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RIESZ HOMOMORPHISMS AND POSITIVE LINEAR MAPS

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It was shown in previous papers [C. T. Tucker, "Homomorphisms of Riesz spaces," Pacific J. Math., 55 (1974), 289–300, and "Concerning σ-homomorphisms of Riesz spaces," Pacific J. Math., 57 (1975), 585–590] that there is a large class β of Riesz spaces with the property that if L belongs to β and ϕ is a Riesz homomorphism of L into an Archimedean Riesz space then ϕ preserves the order limit of sequences. In this paper it is shown that if L belongs to β then every order bounded linear map of L into an Archimedean, directed, partially ordered vector space is sequentially continuous. An application of this is made to the theory of Baire funtions. Further, some properties of those members of β which are also normed Riesz spaces are considered.

This paper is a continuation and extension of Tucker [8] and [9]. The notation of Tucker [8] and [9] will be used.

The following theorem includes Theorem 19.8 of Nakano [5].

THEOREM 1. *Suppose L belongs to β. Then every order bounded linear map of L into an Archimedean, directed, partially ordered vector space is sequentially continuous.*

Proof. Suppose E is a complete Riesz space and let $\mathscr{L}^\sim = \mathscr{L}^\sim(L, E)$ be the complete Riesz space of all order bounded linear transformations of L into E. If $f \in L$ and $T \in \mathscr{L}^\sim$, then denote by $\langle f, T\rangle$ the order bounded bilinear form $\langle f, T\rangle = Tf$. The canonical imbedding of L into $\mathscr{L}^\sim(\mathscr{L}^\sim(L, E), E)$ is a Riesz homomorphism. (A proof when $E = R$ is given in Kelley and Namioka [2], Section 23. The same argument holds when R is replaced by any complete Riesz space.) Since $\mathscr{L}^\sim(\mathscr{L}^\sim(L, E), E)$ is Archimedean, it follows from the hypothesis that the imbedding preserves countable suprema and infima. Hence if $f_n \downarrow \theta$, then for every $T \in \mathscr{L}^\sim$, $Tf_n = \langle f_n, T\rangle \downarrow \theta$ by definition of $\mathscr{L}^\sim$.

Suppose E is only an Archimedean, directed, partially ordered vector space. There exists a one to one order continuous positive linear map λ of E into $\bar{E}$, its completion. If T is an order bounded linear map of L into E, then λT is an order bounded linear map of L into the complete Riesz space $\bar{E}$ and thus λT is sequentially continuous. This implies that T is sequentially continuous.

In view of the previous theorem the elements of β will be said to have the *sequential mapping continuity* property, abbreviated the s.m.c. property.

In Tucker [7], property c was defined. It was shown in Tucker [8] that if a Riesz space contained a point with property c then it belonged to β. The following shows more clearly how property c relates to other properties.

DEFINITION. A point $x \geqq \theta$ has *weak property* c if, whenever $\{h_i\}$ is a sequence such that $h_i \uparrow x$, there exists a subsequence h_{i_1}, $h_{i_2}, h_{i_3}, \cdots$ and a point b such that $b \leqq \sum_{p=1}^{n} h_{i_p}$ for each positive integer n.

Clearly in the hypothesis of Theorem 3 of [8], property c could be replaced by weak property c with only a minor modification of the argument.

THEOREM 2. *Suppose order convergence in* L *is stable and* $f \geqq \theta$. *Then* f *has weak property* c.

Proof. Suppose $h_i \uparrow f$. Then $h_i^- \uparrow \theta$. Since order convergence is stable there exists a point g and a sequence $\{c_i\}$ of real numbers converging to 0 such that $c_i g \leqq h_i^-$. Let $\{c_{i_p}\}$ be a subsequence of $\{c_i\}$ such that $c_{i_p} < 1/2^p$. Then $\sum_{p=1}^{n} h_i^- \geqq \sum_{p=1}^{n} c_{i_p} g \geqq \sum_{p=1}^{n} 1/2^p g \geqq g$.

Suppose each of Ω and Ω' is a linear lattice of functions on a set X containing the constant functions. Denote by $B_1(\Omega)$ (the first Baire class of Ω) the family of all pointwise limits of sequences from Ω and by $LS(\Omega)$ the family of all pointwise limits of nondecreasing sequences from Ω. For a recent survey of the properties of Baire functions, see Mauldin [4].

LEMMA 3. *Pointwise monotone convergence in* $B_1(\Omega)$ *is equivalent to monotone order convergence.* (In the sense that $f_i \downarrow f \in B_1(\Omega)$ in order convergence if and only if it does in pointwise convergence also.)

