $y_0 \in Y$ such that $U(x) \cap F_{y_0} \neq \varnothing$. Choose $z \in U(x) \cap F_{y_0}$. Since $U(z) \subset U(x)$ and $y_0 \in U(z)$, we have $y_0 \in U(x)$. Hence $x \in F_{y_0} \subset \bigcup_{y \in Y} F_y$. This proves that the cover $\mathfrak{F}$ is closure-preserving.

Let us consider the following two conditions (*), (**).
Condition (*): X has an open cover $\mathfrak{U} = \{U(x): x \in X\}$ such that (i) $x \in U(x)$ for each $x \in X$, (ii) $U(x) \subset U(y)$ whenever $x \in U(y)$, and (iii) $\mathfrak{U}$ is point-finite in X.
and
Condition (**): X has an open cover $\mathfrak{V} = \{V(x): x \in X\}$ such that (i) $x \in V(x)$ for each $x \in X$, and (ii) $\mathfrak{V}$ is point-finite in X, and has a countable pairwise disjoint cover $\{X_n: n \in N\}$ such that (iii) each X_n is a discrete subset and (iv) $\bigcup_{i=1}^{n} X_i$ is closed in X for each $n \in N$.
We shall prove the three lemmas below.

LEMMA 3. *The following are equivalent for a space X.*
(a) *X has a closure-preserving cover by finite sets.*
(b) *X satisfies the condition (*).*

Proof. (a) $\rightarrow$ (b): Let $\mathfrak{F} = \{F_\alpha: \alpha \in A\}$ be a closure-preserving cover of X by finite sets. Put $U(x) = X - \bigcup \{F_\alpha: x \notin F_\alpha\}$ for each $x \in X$ and $\mathfrak{U} = \{U(x): x \in X\}$. It is obvious that $\mathfrak{U}$ satisfies (i) and (ii) of (*). Assume now that there is a point $x \in X$ such that $\mathrm{ord}\,(x, U) \geqq \aleph_0$. We can choose an infinite number of points $x_1, x_2, \cdots \in X$ such that $x \in U(x_n)$ for each $n \in N$. Choose $F_\alpha \in \mathfrak{F}$ such that $x \in F_\alpha$. Then we have $x_n \in F_\alpha$ for each $n \in N$. Since F_α is finite, this is a contradiction. Hence $\mathfrak{U}$ is point-finite in X.

(b) $\to$ (a):   It is obvious from Lemma 2 and (iii) of ($*$).

LEMMA 4.   *For every space X, the condition ($*$) and the condition ($**$) are equivalent.*

Proof.   ($*$) $\to$ ($**$):   Put $X_n = \{x \in X\colon \text{ord}\,(x, \mathfrak{U}) = n\}$ for each $n \in N$. It is obvious that $\{X_n\colon n \in N\}$ is a countable pairwise disjoint cover of X.   Since $\bigcup_{i=1}^{n} X_i = \{x \in X\colon \text{ord}\,(x, \mathfrak{U}) \leqq n\}$, $\bigcup_{i=1}^{n} X_i$ is closed in X for each $n \in N$.

Fix $n \in N$ and $x \in X_n$.   We can choose distinct n points $x_1, \cdots, x_n \in X$ such that $\{x' \in X\colon x \in U(x') \in \mathfrak{U}\} = \{x_1, \cdots, x_n\}$.   Let $y \in U(x) \cap X_n$. By (i) and (ii) of ($*$), $y \in U(y) \subset U(x) \subset \bigcap_{i=1}^{n} U(x_i)$.   Since $\text{ord}\,(y, \mathfrak{U}) = n$, $y \in \{x_1, \cdots, x_n\}$.   Thus X_n is a discrete subset of X for each $n \in N$.

($**$) $\to$ ($*$):   We use the induction with respect to n.   Let $P(n)$ be the proposition: There are point-finite (in X) collections $\mathfrak{U}_i = \{U(x)\colon x \in X_i\}$ of open sets for $i = 1, \cdots, n$ such that (a) $U(x) \cap X_i = \{x\}$ for each $x \in X_i$, (b) if $j < i$, $x \in X_i$, $y \in X_j$ and $x \in U(y) \in \mathfrak{U}_j$, then $U(x) \subset U(y)$ and (c) $U(x) \subset V(x)$ and $U(x) \cap \bigcup_{j=1}^{i-1} X_j = \varnothing$ for each $x \in X_i$.

