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BICONTRACTIVE PROJECTIONS AND REORDERING OF L_p-SPACES

S. J. BERNAU AND H. ELTON LACEY

On a Banach space we call a projection, P, bicontractive, if $||P|| \leqq 1$ and $||I - P|| \leqq 1$. In this paper we completely describe bicontractive projections on an L_p-space $(1 \leqq p < \infty)$ by showing that for every such bicontractive projection P, $2P - I$ is an involutive linear isometry. Duality then gives the same result for pre-dual L_1-spaces (in particular for M-spaces). The analysis of bicontractive projections is used, with $p \neq 2$, to describe all Banach lattices which are linearly isometric to an L_p-space.

Such projections on $L_p(\mu)$, when $1 < p < \infty$, $p \neq 2$, and μ is a probability measure, have been considered by Byrne and Sullivan [2]. Their analysis gave the basic result, that $2P - I$ is an isometry. Their methods are different from ours and depend heavily on Lamperti's description [6] of isometries of L_p-spaces; and their approach is weighted much more towards independence of sub σ-algebras rather than the isometry property. Some minor changes in the formulation of their results were made later in Byrne's 1972 Ph. D dissertation at the University of Pittsburg. Our approach relies on our earlier complete description [1] of contractive projections on an L_p-space. We include, in § 3, a rapid survey of some of the Byrne, Sullivan results where their approach is different and outline very simple deductions of their results from ours.

The question of Banach lattice orderings of L_p, under the usual norm, have been considered, with $1 \leqq p < \infty$ and $p \neq 2$, for the separable case by Lacey and Wojtaszczyk [5]. Their results also depend on the Lamperti isometry results and crucially on separability. Our analysis uses our previous discussion of contractive and bicontractive projections and gives a complete generalization of their work.

Throughout the paper we assume $1 \leqq p < \infty$ and $p \neq 2$. We will write $L_p = L_p(X, \Sigma, \mu)$ for the standard real or complex L_p space determined by a set X, a σ-ring, Σ, of subsets of X and a measure μ on Σ. If $f \in L_p$, $S(f) = \{t \in X: f(t) \neq 0\}$, as in [1] the ambiguity of a set of measure zero is irrelevant. Where our results are true for either choice of scalar field the field will not be specified. Where the scalar field is specified the result will be true only for the specified choice. The case $p = 2$ is omitted because the theorems we prove are all trivially true, or trivially false, in this case.

2. **Bicontractive projections.** In this section we prove the following:

THEOREM 2.1. *If P is a bicontractive projection on L_p, then $U = 2P - 1$ is an isometric involution on L_p.*

The proof of this theorem will follow from the equivalent result.

THEOREM 2.2. *If P is a bicontractive projection on L_p, $f = Pf$, and $g = (I - P)g$, then $||f + g|| = ||f - g||$.*

The equivalence of these two is based on the observations $2P - I = P - (I - P)$ and for $f = Pf$, $g = (I - P)g$, $f - g = (2P - I)(f + g)$. We obtain these results in a series of technical lemmas.

LEMMA 2.3. *If P is a bicontractive projection on L_p and J denotes the band projection on $\mathscr{R}(P)^{\perp\perp}$* (the band generated by the range of P), *then $P = PJ = JP$.*

Proof. By [1, Theorem 3.5] this is true, if P is merely contractive, for $p \neq 1$. We assume then that $p = 1$ and put $A = P - PJ$. Suppose $x \perp \mathscr{R}(P)$

$$||x|| + ||Ax|| = ||x - Ax|| = ||x - Px|| \leqq ||x|| ,$$

because P is bicontractive. Hence $Ax = 0$. Thus $A(I - J) = 0$ and $A = A(I - J) + AJ = 0$.

Taking Lemma 2.3 and the uniqueness clause of [1, Theorem 3.5] into account we conclude that for all p, $1 \leqq p < \infty$, $p \neq 2$, if $f = Pf$ and $h \in f^{\perp\perp}$, then

$$Ph = f\mathscr{E}(\Sigma_0, |f|^p\mu)(h/f) ,$$

where $\Sigma_0 = \{S(f): f \in \mathscr{R}(P)\}$ is the sub σ-ring of Σ consisting of supports of functions in $\mathscr{R}(P)$ and $\mathscr{E}(\Sigma_0, |f|^p\mu)$ is the conditional expectation determined by the finite measure $|f|^p\mu$ and the sub σ-ring Σ_0. It follows from this that if $f = Pf$ or if $Pf = 0$ (so that $f = (I - P)f$) and J_f is the band projection on $f^{\perp\perp}$, then $PJ_f = J_fP$. (Again this is automatic for $p > 1$ by [1, Lemma 2.3].)

LEMMA 2.4. *Let P be a bicontractive projection on L_p, suppose $f = Pf$, $g = (I - P)g$, and $J_f = J_g$. If $\Sigma_f = \{S(h): h \in f^{\perp\perp} \cap \mathscr{R}(P)\}$, $\Sigma_g = \{S(k): k \in g^{\perp\perp} \cap \mathscr{R}(I - P)\}$ then the sub σ-rings Σ_f, Σ_g are equal.*

Proof. Since $PJ_f = J_fP$ we have $\Sigma_f = \{A \cap S(f): A \in \Sigma_0\}$ and similarly for Σ_g. If $S(h) \subset S(f)$ and $h = Ph$, then $P_h = J_hP = PJ_h$ as noted above. Hence $(I - P)J_hg = J_h(I - P)g = J_hg$ so that $S(h) = S(J_hg) \in \Sigma_g$. Similarly $\Sigma_g \subset \Sigma_f$ and equality follows.