Proof. Clearly pointwise convergence implies order convergence. Suppose $f_1 \geqq f_2 \geqq f_3 \geqq \cdots \geqq \theta$, $f_i \in B_1(\Omega)$, and $\wedge f_i = \theta$. There exists a sequence $g_1 \geqq g_2 \geqq g_3 \geqq \cdots \geqq \theta$ such that $g_i \in LS(\Omega)$ and the pointwise limit of $\{g_i\}$ is the same as the pointwise limit of $\{f_i\}$. Suppose there exists an $x \in X$ and $\varepsilon > 0$ such that $g_i(x) > \varepsilon$ for every positive integer i. There exists a point $h_i \in \Omega$ such that $g_i \geqq h_i \geqq \theta$ and $h_i(x) = \varepsilon$. Let $k_i = \min_{p \leqq i} \{h_i\}$. Then $k_1 \geqq k_2 \geqq k_3 \geqq \cdots \geqq \theta$, $k_i \in \Omega$, $k_i \leqq g_i$, and $k_i(x) = \varepsilon$. Let j be the pointwise limit of $\{k_i\}$. Then $j \in B_1(\Omega)$, $g_i \geqq j$, and $j(x) = \varepsilon > 0$, so that $j \neq \theta$. This is a contradiction. Thus $\{f_i\}$ converges pointwise to θ.

PROPOSITION 4. *The space $B_1(\Omega)$ contains a point with weak property c.*

Proof. In view of Lemma 3, the proof of Theorem 6 of Tucker [9] holds.

COROLLARY 5. *An order bounded linear mapping from $B_1(\Omega)$ to $B_1(\Omega')$ is sequentially continuous* (and thus preserves bounded pointwise convergence).

If "$\| \ \|$" is a norm on the Riesz space L such that $\|f\| \leqq \|g\|$ if $|f| \leqq |g|$, then "$\| \ \|$" is called a *Riesz norm* on L and L is said to be a *normed Riesz space*. Also L will be said to have *property* (A, i) if $f_i \downarrow \theta$ implies $\|f_i\| \downarrow 0$.

The remainder of this paper will consider those members of β which are also normed Riesz spaces.

THEOREM 6. *Suppose L is a normed Riesz space. Then, of the following conditions,* (1) *implies* (2) *and* (2) *implies* (3). *If L is assumed to be norm complete then each two of the three conditions are equivalent. If L is not assumed to be norm complete then the reverse implications do not hold.*

(1) *Order convergence in L is stable.*
(2) *L has the* s.m.c. *property.*
(3) *L has property (A, i).*

Proof. (1) implies (2) clearly. If L has the s.m.c. property then every positive linear functional is sequentially continuous, thus by Corollary 24.3 of Luxemburg and Zaanen [3], L has property (A, i).

In the event that L is norm complete and has property (A, i), then order convergence implies norm convergence which implies relative uniform convergence and (1) holds.

For an example to show that (3) does not imply (2) if L is not assumed to be norm complete take L^∞ with the L^2 norm.

To show that (2) does not imply (1) in the absence of norm completeness consider the following example: Let S be the set of all ordered pairs of positive integers. Let L be the collection to which f belongs only in case f is a real valued function on S with the property that there is a set ω which includes all but at most a finite number of positive integers such that if k is a positive integer in ω, $f(1, k), f(2, k) f(3, k), \cdots$ is a bounded number sequence and with the property that

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{2^p} \frac{1}{2^q} |f(p, q)| < \infty .$$

The space L is an order complete Riesz space and

$$||f|| = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{2^p} \frac{1}{2^q} |f(p, q)|$$

is a Riesz norm on L.

Suppose M is an ideal which is relatively uniformly closed. Let f be the l.u.b. of a countable subset α of M. The function which is equal to $f(i, j)$ at (i, j) and zero elsewhere is in M. For each positive integer k let $f_k(p, q) = f(p, q)$ if $k = q$ and zero otherwise. There exists a nondecreasing unbounded sequence of positive integers $\{c_i\}$ such that the function g_k defined by $g_k(p, q) = c_p f_k(p, q)$ is in L. Thus f_k is in M. Also, there exists a non-decreasing unbounded sequence of positive integers $\{d_i\}$ such that $h(p, q) = d_q f(p, q)$ is in L. Therefore f is in M. By Corollary 4 of Tucker [9], L has the s.m.c. property.

