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A TOPOLOGICAL CHARACTERIZATION OF BANACH CONTRACTIONS

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A continuous operator T on a metric space (X, ρ) is a Banach contraction with fixed point p under some metric σ topologically equivalent to ρ if, and only if, every orbit $T^n x$ converges to p and the convergence is uniform on some neighborhood of p. For σ to be bounded we demand that the convergence be uniform on X. The latter condition with T uniformly continuous characterizes the case for σ bounded and uniformly equivalent to ρ.

A Banach contraction is an operator T on a metric space (X, σ) such that for some $a < 1$

$$(1) \qquad \sigma(Tx, Ty) \leqq a\sigma(x, y) \qquad \text{for all } x, y \text{ in } X .$$

(T is called nonexpansive if (1) holds with $a = 1$.) By induction (1) extends to

$$(2) \qquad \sigma(T^n x, T^n y) \leqq a^n \sigma(x, y)$$

for all n in the set N of all natural numbers. By the Banach Contraction Theorem a Banach contraction T on a complete metric space has a unique fixed point $p = Tp$. From (2) with $y = p$

$$(3) \qquad \sigma(T^n x, p) \leqq a^n \sigma(x, p) .$$

Since a^n converges to 0 (3) implies

$$(4) \qquad T^n x \longrightarrow p \qquad \text{for all } x \text{ in } X .$$

Moreover, taking any ball about p, we conclude from (3) that

$$(5) \qquad T^n B \longrightarrow p \qquad \text{for some neighborhood } B \text{ of } p .$$

That is, (4) holds uniformly for x in B. We contend that a continuous T satisfying (4) and (5) is topologically equivalent to a Banach contraction. We shall also get topological and uniform characterizations of Banach contractions on bounded spaces. Our proofs depend on two lemmas which extend constructions used by Ludvik Janos [1] to characterize Banach contractions on compact spaces.

For any metric ρ we use the notation $\rho[\ \ ]$ to denote the ρ-diameter of sets.

THEOREM. *Let T operate on a metric space (X, ρ).*

(i) *There exists a metric σ topologically equivalent to ρ on X such that T is a Banach contraction under σ with fixed point p if, and only if, T is continuous and both* (4) *and* (5) *hold.*

(ii) *There exists a bounded metric σ topologically equivalent to ρ on X such that T is a Banach contraction under σ with fixed point p if, and only if, T is continuous and $T^n X \to p$.*

(iii) *There exists a bounded metric σ uniformly equivalent to ρ on X such that T is a Banach contraction under σ if, and only if, T is uniformly continuous and*

$$\rho[T^n X] \longrightarrow 0 . \tag{6}$$

LEMMA 1. *For T an operator on (X, ρ) conditions* (a) *and* (b) *are equivalent in their corresponding versions*:

(a) *T and its iterates are equicontinuous* (equiuniformly continuous).

(b) *T is nonexpansive under some metric $\bar{\rho}$ topologically* (uniformly) *equivalent to ρ. $\bar{\rho}$ may be assumed bounded.*

Proof. That (b) implies (a) is trivial. To prove the converse we may assume $\rho \leqq 1$ since we can replace ρ by the uniformly equivalent metric $\mathrm{Min}\{\rho, 1\}$. Then define

$$\bar{\rho}(x, y) = \sup_{n \geqq 0} \rho(T^n x, T^n y) . \tag{7}$$

$\bar{\rho}$ is readily seen to be a metric with $\rho \leqq \bar{\rho} \leqq 1$. So ρ is uniformly continuous with respect to $\bar{\rho}$. By (a) and (7) $\bar{\rho}$ is (uniformly) continuous with respect to ρ. Finally, (7) implies T is nonexpansive under $\bar{\rho}$.

