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A NOTE ON EDELSTEIN'S ITERATIVE TEST AND SPACES OF CONTINUOUS FUNCTIONS

JACK BRYANT AND T. F. McCABE

In this note a question posed by Nadler is answered. It is shown that if X is a compact Hausdorff space that contains a sequence of distinct points that converge then there exists a linear contractive selfmap f of $C(X)$ such that, for some x, the sequence of iterates $\{f^n(x)\}$ does not converge. In particular, the iterative test is not conclusive for c.

Our setting is a metric space (X, d) and a contractive selfmap $f\colon X \to X$. In [1], Nadler introduces and motivates the following terminology: the *iterative test* (of Edelstein) *is conclusive* (for contractive maps) provided that if f is a contractive selfmap of X with a fixed point then, for all $x \in X$, $\{f^n(x)\}$ converges. Nadler shows that the iterative test is conclusive (ITC) for finite dimensional Banach spaces, but that the iterative test is not conclusive (ITNC) for the spaces $l_p(1 \leqq p < \infty)$ and c_0 (the space of sequences convergent to zero). The technique used there does not seem to apply directly to the space c of convergent sequences, and part of Nadler's Problem 1 is exactly the question of whether c has ITC.

LEMMA 1. *The iterative test is not conclusive for c.*

Proof. Let $\{\alpha_n\}$ be an increasing positive sequence with (infinite) product 1/2. Define $f\colon c \to c$ by $f(\{x_n\}) = \{y_n\}$ where

$$y_1 = 0, \quad y_2 = -y^2 = \alpha_1 x_1 ,$$
$$y_{2n} = -y_{2n+1} = \frac{\alpha_n}{2}(x_{2n-2} - x_{2n-1}) , \qquad n = 2, 3, \cdots .$$

Since f is linear, f has fixed point 0, and it suffices to show f is contractive at 0; if

$$\{z_n\} \in c,\ \{x_n\} \neq 0,\ d(f(\{x_n\}), 0) = \sup\{|y_n|\} = |y_{n_0}| ,$$

since $y_n \to 0$. If $n_0 = 1$ or 2 then it is easy to see that

$$d(f(\{x_n\}), 0) < d(\{y_n\}, 0) .$$

If $n_0 = 2k(k > 1)$, we have

$$|y_{n_0}| = \frac{\alpha_k}{2}|x_{n_0-2} - x_{n_0-1}| \leqq \frac{\alpha_k}{2}\{|x_{n_0-2}| + |x_{n_0-1}|\}$$
$$\leqq \alpha_k d(\{x_n\}, 0) < d(\{x_n\}), 0) .$$

Let e_k be the sequence $\{\delta_{kn}\} = \{0, 0, 0, \cdots, 1, 0, \cdots\}$ (1 in the kth coordinate). We have

$$f^j(e_1) = \left(\prod_{i=1}^{j} \alpha_i\right)(e_{2j} - e_{2j+1}) .$$

In particular, $d(f^j(e_1), 0) = \prod_{i=1}^{j} \alpha_i \rightarrow 1/2$, and so $\{f^j(e_1)\}$ does not converge. (If $\{f^j(e_1)\}$ converges, then, since f is contractive, $f^j(e_1)$ must converge to the fixed point 0 of f.)

It is of definite interest that the map $f: c \rightarrow c$ constructed above is linear. It would seem to be easier to solve Nadler's Problem 1 (if a Banach space has ITC then it is finite dimensional) when restricted to linear maps.

LEMMA 2. *Let Y be a normed space and let X be a subspace of Y. Let P be a projection of norm 1 from Y onto X. Then if the iterative test is not conclusive for X, it is not conclusive for Y.*

Proof. Let $f: X \rightarrow X$ be a contractive map with fixed point such that, for some x_0, $\{f^n(x_0)\}$ does not converge. Define $g: Y \rightarrow Y$ by $g = f \circ P$. Since f is contractive and $||P|| = 1$, then g is contractive. Also, $g^n(x_0) = f^n(x_0)$ (since $P(x_0) = x_0$), and so $\{g^n(x_0)\}$ does not converge.

If X is a compact Hausdorff space with a convergent sequence of distinct points, a projection P of norm 1 can be constructed from $C(X)$ onto a subspace that is linearly isometric to c.

Let $\{x_n\}$ be any sequence of distinct points of X that converges and furthermore $x_n \rightarrow \bar{x}$. Let $P_1: C(X) \rightarrow c$ be defined as follows: if

$$f \in C(X),\ P_1(f) = \{y_n\} \quad \text{where} \quad y_n = f(x_n) .$$

Since f is continuous $y_n \rightarrow f(x)$ and $P_1(f) \in c$. P_1 is nonexpansive for

$$||P_1(f)|| = \sup_n |f(x_n)| \leqq \sup_{x \in X} |f(x)| = ||f|| .$$

An isometric linear map Q is now constructed from c into $C(X)$ such that $P_1 \circ Q(x) = x$. Let $\{U_i\}$ be a sequence of open sets such that $x_i \in U_i$, $U_i \cap U_j = \emptyset$ if $i \neq j$, and $\bar{x} \notin U_i$ for all i. For each i define f_i to be a function such that $f_i(x_i) = 1$, $f_i(X - U_i) = 0$ and $0 \leqq f_i(x) \leqq 1$ for all x. If $\{y_n\} \in c$ and $y_n \rightarrow y$ then define $Q(\{y_n\}) = f$ where

$$f(x) = \sum_{n=1}^{\infty} f_n(x)(y_n - y) + y .$$

It is easily verified that f is continuous, $f(x_i) = y_i$ and $||f|| = ||\{y_n\}||$. Hence $Q: c \rightarrow C(X)$ is a linear isometry and

$$(P_1 \circ Q)(\{y_n\}) = P_1(\{y_n\}) = \{y_n\} .$$

Define $P: C(X) \to C(X)$ as $P = Q \circ P_1$. Since P_1 is onto and Q is an isometry then $||P|| = 1$ and P is a projection, for

$$P^2 = Q \circ P_1 \circ Q \circ P_1 = Q \circ P_1 = P .$$

Thus P is a projection of norm 1 from $C(X)$ onto $Q(c)$.

Combining this construction with Lemmas 1 and 2 we have:

THEOREM. *Let X be a compact Hausdorff space that contains an infinite sequence of distinct points that converge. Then the iterative test is not conclusive for $C(X)$.*

In each of the above, there is a linear selfmap for which the iterative test fails.

REFERENCE

1. S. B. Nadler, Jr., *A note on an iterative test of Edelstein*, Canad. Math. Bull., to appear.

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TEXAS A AND M UNIVERSITY
COLLEGE STATION, TX 77843
AND
PAN AMERICAN UNIVERSITY
EDINBURG, TX 78539