Now we start on the proof of Theorem 2.2. Let $f \in \mathscr{R}(P)$, $g \in \mathscr{R}(I - P)$. Since J_f, J_g commute with each other and with P we have $J_gf \in \mathscr{R}(P)$, $J_fg \in \mathscr{R}(I - P)$. Since $\|f + g\| = \|f - g\|$ if and only if $\|J_gf + J_fg\| = \|J_gf - J_fg\|$ we may, and will, assume that $J_g = J_f$. Writing Σ_1 for $\Sigma_f = \Sigma_g$ (Lemma 2.4) we have

$$PJ_fh = f\mathscr{E}(\Sigma_1, |f|^p\mu)(h/f)$$

and

$$(J_f - PJ_f)h = g\mathscr{E}(\Sigma_1, |g|^p\mu)(h/g) \quad (h \in L_p) .$$

LEMMA 2.5. *With notation as in Lemma* 2.4, $(g/f)^2$ *is* Σ_1-*measurable, and so is* $|g/f|$.

Proof. Suppose first that $1 \leqq p < 2$. Since $Pg = 0$ we have $f\mathscr{E}(\Sigma_1, |f|^p\mu)(g/f) = 0$; but $S(g) = S(f) \in \Sigma_1$ so $\mathscr{E}(\Sigma_1, |f|^p\mu)(g/f) = 0$. If $A \in \Sigma_1$

$$\int_A (|f|/|g|)^p(g/f) \cdot |g|^p d\mu = \int_A (g/f) \cdot |f|^p d\mu = 0 .$$

This gives $\mathscr{E}(\Sigma_1, |g|^p\mu)[(|f|/|g|)^p(g/f)] = 0$. Since $1 \leqq p < 2$,

$$|(|f|/|g|)^p(g^2/f)| = |f|^{p-1}|g|^{2-p} \leqq (|f| + |g|)^{p-1+2-p} = |f| + |g| \in L_p .$$

We conclude that $(J_f - PJ_f)((|f|/|g|)^p(g^2/f) = 0$ so that $((|f|/|g|)^p(g^2/f)) \in \mathscr{R}(PJ_f)$. In particular $(|f|/|g|)^p(g^2/f^2)$ is Σ_1-measurable.

If $2 < p < \infty$ we consider the bicontractive projection $(PJ_f)^*$ on $L_{p'}$ where $1/p + 1/p' = 1$. By [1, Lemma 2.2] $f^* = |f|^{p-1} \operatorname{sgn} \bar{f} \in \mathscr{R}(PJ_f^*)$ and $g^* = |g|^{p-1} \operatorname{sgn} \bar{g} \in \mathscr{R}((J_f - PJ_f)^*)$. Since the map $h \mapsto |h|^{p-1} \operatorname{sgn} \bar{h}$ does not change supports we conclude from what we have just proved that $(g^*/f^*)^2$ is Σ_1-measurable, hence, so is $(g/f)^2$.

Our lemma follows:

LEMMA 2.6. *With notation as in Lemma* 2.4, *define* $B = \{t \in S(f): 0 \leqq \arg(g/f) < \pi\}$ *and* $B' = S(f) \sim B$.

(i) *The map* $h \mapsto h(\chi_B - \chi_{B'})$ *is an involutive linear isometry of* $\mathscr{R}(PJ_f)$, *onto* $\mathscr{R}(J_f - PJ_f)$.

(ii) *If* $h \in \mathscr{R}(PJ_f)$, $\|h\chi_B\|^p = 1/2\|h\|^p = \|h\chi_{B'}\|^p$, *and* $P(h\chi_B) = (1/2)h = P(h\chi_{B'})$.

Proof. Consider $g(\chi_B - \chi_{B'})/f$. We have $|g(\chi_B - \chi_{B'})/f| = |g/f|$ which is Σ_1-measurable by Lemma 2.5. By definition of B, $\arg g(\chi_B - \chi_{B'})/f \in [0, \pi)$ so Σ_1-measurability of $g(\chi_B - \chi_{B'})/f$ follows from that of g^2/f^2 (Lemma 2.5 again). Hence $g(\chi_B - \chi_{B'})/f$ is Σ_1-measurable so that $P(g(\chi_B - \chi_{B'})) = f\mathscr{E}(\Sigma_1, |f|^p\mu)(g(\chi_B - \chi_{B'})/f) = g(\chi_B - \chi_{B'})$.

Now if $h \in \mathscr{R}(J_f - PJ_f)$, h/g is Σ_1-measurable and $h(\chi_B - \chi_{B'})/f = (h/g)(g(\chi_B - \chi_{B'})/f)$ is Σ_1-measurable. It follows as above that $h(\chi_B - \chi_{B'}) \in \mathscr{R}(PJ_f)$. Similarly, if $h \in \mathscr{R}(PJ_f)$, $h(\chi_B - \chi_{B'}) \in \mathscr{R}(J_f - PJ_f)$. This proves (i).