For each positive integer i, let g_i be the function such that $g_i(p, q) = 1$ if $p = i$ and $g_i(p, q) = 0$ if $p \neq i$. Then $\{g_i\}$ is an orthogonal subset of L whose supremum is the constant function 1, but there is no nondecreasing unbounded positive number sequence $\{k_i\}$ such that $\{k_i g_i\}$ is bounded above. Thus order convergence is not stable in L.

The Riesz space L is said to be *almost σ-complete* if L is a Riesz subspace of a σ-complete space K such that for every $\theta \leqq u \in K$ there exists a sequence $\{u_n\} \subseteq L$ with $\theta \leqq u_n \uparrow u$ in K. See Aliprantis and Langford [1] or Quinn [6] for some properties of almost σ-complete spaces.

COROLLARY 7. *Suppose the normed Riesz space L has* s.m.c. *property. Then every order bounded linear mapping of L into an Archimedean, directed, partially ordered vector space preserves order convergence of nets if and only if L is almost σ-complete.*

Proof. Suppose L is almost σ-complete. By Theorem 9.1 of Quinn [6], L is order separable which implies that sequentially continuous maps are net continuous. On the other hand, if every sequentially continuous map is net continuous, then L is order separable and therefore almost σ-complete.

The following theorem includes Theorem 5.1 of Zaanen [10] and Theorem 3 of Tucker [7].

THEOREM 8. *Suppose L has the* s.m.c. *property, is almost σ-complete, and has a strong unit. Then L is finite dimensional.*

Proof. Let $\{h_i\}$ be a countable orthogonal subset of L^+. Suppose e is a strong unit of L. It may be assumed that for each positive integer $i, e \geqq h_i$. By Corollary 8.5 of Quinn [6], there exists a sequence $f_n \downarrow \theta$ such that $f_n \geqq |\sum_{i=1}^{n+p} h_i - \sum_{i=1}^{n} h_i|$ for each positive integer p. Thus f_n is an upper bound of $\{h_i\}_{i=n+1}^{\infty}$. It may be assumed that for each positive integer $n, f_n \leqq e$. Let $g_n = e - (f_n \vee (\bigvee_{i=1}^{n} h_i))$. Then $e \geqq g_{n+1} \geqq g_n$.

Let Q be the set to which f belongs only in case $f \in L^+$, $f \leqq e$, and if $\varepsilon > 0$ there exists a positive integer n such that

$$\bigwedge_{p=1}^{\infty} (f - \varepsilon e - \sum_{i=1}^{n} h_i - g_p)^+ = \theta .$$

Let M be the set to which f belongs only in case there is a positive number c such that $c|f| \in Q$. To show that M is an ideal, suppose that f and g are in M. Now $|f+g| \leqq |f| + |g| = |f| \vee |g| + |f| \wedge |g| \leqq 2(|f| \vee |g|) \in M$. So that $f+g$ is in M. The other properties of an ideal follow easily.

Note that $e = \vee h_i + \vee g_i$, but since $e - \varepsilon e - g_p = (f_p \vee (\bigvee_{i=1}^{p} h_i)) - \varepsilon e \geqq \sum_{i=1}^{n} h_i - \varepsilon e$ for each positive integer n, e is not in M and M is not a σ-ideal.

Suppose $d_i \downarrow 0$, $\{f_i\}$ is a sequence of points of M, and f is a point of L such that $|f - f_i| \leqq d_i e$. Let $\varepsilon > 0$ and $d_i < \varepsilon/2$. Then $f_i + \varepsilon/2e \geqq f$ and $f_i - \varepsilon/2e \geqq f - \varepsilon e$. There exists a positive integer n such that $\bigwedge_{p=1}^{\infty} (f_i - \varepsilon/2e - \sum_{i=1}^{n} h_i - g_p)^+ = \theta$. Thus,

$$\bigwedge_{p=1}^{\infty} (f - \varepsilon e - \sum_{i=1}^{n} h_i - g_p)^+ = \theta ,$$

f belongs to M, and M is uniformly closed. This is a contradiction.

In Theorem 8, the s.m.c. property cannot be replaced by property (A, i) as the first example in Theorem 6 shows. The following example shows that the almost σ-completeness can not be dropped.

EXAMPLE 9. Let X be an uncountable set and let L be the space of all real valued functions on X that are constant except possibly on a finite subset of X. Then L has the principal projection property but is not almost σ-complete. Also L is infinite dimensional, has a strong unit, and has the s.m.c. property since the constant function 1 has weak property c.

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