PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

PRODUCTS OF COMPACT SPACES WITH $bi-k$ AND RELATED SPACES

ANDREW J. BERNER

The main theorem of this paper characterizes $bi-k$ spaces as those spaces whose product with every compact spaces is sequentially k.

1. Introduction.

The classes of $bi-k$ spaces, countably $bi-k$ spaces and singly $bi-k$ spaces were studied in [5], and the class of sequentially k spaces was introduced in [3]. The following implications hold among these spaces, without the assumption of any separation axioms: $bi-k \rightarrow$ countably $bi-k \rightarrow$ singly $bi-k \rightarrow$ sequentially k. Also, all k spaces are sequentially k, and all Hausdorff sequentially k spaces are k spaces. (These classes will be defined at the end of this introduction.)

THEOREM 1.1. *The following are equivalent:*
(a) *X is a $bi-k$ space.*
(b) *$X \times Y$ is a singly $bi-k$ space for every compact Hausdorff space Y.*
(c) *$X \times Y$ is sequentially k for every compact space Y.*

This theorem is proved in § 2.

REMARK 1.2. Cohen [4] proved that the product of a k space with a (locally) compact Hausdorff space is a k space. Noble [6] showed this is false without the Hausdorff assumption, but in Noble's example, the product was a $bi-k$ space. Theorem 1.1 ($c \leftrightarrow a$) shows that the product of a k space with a compact space need not even be sequentially k.

REMARK 1.3. Michael [5] has asked whether the product of two countably $bi-k$ spaces must be countably $bi-k$. Examples have been given showing it is consistent with Zermelo-Fraenkel set theory that this is false. Theorem 1.1 can be used to give an absolute counterexample. All we need is a countably $bi-k$ space that is not $bi-k$. Let X be the subspace of the product of uncountably many copies of $\{0, 1\}$ consisting of points that are 1 on only countably many coordinates (i.e., a Σ-product centered at the point all of whose coordinates are 0). Arhangel'skii proved that this space is countably $bi-k$ (in fact, countably bi-sequential) but not $bi-k$ [2]. Thus, by Theorem 1.1, there is a compact Hausdorff space Y_1 and a compact T_1 space Y_2 such that $X \times Y_1$ is not singly $bi - k$, thus not countably $bi - k$, and $X \times Y_2$ is not even sequentially k.

DEFINITION 1.4. [3] If S is a subset of a topological space X and (S_i) is a nested sequence of subsets of X, then (S_i) is an S-sequence if whenever (s_i) is a sequence of points with $s_i \in S_i$ for each i, then (s_i) has an accumulation point in S.

DEFINITION 1.5. A space X is a $bi - k$ space if whenever $\mathscr{F}$ is a filter base containing the open sets around a point $p \in X$, there is a compact set $S \subset X$ and a nested sequence of sets (S_i) such that (S_i) meshes with $\mathscr{F}$ and (S_i) is an S-sequence.

DEFINITION 1.6. A space X is a countably $bi - k$ space if whenever (F_i) is a nested sequence of sets accumulating at a point p (i.e., $p \in \mathrm{cl}(F_i)$ for each i) there is a nested sequence of sets (S_i) accumulating at p and a compact set S such that $S_i \subset F_i$ for each i and (S_i) is an S-sequence.

DEFINITION 1.7. A space X is a singly $bi - k$ space if whenever $p \in \mathrm{cl}(F)$, there is a compact set S and a nested sequence of sets (S_i) accumulating at p such that $S_i \subset F$ for each i and (S_i) is an S-sequence.

DEFINITION 1.8. A space X is sequentially k if whenever a set F is not closed there is a point $p \in \mathrm{cl}(F) - F$, a compact set S and a nested sequence of sets (S_i) accumulating at p such that $S_i \subset F$ for each i and (S_i) is an S-sequence.

2. **Proof of Theorem 1.1.** In [2], Michael proved that a space X is countably $bi - k$ if and only if $X \times I$ is singly $bi - k$ (where I is the unit interval). The heart of the proof (in one direction) involves coding a bad nested sequence (S_i) of subsets of X (i.e., a witnessing sequence to the statement 'X is not countably $bi - k$') as a single bad subset of $X \times I$. This idea of coding is hinted at in the following proof of Theorem 1.1.

$$a \longrightarrow b \quad \text{and} \quad a \longrightarrow c:$$

The product of two (or even countably many) $bi - k$ spaces is $bi - k$ [5]. Since compact spaces are $bi - k$, the product of a $bi - k$ space and a compact space is $bi - k$, and thus singly $bi - k$ and sequentially k.

$$\text{not } a \longrightarrow \text{not } b \quad \text{and} \quad \text{not } a \longrightarrow \text{not } c:$$

Both implications make use of the following construction.

Suppose X is not $bi-k$. Then there is a point $p \in X$ and a filter base $\mathscr{F}$ of subsets of X such that $\mathscr{F}$ contains the open sets around p, but there is no compact $S \subset X$ and nested sequence of sets (S_i) such that (S_i) meshes with $\mathscr{F}$ and is an S-sequence. Thus, in particular, there is an $F \in \mathscr{F}$ such that $p \notin F$ and therefore if $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}$, then $F_1 \cap F_2 - \{p\} \neq \varnothing$.

Define a base for a new topology on X as follows. If $x \in X - \{p\}$, then $\{x\}$ is open, and if $F \in \mathscr{F}$, then $F \cup \{p\}$ is a neighborhood of p. This refines the original topology on X. Let X' be X with this new topology. Note that X' is completely regular.