For (ii) take $h \in \mathscr{R}(PJ_f)$; by (i), $h(\chi_B - \chi_{B'}) \in \mathscr{R}(J_f - PJ_f)$ so $P(h\chi_B) = PJ_f(h\chi_B) = P(h\chi_{B'}) = (1/2)P(h(\chi_B + \chi_{B'})) = (1/2)Ph$.

Apply this to the special case when $h = \chi_A f$ with $A \in \Sigma_1$. We have

$$\begin{aligned}\int_{A\cap B} |f|^p d\mu &= \int_A (h\chi_B/f)|f|^p d\mu \\ &= \int_A \mathscr{E}(\Sigma_1, |f|^p d\mu)(h\chi_B/f)|f|^p d\mu \\ &= \int_A (P(h\chi_B)/f)|f|^p d\mu \\ &= \frac{1}{2}\int_A (Ph/f)|f|^p d\mu \\ &= \frac{1}{2}\int_A |f|^p d\mu\,.\end{aligned}$$

Hence, $\int_B \chi_A |f|^p d\mu = \int_{B'} \chi_A |f|^p d\mu = 1/2 \int |f|^p d\mu$ $(A \in \Sigma_1)$. This extends to Σ_1-simple functions and hence to all elements of $L_1(\Sigma_1, |f|^p\mu)$. In particular, if $h \in \mathscr{R}(PJ_f)$, $|h/f| \in L_1(\Sigma_1, |f|^p\mu)$ and

$$||h\chi_B||^p = \int_B |h/f|^p |f|^p d\mu = ||h\chi_{B'}||^p = 1/2\,||h||^p\,.$$

Proof of Theorem 2.2. We apply Lemma 2.6. By (i), $h = f \pm g(\chi_B - \chi_{B'}) \in \mathscr{R}(PJ_f)$. By (ii)

$$\int_B |f \pm g|^p d\mu = ||h\chi_B||^p = ||h\chi_{B'}||^p = \int_{B'} |f \mp g|^p d\mu\,.$$

Hence

$$\begin{aligned}||f + g||^p &= \int_B |f + g|^p d\mu + \int_{B'} |f + g|^p d\mu \\ &= \int_{B'} |f - g|^p d\mu + \int_B |f - g|^p d\mu \\ &= ||f - g||^p\,.\end{aligned}$$

Our next lemma will be used in §§ 3 and 4.

LEMMA 2.7. *Let P be a bicontractive projection on L_p such that $f \in \mathscr{R}(P)$, $g \in \mathscr{R}(I-P)$ and $f^{\perp\perp} = g^{\perp\perp}$, and suppose $B \in \Sigma$ is chosen as in Lemma 2.6. If $h \in L_p$, and $S(h) \subset B$, then $h = 2\chi_B Ph$; while if $S(h) \subset B'$, $h = 2\chi_{B'}Ph$.*

Proof. By Lemma 2.6, $h - (\chi_B - \chi_{B'})Ph = (\chi_B - \chi_{B'})(I-P)h \in \mathscr{R}(P)$. Hence, $h - (\chi_B - \chi_{B'})Ph = P(h - (\chi_B - \chi_{B'})Ph) = Ph$ so that $h = 2\chi_B Ph$ as required. The case $S(h) \subset B'$ is similar.

We now answer a question raised in conversation with David Dean and Bill Johnson.

THEOREM 2.8. *Let X be a predual L_1-space and P a bicontractive projection on X; then $2P - I$ is an isometry on X.*

Proof. The dual space X^* of X is an L_1-space and the adjoint operator P^* is a bicontractive projection on X^*. By Theorem 2.2, $2P^* - I^*$ is an isometry on X^*. A routine computation shows that any linear operator whose adjoint is an isometry of X^* onto X^*, is itself an isometry. Our theorem is proved.

3. **The results of Byrne and Sullivan.** We first summarise the main definitions and results from [2]. For this discussion $1 < p < \infty$ $p \neq 2$ and μ is a probability measure.

An isometry U of L_p is *reduced* if for every $A \in \Sigma$ with $\mu A > 0$, there exists $E \in \Sigma$ such that $E \subset A$ and $S(U\chi_E) \neq E$ (meaning $\mu(E \Delta S(U\chi_E)) > 0$). A bicontractive projection is *total* if $\mathscr{R}(P)^{\perp\perp} = L_p = \mathscr{R}(I-P)^{\perp\perp}$, and *independent* if it is total and the σ-ring $\Sigma_1 = \{S(f) : f \in \mathscr{R}(P)\}$ and the ratio g/f, for some $f \in \mathscr{R}(P)$, $g \in \mathscr{R}(I-P)$, are independent for the measure $|f|^p\mu$. The theorems concerned follow.

(A) A total bicontractive projection is independent.

(B) The following are equivalent.

(a) There is a reduced reflection U with invariant subspace M.

(b) There is an independent bicontractive projection P with range M.