LEMMA 2. *Let $\langle B_n \rangle$ be a sequence of subsets of (X, ρ) indexed by the set Z of all integers* (positive, negative, and zero) *so that the conditions* (8), (9), *and* (10) *hold*:

$$B_{n+1} \subseteqq B_n \quad \text{for all } n \text{ in } Z , \tag{8}$$

$$U_{n \in Z} B_n^0 = X , \tag{9}$$

where the superscript denotes interior, and

$$\rho[B_n] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty . \tag{10}$$

Given $0 < a < 1$ let σ be the largest pseudometric on X such that

$$\sigma \leqq a^n \rho \qquad \text{on } B_n \text{ for all } n \text{ in } Z . \tag{11}$$

Then

(12) $$\rho \leqq a^{-n}\sigma + \rho[B_n] \quad \textit{for all } n \textit{ in } Z .$$

So ρ is uniformly continuous with respect to σ. Hence σ is a metric. Moreover, σ is topologically equivalent to ρ. If $B_0 = X$ then σ is uniformly equivalent to ρ. Finally, if T is a nonexpansive operator on (X, ρ) such that

(13) $$TB_n \subseteqq B_{n+1} \quad \textit{for all } n \textit{ in } Z$$

then (1) *holds.*

Proof. Let $S(x, y)$ be the set of all finite sequences

(14) $$\begin{cases} \langle x_i, n_i \rangle_{i=0,1,\cdots,m} \text{ in } X \times Z \text{ such that } x_0 = x, \\ x_m = y, \text{ and both } x_{i-1} \text{ and } x_i \text{ belong to } B_{n_i} \text{ for} \\ i = 1, \cdots, m . \end{cases}$$

Then the largest pseudometric σ satisfying (11) is

(15) $$\sigma(x, y) = \inf_{S(x,y)} \sum_{i=1}^{m} a^{n_i}\rho(x_{i-1}, x_i)$$

as the reader can routinely verify. Given n in Z we contend that for any member (14) of $S(x, y)$

(16) $$a^n\rho(x, y) \leqq \sum_{i=1}^{m} a^{n_i}\rho(x_{i-1}, x_i) + a^n\rho[B_n] .$$

Case 1. No $x_i \in B_n$. Then $n_i < n$ by (8) and the last condition in (14). Therefore

(17) $$a^n < a^{n_i}$$

for $i = 1, \cdots, m$. By the triangle inequality and (17),

$$a^n\rho(x, y) \leqq a^n \sum_{i=1}^{m} \rho(x_{i-1}, x_i) \leqq \sum_{i=1}^{m} a^{n_i}\rho(x_{i-1}, x_i)$$

which gives (16) in Case 1.

Case 2. Some $x_i \in B_n$. Let x_j be the first and x_k the last such x_i. Let $J = [1, j] \cup [k + 1, m]$ in N. Then $\rho(x, y) \leqq \sum_{i \in J} \rho(x_{i-1}, x_i) + \rho(x_j, x_k) \leqq \sum_{i \in J} \rho(x_{i-1}, x_i) + \rho[B_n]$. Multiplying by a^n and noting that (17) holds for all i in J, we get $a^n\rho(x, y) \leqq \sum_{i \in J} a^{n_i}\rho(x_{i-1}, x_i) + a^n\rho[B_n]$ which implies (16) in Case 2.

Now (15) and (16) imply $a^n\rho \leqq \sigma + a^n\rho[B_n]$ which gives (12).

Given $\varepsilon > 0$ use (10) to get n such that $\rho[B_n] < \varepsilon/2$. Take $\delta = a^n(\varepsilon/2)$. Then by (12), $\sigma < \delta$ implies $\rho < \varepsilon$. So ρ is uniformly continuous with respect to σ, which implies σ is a metric. (9) implies

σ is continuous with respect to ρ since continuity holds on each B_n by (11). If $B_0 = X$ then $\sigma \leqq \rho$ by (11) at $n = 0$, so σ is uniformly continuous with respect to ρ.

Finally, if T is nonexpansive and satisfies (13) consider any member (14) of $S(x, y)$. Then $\langle Tx_i, n_i + 1\rangle$ belongs to $S(Tx, Ty)$ by (13). Therefore, since T is nonexpansive under ρ, (15) yields $\sigma(Tx, Ty) \leqq \sum_{i=1}^{m} a^{n_i+1}\rho(Tx_{i-1}, Tx_i) \leqq a \sum_{i=1}^{m} a^{n_i}\rho(x_{i-1}, x_i)$ which gives (1) by (15).