It is obvious that $P(1)$ is true.   Assume now that $P(n)$ is true. By (iii) and (iv) of ($**$), we can choose an open neighborhood $U(x)$ of x for each $x \in X_{n+1}$ such that $U(x) \subset V(x)$, $U(x) \cap X_{n+1} = \{x\}$ and $U(x) \cap \bigcup_{i=1}^{n} X_i = \varnothing$.   Moreover, if $x \in \bigcup_{i=1}^{n} \{U\colon U \in \mathfrak{U}_i\} \cap X_{n+1}$, then we choose $U(x)$ such that $U(x) \subset \bigcap \{U\colon x \in U \in \bigcup_{i=1}^{n} \mathfrak{U}_i\}$.   It is possible since $\{U\colon x \in U \in \bigcup_{i=1}^{n} \mathfrak{U}_i\}$ is a finite collection.   Put $\mathfrak{U}_{n+1} = \{U(x)\colon x \in X_{n+1}\}$.   By our construction and $P(n)$, it is easy to see that $\mathfrak{U}_1, \cdots, \mathfrak{U}_{n+1}$ satisfy the conditions (a), (b) and (c) of $P(n+1)$ and $\mathfrak{U}_{n+1}$ is point-finite in X.   So, X has an open cover $\bigcup_{n=1}^{\infty} \mathfrak{U}_n$ such that $\mathfrak{U}_1, \cdots, \mathfrak{U}_n$ satisfy the conditions of $P(n)$ for each $n \in N$.   Put $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n$.   By the pairwise disjointness of $\{X_n\colon n \in N\}$, $\mathfrak{U} = \{U(x)\colon x \in X\}$. The open cover $\mathfrak{U}$ satisfies (i), (ii) and (iii) of ($*$).

LEMMA 5.   *The following are equivalent for a space X.*
(a)   *X satisfies the condition ($**$).*
(b)   *X is hereditarily metacompact and weakly σ-discrete.*

Proof.   (a) $\to$ (b):   It follows from (i) and (ii) of ($**$) that X is hereditarily metacompact. X is obviously weakly σ-discrete.

(b) $\to$ (a):   Let $X = \bigcup_{n=1}^{\infty} X_n$ such that X_n is a discrete subset, $\bigcup_{i=1}^{n} X_i$ is closed in X and $\bigcup_{i=n}^{\infty} X_i$ is metacompact for each $n \in N$. We may assume, without loss of generality, $X_m \cap X_n = \varnothing$ for $m \neq n$. Since each X_n is a discrete closed subset in $\bigcup_{i=n}^{\infty} X_i$, there is a point-finite collection $\mathfrak{V}_n = \{V(x)\colon x \in X_n\}$ of open sets for each $n \in N$ such that $V(x) \cap X_n = \{x\}$ and $V(x) \subset \bigcup_{i=n}^{\infty} X_i$ for each $x \in X_n$.   Put $\mathfrak{V} =

$\bigcup_{n=1}^{\infty} \mathfrak{V}_n$. Then $\mathfrak{V} = \{V(x) \colon x \in X\}$. It is easy to prove that $\mathfrak{V}$ is point-finite in X. Hence X satisfies the condition ($\ast\ast$).

By Lemmas 3, 4 and 5, the proof of Theorem 1 is complete.

4. Applications.

LEMMA 6. *Each metacompact σ-discrete space is hereditarily metacompact.*

Proof. Let X be a metacompact σ-discrete space. We may assume, without loss of generality, that $X = \bigcup_{n=1}^{\infty} X_n$ such that each X_n is a discrete closed subset of X and $X_m \cap X_n = \varnothing$ for $m \neq n$. Let Y be an open subset of X. Let $\mathfrak{U}$ be an open cover of Y. Since $X_n \cap Y$ is a discrete closed subset of X, there is a point-finite (in X) collection $\mathfrak{V}_n = \{V(x) \colon x \in X_n \cap Y\}$ of open sets of X for each $n \in N$ such that $x \in V(x)$, $V(x) \cap \bigcup_{i=1}^{n-1} X_i = \varnothing$ and $V(x)$ is a subset of some element of $\mathfrak{U}$ for each $x \in X_n \cap Y$. Put $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$. Then $\mathfrak{V} = \{V(x) \colon x \in Y\}$ and $\mathfrak{V}$ is a point-finite open refinement of $\mathfrak{U}$ in Y. Hence X is hereditarily metacompact.

COROLLARY 7. *Each metacompact σ-discrete space has a closure-preserving cover by finite sets.*

This is an immediate consequence of Theorem 1 and Lemma 6.