(c) There is a sub σ-ring, Σ_1, of Σ and a set $B \in \Sigma$ such that for every $E \in \Sigma$ there exist unique $A, C \in \Sigma_1$ (up to sets of measure zero) such that $E = (A \cap B) \cup (C \cap B')$. (One way to achieve uniqueness is to require that B satisfies the condition, if $A \in \Sigma_1$ and $\mu(A \cap B) = 0$ or $\mu(A \cap B') = 0$, then $\mu A = 0$.)

For (A) our analysis in § 2 applies directly. The totality hypothesis lets us choose $f \in \mathscr{R}(P)$, $g \in \mathscr{R}(I-P)$ such that $J_f = J_g = I$. Then we use Lemma 2.6 to find B such that $f(\chi_B - \chi_{B'}) \in \mathscr{R}(I-P)$ and check that Σ_1 and $\chi_B - \chi_{B'}$ $(= f(\chi_B - \chi_{B'})/f)$ are independent for $|f|^p\mu$.

To show that (b) implies (c) in (B) we use Lemma 2.7. We have f, g, B as above and take $h = f\chi_{E\cap B}(E \in \Sigma)$. Then $f\chi_{E\cap B} = h = 2\chi_B Ph$. Hence $A = S(Ph) \in \Sigma_1$ and $E \cap B = A \cap B$. Similarly $C = S(P(f\chi_{E\cap B'}))$ satisfies $E \cap B = C \cap B$.

To show that (a) implies (b) put $P = (1/2)(I + U)$. Since U is isometric and $U^2 = I$, P is a bicontractive projection with range M. If $g \perp \mathscr{R}(P)$ put $A = S(g)$ and find $E \in \Sigma$ such that $E \subset A$ and $S(U\chi_E) \neq E$. Then $\chi_E \in \mathscr{R}(P)^\perp \subset \mathscr{R}(I-P)$ and $U\chi_E = -\chi_E$. This gives $S(U\chi_E) = E$ contrary to our choice of E. We conclude that $L_p = \mathscr{R}(P)^{\perp\perp}$. Since U reduced implies $-U$ reduced we see also that $L_p = \mathscr{R}(I-P)^{\perp\perp}$; thus P is total and (b) follows from (A).

Finally for (c) implies (a) we can argue as in [2]. For $E \in \Sigma$ define $T(E) = (A \cap B') \cup (C \cap B)$ for $E = (A \cap B) \cup (C \cap B)$ with $A, C \in \Sigma_1$). Now set $\mu^*(E) = \mu(T(E))$ and let f^p be the Radon Nikodym derivative of μ^* with respect to μ. Then define U by $U\chi_E = f \cdot \chi_{T(E)}$ $(E \in \Sigma)$ and extend the definition to L_p in the obvious way. It is easy to check that U is an isometry. If $A \in \Sigma$ take $E = A \cap B$, then $T(E) \subset B'$ and $S(U\chi_E) = S(f\chi_{T(E)}) \subset B'$ so $S(U\chi_E) \neq E$.

4. Reordering of L_p. Here we consider the question, what are the vector-lattice orderings on L_p such that, under its usual norm, L_p is a Banach lattice. The real separable case has been considered by Lacey and Wojtaszczyk [5] who show that up to linear isometry and lattice isomorphism all such are obtained as $\tilde{L}_p(X, \Sigma, \mu) = L_p(A, \Sigma, \mu) \oplus L_p(B, \Sigma, E_p(2), \mu)$. Hence $\tilde{L}_p$ denotes L_p with its new Banach lattice ordering. The direct sum is in the sense of Banach lattices, and $A, B \in \Sigma$, $A \cap B = \varnothing$. Finally $L_p(B, \Sigma, E_p(2), \mu)$ is the $E_p(2)$ valued L_p-space on B, where $E_p(2)$ denotes R^2 with its natural L_p-norm but ordered by $(\xi_1, \xi_2) \geqq 0$ if and only if $\xi_1 + \xi_2 \geqq 0$ and $\xi_1 - \xi_2 \geqq 0$ (equivalently $\xi_1 \geqq |\xi_2|$). We shall show that, apart from measurability of A, B their result is true in general for real L_p. Also, for complex L_p spaces the natural complex Banach lattice structure is unique.

We now begin our analysis. We write, again $\tilde{L}_p$ to denote the space L_p with its usual norm and some vector lattice structure such that $\tilde{L}_p$ is a Banach lattice. Since L_p is weakly complete and c_0 is not [3], L_p contains no linearly isomorphic copy of c_0, and we see by a result of Meyer-Nieberg [7] that $\tilde{L}_p$ has order continuous norm. Hence every band M in $\tilde{L}_p$ has associated with it a natural band

projection P_M. (If $M = \{f\}$ we write P_f.) The letters P, Q with and without subscripts, will usually denote $\tilde{L}_p$-band projections. The letters J, K with or without subscripts will denote natural (i.e., L_p-) band projections. The symbol, $^\perp$, refers to disjointness or polar sets (bands) in L_p.

LEMMA 4.1. *Let P be an $\tilde{L}_p$-band projection, and suppose $M \subset \mathscr{R}(P)$, $M \neq \emptyset$; then, if J_M is the natural band projection on $M^{\perp\perp}$, $PJ_M = J_M P$.*

(In fact Lemma 4.1 is valid for any bicontractive projective if $p = 1$ and any contractive projection if $p > 1$.)