Let $Y_1 = \beta(X')$, the Stone-Čech compactification of X' (actually, any Hausdorff compactification will do), and let Y_2 be the one point compactification of X'. Note that Y_2 is a T_1 space, but is definitely not Hausdorff.

Claim 1. $X \times Y_1$ is not singly $bi-k$.

Proof. Let $C = \{(x, x): x \in X - \{p\}\}$. C is a subset of $X \times X' \subset X \times Y_1$. If $U \times V$ is a basic open set around (p, p), then U and $V \cap X'$ are both in $\mathscr{F}$, so $U \cap V \cap X' - \{p\} \neq \varnothing$. Thus $(U \times V) \cap C \neq \varnothing$, i.e., $(p, p) \in \mathrm{cl}\,(C) - C$. Suppose $X \times Y_1$ is singly $bi-k$. Then there is a compact set $K \subset X \times Y_1$, and a nested sequence (K_i) of subsets of C such that $(p, p) \in \mathrm{cl}\,(K_i)$ for each i, and (K_i) is a K-sequence. Then, if $\pi_X: X \times Y_1 \rightarrow X$ is the projection map, the sequence $(\pi_X(K_i))$ is a $\pi_X(K)$-sequence and $\pi_X(K)$ is compact in X. Suppose $F \in \mathscr{F}$. Then there is an open set $V \subset Y_1$ such that $V \cap X' = F \cup \{p\}$. But then for each i, there is an x_i such that $(x_i, x_i) \in K_i \cap (X \times V)$. Then $x_i \in F$, so $\pi_x(K_i) \cap F \neq \varnothing$. Thus $(\pi_x(K_i))$ is a $\pi_X(K)$-sequence meshing with $\mathscr{F}$. This violates the choice of $\mathscr{F}$, so $X \times Y_1$ is not singly $bi-k$.

Claim 2. $X \times Y_2$ is not sequentially k.

Proof. Let $Y_2 = X' \cup \{\alpha\}$.

Let $C = (X \times \{\alpha\}) \cup \{(y, x): y \in \mathrm{cl}_X\,(\{x\})$ and $x \in X' - \{p\}\}$. (Nobody said X was a T_1 space!) As in the proof of Claim 1, $(p, p) \in \mathrm{cl}\,(C) - C$. Suppose $(x, y) \in \mathrm{cl}\,(C) - C$. Then $y \neq \alpha$. Suppose $y \neq p$. Then since $(x, y) \notin C$, it follows that $x \notin \mathrm{cl}_X\,(\{y\})$ so there is an open (in X) set U such that $x \in U$ but $y \notin U$. But then $U \times \{y\}$ is open (in $X \times Y_2$) and $(U \times \{y\}) \cap C = \varnothing$. Thus $y = p$.

Could $X \times Y_2$ be sequentially k? If so, then there is a point $(z, p) \in \mathrm{cl}\,(C) - C$, a compact set $K \subset K \times Y_2$ and a sequence (K_i)

such that $(z, p) \in \mathrm{cl}(K_i)$ and $K_i \subset C$ for each i, and (K_i) is a K-sequence. Again, let $\pi_X \colon X \times Y_2 \to X$ be the projection map. Let $D_i = \{x\colon$ there is a $y \in \pi_X(K_i)$ such that $y \in \mathrm{cl}_X(\{x\})\}$. Suppose $x_i \in D_i$ for each i. There is, for each i, a point $y_i \in \pi_X(K_i) \cap \mathrm{cl}_X(\{x_i\})$. Since $(\pi_X(K_i))$ is a $\pi_X(K)$-sequence, the sequence (y_i) has an accumulation point $k \in \pi_X(K)$. But, since any open set containing y_i also contains x_i, k is an accumulation point of (x_i). Thus (D_i) is a $\pi_X(K)$-sequence.

Suppose $F \in \mathscr{F}$. Since $F \cup \{p\}$ is open in Y_2, for each i $(X \times (F \cup \{p\})) \cap K_i \neq \emptyset$. But $\alpha \notin F \cup \{p\}$ so there is a point $(y, x) \in K_i$ with $x \in F$ and $y \in \mathrm{cl}_X(\{x\})$. Thus $x \in D_i \cap F$. Therefore D_i is a $\pi_X(K)$-sequence meshing with $\mathscr{F}$, and $\pi_X(K)$ is compact. As in the case of Claim 1, this contradicts the choice of $\mathscr{F}$, so $X \times Y_2$ is not sequentially-k.

REFERENCES

1. A. V. Arhangel'skii, *Bicompact sets and the topology of spaces*, Trudy Moskov. Mat. Obsc., **13** (1965), 3-55 (Trans, Moscow Math. Soc., (1965), 1-62).
2. ———, *The frequency spectrum of a topological space and the classification of spaces*, Soviet Math. Dokl., **13** (1972), 1185-1189.
3. A. J. Berner, *Spaces defined by sequences*, Proc. Amer. Math. Soc., **55** (1976), 193-200.
4. D. E. Cohen, *Spaces with weak topology*, Quart. J. Math. Oxford Ser., (**2**), **5** (1954), 77-80.
5. E. A. Michael, *A quintuple quotient quest*, General Topology Appl., **2** (1972), 91-138.
6. N. Noble, *Two examples on preimages of metric spaces*, Proc. Amer. Math. Soc., **36** (1972), 586-590.

Received January 15, 1976 and in revised form October 27, 1976.

UNIVERSITY OF WISCONSIN
MADISON, WI 53706