REMARK. R. Telgársky proved in [12] (Theorem 5) that each paracompact σ-discrete space has a closure-preserving cover by finite sets. Corollary 7 generalizes this result. We shall show in §5 (Example 3) that there is a metacompact σ-discrete space which is not paracompact.

Recall that Y is said to be *scattered*, if each subset Z of Y has an isolated point in Z.

COROLLARY 8 (R. Telgársky [12]). *If X has a closure-preserving cover by finite sets, then X has a countable cover $\{Y_n \colon n \in N\}$, where Y_n is a scattered closed subset of X and it is the union of n discrete subsets for each $n \in N$.*

This an immediate consequence of Theorem 1.

COROLLARY 9. *The following are equivalent for a space X.*
(a) *X has a closure-preserving cover by finite sets.*
(b) *X is hereditarily metacompact and has a σ-closure-pre-*

serving cover by finite sets.

Proof. (a) → (b): Obvious.

(b) → (a): It is easy to show that the countable union of weakly σ-discrete closed subsets is weakly σ-discrete. Hence X is hereditarily metacompact and weakly σ-discrete.

REMARK. Our Corollary 9 has been proved earlier, in [13] (Theorem 12.1), by R. Telgársky for the case of X being a metrizable space.

THEOREM 10. *The following are equivalent for a space X.*

(a) X *has a closure-preserving closed cover by compact sets.*

(b) X *has an open cover* $\mathfrak{U} = \{U(x) : x \in X\}$ *such that* (i) $x \in U(x)$ *for each* $x \in X$, (ii) $U(x) \subset U(y)$ *whenever* $x \in U(y)$ *and* (iii) $C(x) = \{y \in X : x \in U(y)\}$ *is a compact closed set for each* $x \in X$.

Proof. (a) → (b): Let $\mathfrak{C} = \{C_\alpha : \alpha \in A\}$ be a closure-preserving closed cover by compact sets. Put $U(x) = X - \bigcup \{C_\alpha : x \notin C_\alpha\}$ for each $x \in X$ and $\mathfrak{U} = \{U(x) : x \in X\}$. It is obvious that $\mathfrak{U}$ satisfies (i) and (ii) of (b). Let $y \notin C(x)$ and $z \in U(y)$. Since $x \notin U(y)$ and $U(z) \subset U(y)$, $x \notin U(z)$. Thus, we have $z \notin C(x)$. Therefore $U(y) \cap C(x) = \varnothing$. This proves that each $C(x)$ is closed in X. Let $y \in C(x)$. Since $x \in U(y)$, $y \in C_\alpha$ whenever $x \in C_\alpha$. Thus, $C(x)$ is contained in $\bigcap \{C_\alpha : x \in C_\alpha\}$ which is compact and closed. Hence $C(x)$ is a compact closed set for each $x \in X$.

(b) → (a): It is obvious from Lemma 2 and (iii) of (b) in this theorem.

REMARK. For a space to have a closure-preserving closed cover by compact sets, R. Telgársky gave, in [12], a sufficient condition which is not necessary.

LEMMA 11. *If a first countable space X has a closure-preserving cover by finite sets, then X is developable.*

Proof. By Lemma 3, X has an open cover $\mathfrak{U} = \{U(x) : x \in X\}$ satisfying (i), (ii) and (iii) of (∗). So, X has a countable number of open covers $\mathfrak{V}_n = \{V_n(x) : x \in X\}$ of X such that $\{V_n(x) : n \in N\}$ is a local base at x and $V_{n+1}(x) \subset V_n(x) \subset U(x)$ for each $x \in X$ and $n \in N$. We shall show that $\{\mathfrak{V}_n : n \in N\}$ is a development of X. Let $x \in X$ and let W be an open neighborhood of x. There is a finite number of points $x_1, \cdots, x_m \in X$ such that $\{U \in \mathfrak{U} : x \in U\} = \{U(x_1), \cdots, U(x_m)\}$. For $i = 1, \cdots, m$, we choose some $n_i \in N$ such that $V_{n_i}(x_i) \subset W$ if

$x_i \in W$ and $x \notin V_{n_i}(x_i)$ if $x_i \notin W$. Put $n_0 = \max\{n_i \colon i = 1, \cdots, m\}$. Assume that $x \in V_{n_0}(y) \in \mathfrak{V}_{n_0}$. By $x \in V_{n_0}(y) \subset U(y)$, $y = x_k$ for some $k \leq m$. Since $x \in V_{n_k}(x_k)$, $x_k \in W$. So, we have $V_{n_0}(y) \subset V_{n_k}(x_k) \subset W$. Hence $\mathrm{St}(x, \mathfrak{V}_{n_0}) \subset W$.