Proof. From the discussion preceding Lemma 2.4 $PJ_f = J_f P$ for all $f \in \mathscr{R}(P)$. Since the set of supports of elements of $\mathscr{R}(P)$ is a σ-ring [1, Lemma 3.1], the set $\{J_f : f \in \mathscr{R}(P)\}$ is upwards directed. (We only need a subspace of $\mathscr{R}(P)$, and the first paragraph of the proof of [1, Lemma 6.1] for this.) Since the norm in L_p is order continuous, J_M is the strong limit of a set of band projections each of which commutes with P. It follows that $PJ_M = J_M P$.

LEMMA 4.2. *Let P, Q be bicontractive projections on L_p such that $\mathscr{R}(Q) \subset \mathscr{R}(P)$, then $\mathscr{R}(P) \cap \mathscr{R}(Q)^{\perp\perp} \cap \mathscr{R}(I - P)^{\perp\perp} \subset \mathscr{R}(Q)$.*

Proof. Suppose $f \in \mathscr{R}(P) \cap \mathscr{R}(Q)^{\perp\perp} \cap \mathscr{R}(I - P)^{\perp\perp}$. By [1, Corollary 3.2] there exist $g \in \mathscr{R}(I - P)$ and $h \in \mathscr{R}(Q)$ such that $f \in g^{\perp\perp} \cap h^{\perp\perp}$. Since $PJ_f = J_f P$ we may assume $g^{\perp\perp} = f^{\perp\perp}$. Since $g \in \mathscr{R}(I - P) \subset \mathscr{R}(I - Q)$, $(I - Q)J_g = J_g(I - Q)$ so $J_f Q = Q J_f$ and we may also assume $h^{\perp\perp} = f^{\perp\perp}$.

Write $\Sigma_P = \{S(k) : k \in f^{\perp\perp} \cap \mathscr{R}(P)\}$ and $\Sigma_Q = \{S(k) : k \in f^{\perp\perp} \cap \mathscr{R}(Q)\}$. Clearly $\Sigma_Q \subset \Sigma_P$. By Lemma 2.4 $\Sigma_P = \Sigma_{I-P} \subset \Sigma_{I-Q} = \Sigma_Q$ so $\Sigma_P = \Sigma_Q$.

Now, since $f, h \in \mathscr{R}(P)$, $h^{\perp\perp} = f^{\perp\perp}$, and $h \in \mathscr{R}(Q)$, we have

$$\begin{aligned} f = Pf &= h\mathscr{E}(\Sigma_P, |h|^p\mu)(f/h) \\ &= h\mathscr{E}(\Sigma_Q, |h|^p\mu)(f/h) \\ &= Qf \end{aligned}$$

Our lemma is proved.

LEMMA 4.3. *There is a minimal $\tilde{L}_p$-band projection P such that $\mathscr{R}(P)^{\perp\perp} = L_p$.*

(Minimality is in the natural ordering of $\tilde{L}_p$-band projections.)

Proof. Let $\mathscr{S}$ denote the set of $\tilde{L}_p$-band projections P such that $\mathscr{R}(P)^{\perp\perp} = L_p$ and let $\mathscr{P}$ be a decreasing chain in $\mathscr{S}$. The infimum P_0, of $\mathscr{P}$ in the set of $\tilde{L}_p$-band projections has range $\bigcap \{\mathscr{R}(P): P \in \mathscr{P}\}$ Suppose $g \perp \mathscr{R}(P_0)$. Choose $P \in \mathscr{P}$ and put $h = g - Pg$. By [1, Corollary 3.2], using $\mathscr{R}(P)^{\perp\perp} = L_p$, there is $f \in \mathscr{R}(P)$ such that $f^{\perp\perp} = h^{\perp\perp}$. Hence $f \in \mathscr{R}(Q)^{\perp\perp} \cap \mathscr{R}(P) \subset \mathscr{R}(I - P)^{\perp\perp}$ ($Q \in \mathscr{P}$, $\mathscr{R}(Q) \subset \mathscr{R}(P)$). By Lemma 4.2, $f \in \bigcap \{\mathscr{R}(Q): Q \in \mathscr{P}\} = \mathscr{R}(P_0)$ and $g \in f^{\perp} = h^{\perp} = (g - Pg)^{\perp}$. Hence,

$$||g||^p \geqq ||Pg||^p = ||Pg - g||^p + ||g||^p ,$$

so that $g = Pg$ ($P \in \mathscr{P}$). Thus,

$$g \in \bigcap \{\mathscr{R}(P): P \in \mathscr{P}\} = \mathscr{R}(P_0) \perp g ,$$

and $g = 0$ so that $P_0 \in \mathscr{S}$. Zorn's lemma finishes our proof.

Now we must distinguish the real and complex cases.