We can now prove the theorem. The direct implications in (i), (ii), and (iii) follow from the Banach Contraction Theorem and the topological (uniform) invariance of the conclusions. So we need only prove the converses.

To prove (i) let T be continuous and satisfy (4) and (5). We contend first that the T^n are equicontinuous for all n in N. That is, given q in X and $\varepsilon > 0$ there exists a neighborhood D of q such that

$$\rho[T^n D] < \varepsilon \tag{18}$$

for all n in N. To get such a D choose m in N large enough to ensure that for B some neighborhood of p satisfying (5)

$$T^m q \in B^0 \tag{19}$$

by (4) and

$$\rho[T^k B] < \varepsilon \qquad \text{for all} \quad k > m \tag{20}$$

by (5). Since T, and hence each interate of T, is continuous there exists a neighborhood D of q such that (18) holds for all $n \leqq 2m$. By (19) $T^{-m}B$ is a neighborhood of q since T^m is continuous. So we may assume $D \subseteqq T^{-m}B$. For $n > 2m$ take $k = n - m$ to get $k > m$ and $T^n D \subseteqq T^k B$. So (20) implies (18) for $n > 2m$. Thus (18) holds for all n in N.

Hence we may assume by Lemma 1 that T is nonexpansive under ρ. Consequently each open ball about p is mapped into itself by T since $p = Tp$ by (4) and the continuity of T. Hence in (5) we may assume B is open and

$$TB \subseteqq B\,. \tag{21}$$

Apply Lemma 2 with

$$B_n = T^n B \qquad \text{for all } n \text{ in } Z\,. \tag{22}$$

Indeed, (21) and (22) imply (8) and (13). (10) follows from (22) and (5). Since B is open and T is continuous, B_n is open for $n < 0$ by

(22). So (9) follows from (4) and (22). Hence Lemma 2 applies and gives (i).

To prove (ii) let T be continuous and $T^n X \to p$. These conditions are just those in (i) with $B = X$. So we get σ from Lemmas 1 and 2 exactly as in the proof of (i). But now $\sigma \leqq \rho$ from (11) at $n = 0$ since $B_0 = X$ under (22). Thus, since we may assume ρ is bounded, σ is bounded.

To prove (iii) let T be uniformly continuous and satisfy (6). We contend first that the T^n are equiuniformly continuous for all n in N. Given $\varepsilon > 0$ choose m by (6) so that

$$\rho[T^m X] < \varepsilon . \tag{23}$$

Since T, and hence each of its iterates, is uniformly continuous we can choose $\delta > 0$ such that every subset D of X with $\rho[D] < \delta$ must satisfy

$$\rho[T^n D] < \varepsilon \tag{24}$$

for $n = 1, \cdots, m$. But (24) holds for $n > m$ by (23). So (24) holds for all n in N. Therefore, by the uniform version of Lemma 1 we may assume T is nonexpansive under a bounded ρ.

Let $B_n = T^n X$. Then (8), (9), and (13) are trivial while (10) is just (6). So Lemma 2 applies and gives (iii), which completes the proof of the theorem.

Note that (iii) is the uniform analogue of (ii). However, we have no uniform analogue of (i).

For X a compact metric space each of our results (i), (ii), and (iii) reduces to the theorem of Ludvik Janos [1]: An operator T on a compact metric space (X, ρ) is a Banach contraction under some metric σ topologically equivalent to ρ if, and only if, T is continuous and the $T^n X$ intersect at only one point.

To get this theorem from (i), (ii), or (iii) one can use the following remarks about a continuous T operating on a compact (X, ρ):

(a) Continuity of T is equivalent to uniform continuity,

(b) Every continuous metric σ on X is bounded,

(c) σ is topologically equivalent to ρ if, and only if, σ is uniformly equivalent to ρ,

(d) The intersection I of the compact nested after-images $T^n X$ is nonempty and $\rho[T^n X] \to \rho[I]$. (See Lemma A in [2].)

REFERENCES

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