LEMMA 12. *If a collectionwise Hausdorff, regular space X has a closure-preserving cover by finite sets, then X is paracompact.*

The proof of Lemma 12 is quite parallel to that of H. B. Potoczny's theorem in [8]. So, we omit it.

THEOREM 13. *If a first countable normal space X has a closure-preserving cover by finite sets, then*

(a) *X is metrizable under $V = L$ (i.e., the axiom of constructibility), and* (b) *X is not necessarily metrizable under Martin's Axiom and $2^{\aleph_0} > \aleph_1$.*

Proof. (a): W. Fleissner showed in [2] that a first countable normal space is collectionwise Hausdorff under $V = L$. So, X is paracompact and developable by Lemmas 11 and 12. Hence X is metrizable.

(b): It is well-known that there is a subset A of the real-line such that every subset of A is a relative G_δ-set, assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$. S. A. Peregudov showed in [6] that Heath's space in [4] (Theorem 3) has the desired property. Here it is to be noted the existence of Heath's space is guaranteed under such an assumption.

THEOREM 14. *If a first countable normal space X has a closure-preserving cover by compact sets, then*

(a) *X is paracompact under $V = L$, and* (b) *X is not necessarily paracompact under Martin's Axiom and $2^{\aleph_0} > \aleph_1$.*

The proof of (a) of Theorem 14 is similar to that of H. B. Potoczny [8] by using the techniques of W. Fleissner [3]. The detail of the proof is left to the reader. When we consider above Heath's space, (b) is obvious by Lemma 11.

5. **Examples.**

EXAMPLE 1. There is a compact T_2-space which has a closure-preserving cover by finite sets and is not σ-discrete.

Let X be the one-point-compactification of an uncountable dis-

crete space. The space X has only one accumulation point x_0. If F is a closed subset of X such that $x_0 \notin F$, then F is a finite set. If D is a discrete subset of X such that $x_0 \in D$, then D is also a finite set. Since X is uncountable, the countable union of discrete closed sets is not X. Hence X is not σ-discrete. It is easily verified that X has a closure-preserving cover consisting of two-point sets.

REMARK. By our Example 1, for a space which has a closure-preserving cover by finite sets, σ-discreteness is not a necessary condition. Hence the sufficient condition given in our Corollary 7 and R. Telgársky's Theorem 5 in]12] for such a space is not necessary.

EXAMPLE 2.   There is a paracompact weakly σ-discrete T_2-space which is not hereditarily metacompact and not σ-discrete.

Let $X_1 = \{(x, 0): x \in \mathbf{R}^1\}$ and $X_2 = \{(x, y): (x, y) \in \mathbf{R}^2, y > 0\}$. Let $p_0 = (0, -1)$ and $X = X_1 \cup X_2 \cup \{p_0\}$. The topology for the space X is defined as follows: For each $(x, 0) \in X_1$ a local base at $(x, 0)$ is $\{U_n(x, 0): n \in N\}$, where $U_n(x, 0) = \{(x', y): (x' - x)^2 + (y - 1/n)^2 < 1/n^2\} \cup \{(x, 0)\}$, for each $(x, y) \in X_2$ $\{(x, y)\}$ is open in X and a local base at p_0 is $\{X - U_{n_1}(q_1) \cup \cdots \cup U_{n_i}(q_i) \cup \{r_1, \cdots, r_j\}: q_1, \cdots, q_i \in X_1, r_1, \cdots, r_j \in X_2 \text{ for each } i, j \in N\}$.

It is obvious that X is a weakly σ-discretè Tychonoff space. Since every open cover of X has a pairwise disjoint open refinement, X is paracompact with dim $X = 0$. It follows from Baire's category theorem that $X_1 \cup X_2$ is not metacompact, which is a well-known fact. By Theorem 1 and Corollary 7, X is not σ-discrete.

REMARK. Our Example 2 shows that we can not replace hereditary metacompactness in Theorem 1 by metacompactness and that the necessary condition in Corollary 8 proved by R. Telgársky is not sufficient.

EXAMPLE 3.   There is a metacompact σ-discrete normal space which is not paracompact.

Let G be Bing's space modified by E. Michael in [5] (Example 2). We define F_p as in [1] (Example H). Then the space G is perfectly normal, metacompact, and non-paracompact ([5]). Since every point of $G - F_p$ is an isolated point of G and the discrete closed subset F_p is a G_δ-set, G is σ-discrete.

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