LEMMA 4.4. *In the complex case, if $f, g \in L_p$ and $P_f g = 0$, then $f \perp g$.*

Proof. Let $h = J_g f$, $k = J_f g$. Then since J_f, J_g commute with P_f, P_g we have $h = P_f h$ and $P_f k = 0$. Thus $P_h k = 0$. Hence we may, and do assume that $f^{\perp\perp} = g^{\perp\perp}$. We have $f \in \mathscr{R}(P_f)$, $g \in \mathscr{R}(I - P_f)$ and $f^{\perp\perp} = g^{\perp\perp}$. By Lemma 2.6, there is a set $B \in \Sigma$, such that $f(\chi_B - \chi_{B'}) \in \mathscr{R}(I - P_f)$ (where $B' = S(f) \sim B$) and

$$\int_B |f|^p d\mu = \int_{B'} |f|^p d\mu = 1/2 \int |f|^p d\mu = 1/2 ||f||^p$$

Since P_f is an $\tilde{L}_p$-band projection,

$||f + f(\chi_B - \chi_{B'})||^p = ||f + if(\chi_B - \chi_{B'})||^p$. Now, by Lemma 2.6 again,

$||f + f(\chi_B - \chi_{B'})||^p = \int_B |2f|^p d\mu = 2^{p-1} ||f||^p$; and

$$\begin{aligned} ||f + if(\chi_B - \chi_{B'})||^p &= \int_B |(1 + i)f|^p d\mu + \int_{B'} |(1 - i)f|^p d\mu \\ &= 2^{p/2} ||f||^p . \end{aligned}$$

Since $p \neq 2$, we have $f = 0$. This proves our lemma.

COROLLARY 4.5. *In the complex case the minimal $\tilde{L}_p$-band projection P such that $\mathscr{R}(P)^{\perp\perp} = L_p$ is the identity on L_p, and every $\tilde{L}_p$-band is an L_p-band.*

LEMMA 4.6. *Let P be an $\tilde{L}_p$-band projection, then*

$\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$ is an $\tilde{L}_p$-band with band projection PK where K is the L_p-band projection on $\mathscr{R}(I-P)^{\perp\perp}$. In addition K is an $\tilde{L}_p$-band projection.

Proof. Since the result is trivial in the complex case, by Corollary 4.5, we assume we are working with real scalars. Since the norm in $\tilde{L}_p$ is order continuous and $\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$ is closed it is enough to show that $\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$ is a solid $\tilde{L}_p$-sublattice of $\tilde{L}_p$.

For this it is sufficient [5, Lemma 1] to show that if Q is an $\tilde{L}_p$-band projection such that $\mathscr{R}(Q) \subset \mathscr{R}(P)$, then $Q(\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}) \subset \mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$. Let J be the L_p-band projection on $\mathscr{R}(Q)^{\perp\perp}$. By Lemma 4.1 $JP = PJ$. Hence, if $x \in \mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$, we have $Jx = PJx = JPx \in \mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$, and by Lemma 4.2,

$$Qx = QJx = Jx \in \mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$$

Now $(I-P)K = I-P = K(I-P)$ so $PK = KP$ is a contractive projection with range $\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$. The uniqueness conditions of [1, Theorem 3.5], combined with Lemma 4.1, show that PK is the $\tilde{L}_p$-band projection on $\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$.

Finally $I-K = P-PK$ so that $I-K$ is an $\tilde{L}_p$-band projection and hence, so is K.

LEMMA 4.7. *Let P be a minimal $\tilde{L}_p$-band projection such that $\mathscr{R}(P)^{\perp\perp} = L_p$, and suppose $f, g \in \mathscr{R}(P)$ or $f, g \in \mathscr{R}(I-P)$, and $P_f g = 0$, then $f \perp g$.* (i.e., in the ranges of P and of $I-P$, $\tilde{L}_p$-disjointness implies L_p-disjointness.)

Proof. Let J, K be the L_p-band projections on $\mathscr{R}(P_f)^{\perp\perp}$ and $\mathscr{R}(I-P_f)^{\perp\perp}$ respectively. By Lemma 4.6, J, K are $\tilde{L}_p$-band projections. In particular J, K commute with all $\tilde{L}_p$-band projections.

Suppose first that $f, g \in \mathscr{R}(P)$ and consider

$$P_0 = P(I-K) + P_f K + P(I-J) .$$

The summands are $\tilde{L}_p$-band projections whose products in pairs are zero. Hence P_0 is an $\tilde{L}_p$-band projection with $\mathscr{R}(P_0) \subset \mathscr{R}(P)$. Suppose $x \in L_p$ and $x \perp \mathscr{R}(P_0)$. Since $P_f P_0 = P_f$ we have $\mathscr{R}(P_f) \subset \mathscr{R}(P_0)$ so $Jx = 0$ and $x = (I-J)x$. Let $y \in L_p$, since $PJ = JP$, $JPy = PJy \perp x$ and $(I-J)Py = P(I-J)y \in \mathscr{R}(P_0) \perp x$. Hence $x \in \mathscr{R}(P)^{\perp} = \{0\}$ and $x = 0$.

By minimality, $P_0 = P$, so that $JPK = PJK = P_fK$. Now we have $g \in \mathscr{R}(I - P_f) \cap \mathscr{R}(P)$ so $g = Kg = PKg$. Hence $Jg = JPKg = P_fKg = P_fg = 0$. Since $f = Jf$ we have $f \perp g$ as required.

If $f, g \in \mathscr{R}(I - P)$ then by Lemma 4.6, $J(I - P)$ is an $\tilde{L}_p$-band projection with range $\mathscr{R}(P_f)^{\perp\perp} \cap \mathscr{R}(I - P) = \mathscr{R}(P_f)^{\perp\perp} \cap \mathscr{R}(I - P) \cap \mathscr{R}(P)^{\perp\perp}$. Since $\mathscr{R}(P_f) \subset \mathscr{R}(I - P)$, Lemma 4.2 shows that $\mathscr{R}(P_f) = \mathscr{R}(P_f)^{\perp\perp} \cap \mathscr{R}(I - P)$. Hence $P_f = J(I - P)$. Since $g \in \mathscr{R}(I - P)$, $Jg = J(I - P)g = P_fg = 0$. Since $f = Jf$, we have $f \perp g$ as required;

For the rest of this section we consider a fixed minimal $\tilde{L}_p$-band projection P such that $\mathscr{R}(P)^{\perp\perp} = L_p$. We write K for the L_p-band projection on $\mathscr{R}(I - P)^{\perp\perp}$. By Lemma 4.6, K is also an $\tilde{L}_p$-band projection and by Corollary 4.5, $K = 0$ in the case of complex scalars.

LEMMA 4.8. *If $f, g \in \mathscr{R}(P)$ or if $f, g \in \mathscr{R}(I - P)$, then f and g are disjoint in L_p if and only if they are disjoint in $\tilde{L}_p$. Consequently, $P_f = J_fP$ or $P_f = J_f(I - P)$ according as $f \in \mathscr{R}(P)$ or $f \in \mathscr{R}(I - P)$ and in either case J_f is an $\tilde{L}_p$-band projection.*

Proof. We consider $\mathscr{R}(P)$. By Lemma 4.7, the norm in $\mathscr{R}(P)$ is p-additive for the $\tilde{L}_p$-ordering; i.e., if, $x, y \in \mathscr{R}(P)$ and $P_xy = 0$ then

$$||x + y||^p = ||x||^p + ||y||^p , \quad \text{(because } P_xy = 0 \text{ implies } x \perp y).$$

By the well known characterisation of L_p-spaces, $\mathscr{R}(P)$, with its $\tilde{L}_p$-order, is linearly isometric and lattice isomorphic to some L_p space [4, § 15, Theorem 3]. Now if $f, g \in \mathscr{R}(P)$ and $f \perp g$ then $||f + g||^p + ||f - g||^p = 2(||f||^p + ||g||^p)$ and the equality condition for Clarkson's inequality [6, Corollary 2.1] shows that f, g are $\tilde{L}_p$-disjoint. Continuing with $f \in \mathscr{R}(P)$ we see that J_f commutes with P_f and P so that $\mathscr{R}(P_f) \supset \mathscr{R}(P) \cap f^{\perp\perp}$ and $\mathscr{R}(P - P_f) \supset \mathscr{R}(P) \cap f^{\perp}$. It follows that $P_f = PJ_f$, that $\mathscr{R}(P_f)^{\perp\perp} = f^{\perp\perp}$ and, by Lemma 4.6, that J_f is an $\tilde{L}_p$-band projection.

The same argument works for $f, g \in \mathscr{R}(I - P)$.

LEMMA 4.9. *Let $f = PKf$, then there is a set $B_f \in \Sigma$ such that $f^{\perp\perp}$ with its $\tilde{L}_p$-ordering is linearly isometric and lattice isomorphic to $L_p(B_f, \Sigma, E_p(2), \mu)$.*

Proof. By Lemma 4.8, $f^{\perp\perp}$ is an $\tilde{L}_p$-band. Since $f \in \mathscr{R}(I - P)^{\perp\perp}$ we can find $g \in \mathscr{R}(I - P)$ such that $g^{\perp\perp} = f^{\perp\perp}$. Define $B_f = \{t \in S_f: 0 \leqq \arg g/f < \pi\}$; $\Sigma_1 = \Sigma_f = \Sigma_g$ as in Lemmas 2.4, 2.5, 2.6, 2.7; write f_1, g_1 for the $\tilde{L}_p$-absolute values of f, g respectively, and for $h \in f^{\perp\perp}$

define

$$T_f h = \chi_B \cdot (\operatorname{sgn} \bar{f}_1 \cdot Ph \cdot u_1 + \operatorname{sgn} \bar{g}_1 \cdot (I - P)h \cdot u_2)$$

with $u_1 = (1, 1)$, $u_2 = (1, -1) \in E_p(2)$. Since $K = 0$ in the complex case we can assume we are in the real situation. Hence $\operatorname{sgn} \bar{f}_1 + \pm 1$, $\operatorname{sgn} \bar{g}_1 = \pm 1$ and we have, using Lemma 2.6 as in the proof of Theorem 2.2.

$$\begin{aligned}
&\int |T_f h|^p d\mu \\
&= \int_B [|\operatorname{sgn} \bar{f}_1 \cdot Ph + \operatorname{sgn} \bar{g}_1 \cdot (I-P)h|^p + |\operatorname{sgn} \bar{f}_1 \cdot Ph - \operatorname{sgn} \bar{g}_1 (I-P)h|^p] d\mu \\
&= \int_B [|Ph + (I - P)h|^p + |Ph - (I - P)h|^p] d\mu \\
&= \int_B |Ph + (I - P)h|^p d\mu + \int_{B'} |Ph + (I - P)h|^p d\mu \\
&= \|h\|^p .
\end{aligned}$$

Hence T_f is an isometry of $f^{\perp\perp}$ into $L_p(B_f, \Sigma, E_p(2), \mu)$.

If $h \in f^{\perp\perp}$ and h is $\tilde{L}_p$-positive, then Ph and $(I - P)h$ are $\tilde{L}_p$-positive. Since $\mathscr{R}(P)$ and $\mathscr{R}(I - P)$ are abstract L_p-spaces and f_1, g_1 are $\tilde{L}_p$-positive, it follows as in the proof of [1, Theorem 4.1] that Ph and $(I - P)h$ are $\tilde{L}_p$-positive if and only if $\operatorname{sgn} \bar{f}_1 \cdot Ph$ and $\operatorname{sgn} \bar{g}_1 \cdot (I - P)h$ are positive in L_p. Thus T_f and T_f^{-1} are positive. To complete the proof it is sufficient to show that T_f is onto.

Suppose $u \in L_p(B_f, \Sigma, E_p(2), \mu)$, then $u = \operatorname{sgn} \bar{f}_1 \cdot h_1 u_1 + \operatorname{sgn} \bar{f}_2 \cdot h_2 u_2$ with $h_1, h_2 \in L_p(B_f, \Sigma, \mu)$.

Since $S(h_1) \cup S(h_2) \subset B_f$, Lemma 2.7 gives $h_1 = 2\chi_B Ph_1$, $h_2 = 2\chi_B Ph_2$. Put, $g = 2Ph_1 + 2(\chi_B - \chi_{B'})Ph_2$, then by Lemma 2.6, $(\chi_B - \chi_{B'})Ph_2 \in \mathscr{R}(I - P)$ so that $T_f g = u$. Our lemma is proved.

THEOREM 4.11. *There are subsets A, B of X such that any σ-finite subset of $A \cap B$ has measure zero and $\tilde{L}_p$ is linearly isometric and lattice isomorphic to $L_p(A, \Sigma, \mu) \oplus L_p(B, \Sigma, E_p(2), \mu)$. In the complex case $A = X$ and $B = \varnothing$.*

Proof. $\tilde{L}_p$ decomposes into complementary $\tilde{L}_p$-bands $\mathscr{R}(I - P)^\perp$ and $\mathscr{R}(I - P)^{\perp\perp}$. Since $\mathscr{R}(I - P)^\perp \subset R(P)$, the $\tilde{L}_p$-order on $\mathscr{R}(I - P)^\perp$ is that of an abstract L_p-space and $\tilde{L}_p$- and L_p-disjointness coincide in $\mathscr{R}(I - P)^\perp$.

Choose in $\mathscr{R}(I - P)^\perp$ a maximal $\tilde{L}_p$-disjoint set $\{h_\gamma : \gamma \in \Gamma\}$ of $\tilde{L}_p$-positive elements, set $A = \bigcup \{S(h_\gamma) : \gamma \in \Gamma\}$. Then as in the proof of [1, Theorem 4.1] the map $f \mapsto \sum_{\gamma \in \Gamma} \operatorname{sgn} \bar{h}_\gamma \cdot f$ is a linear isometry

and lattice isomorphism of $\mathscr{R}(I-P)^{\perp}$ with its $\tilde{L}_p$-order to $\mathscr{R}(I-P)^{\perp} = L_p(A, \Sigma, \mu)$ with its natural order. (σ-finiteness of supports of integrable functions ensures that the summations are over countable sets and that all the relevant sums converge.)

For the $\tilde{L}_p$-band, $\mathscr{R}(I-P)^{\perp\perp}$ we choose a maximal $\tilde{L}_p$-disjoint subset $\{f_\delta : \delta \in \Delta\}$ in $\mathscr{R}(P) \cap \mathscr{R}(I-P)^{\perp\perp}$ and apply Lemma 4.10 to get sets $B_\delta = B_{f_\delta} \subset S(f_\delta)$ and isometric isomorphisms $T_\delta : f_\delta^{\perp\perp} \to L_p(B_\delta, \Sigma, E_p(2), \mu)$. Then we put $B = \bigcup_x \{B_\delta : \delta \in \Delta\}$ and check that $f \mapsto \sum_{\delta \in \Delta} T_\delta f$ is a linear isometry and lattice isomorphism of $\mathscr{R}(I-P)^{\perp\perp}$ with its $\tilde{L}_p$-ordering onto $L_p(B, \Sigma, E_p(2), \mu)$.

This shows that $\tilde{L}_p$ is linearly isometric and lattice isomorphic to $L_p(A, \Sigma, \mu) \oplus L_p(B, \Sigma, E_p(2), \mu)$ as claimed. In the complex case $P = I$ by Corollary 4.5.

Suppose $D \in \Sigma$, $D \subset A \cap B$ and $\mu(D) < \infty$, then $\chi_D \in L_p$. Because $D \subset A$, $\chi_D \in \mathscr{R}(I-P)$ and because $D \subset B$, $\chi_D \in \mathscr{R}(I-P)^{\perp\perp}$. Thus $\chi_D = 0$, $\mu(D) = 0$.

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UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TX 78712