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UNIMODALITY OF THE LÉVY SPECTRAL FUNCTION

CAROL ALF AND THOMAS A. O'CONNOR

A. Ya. Khinchin proved that if Φ and Ψ are characteristic functions and $\Phi(t) = t^{-1}\int_0^t \Psi(u)du$, then the distribution function of Φ is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. A similar theorem is proved here for logarithms of infinitely divisible characteristic functions and their Lévy spectral functions.

Suppose $\Phi(t)$ is a characteristic function (ch. f) of a distribution function (df), F, so that $\Phi(t) = \int_R e^{ixt}dF(x)$. An application of Bochner's theorem (see [2]) shows that $\tilde{\Phi}(t) = t^{-1}\int_0^t \Phi(u)du$ is also a ch. f. Khinchin proved that $\tilde{\Phi}$ is a ch. f by constructing its df. In fact, he showed that a ch. f is of the form $\tilde{\Phi}$ if and only if its df is unimodal at 0; that is, the df is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. We shall prove a "unimodal theorem" for the function $\tilde{\phi}(t) = t^{-1}\int_0^t \phi(u)du$ under the assumptions that $\Phi(t)$ is infinitely divisible and $\phi(t) = \ln \Phi(t)$. Johansen's characterization of infinitely divisible ch. fs. ([1], Theorem 2) insures that $\tilde{\phi}$, defined above, may also be written $\tilde{\phi}(t) = \ln \Psi(t)$, for some infinitely divisible ch. f Ψ, and hence provided the motivation for our work. To begin with, we state Lévy's form of infinitely divisible ch. fs. (See [2].)

THEOREM 1. *A* ch. f Φ *is infinitely divisible if and only if* $\phi(t) = \ln \Phi(t)$ *may be uniquely represented as*

$$\phi(t) = i\mu t - \sigma^2 t^2 + \int_R\left(e^{ixt} - 1 - \frac{ixt}{1+x^2}\right)dM(x) \tag{1}$$

where $\mu \in R$, $\sigma^2 \geqq 0$, *and the function* M *has the following properties*:

(i) M *is defined on* $R\backslash\{0\}$

(ii) M *is nondecreasing on* $(-\infty, 0)$ *and on* $(0, +\infty)$ *and is right continuous*

(iii) $M(-\infty) = 0 = M(+\infty)$

(iv) $\int_{(-\varepsilon,\varepsilon)} x^2 dM(x)$ *is finite for all* $\varepsilon > 0$.

When (1) is in force, M and (μ, σ^2, M) are respectively called the Lévy spectral function and the Lévy triple of Φ. Moreover, every function which satisfies (i)-(iv) is a Lévy spectral function of some infinitely divisible ch. f. The main result of this article is Theorem 2 below; two preliminary lemmas are proven first.

LEMMA 1. *For every Lévy spectral function, M, the following relations hold*:

$$\text{(i)} \qquad \lim_{x\to+\infty} x\int_x^{+\infty} \frac{dM(z)}{z} = 0 = \lim_{x\to-\infty} x\int_{-\infty}^{x} \frac{dM(z)}{z}$$

$$\text{(ii)} \qquad \lim_{x\to0+} x^3\int_x^{+\infty} \frac{dM(z)}{z} = 0 = \lim_{x\to0-} x^3\int_{-\infty}^{x} \frac{dM(z)}{z}\,.$$

Proof. It is known that to each Lévy spectral function, M, there exists a df, G, and nonneagative number c such that

$$(2) \qquad M(x) = \begin{cases} c\int_{-\infty}^{x} u^{-2}(1+u^2)dG(u) & \text{if} \quad x<0 \\ -c\int_{x}^{+\infty} u^{-2}(1+u^2)dG(u) & \text{if} \quad x>0\,. \end{cases}$$

Then, according as $x>1$ or $0<x<1$, we have $x\int_x^{+\infty} u^{-1}dM(u) \leqq 2cx\int_x^{+\infty} u^{-1}dG(u)$ or $x^3\int_x^{+\infty} u^{-1}dG(u) \leqq 2cx\int_x^{+\infty} u^{-1}dG(u)$. Similar statements hold for negative x. Now, if we apply Lemma 4.5.1 of [2] to the integrals involving G, the assertions of Lemma 1 follow at once.

LEMMA 2. *Let M_1 and M_2 be two Lévy spectral functions and assume they are related by*

$$(3) \qquad M_2(x) = \begin{cases} -\int_{-\infty}^{x}\int_{-\infty}^{y} \frac{dM_1(z)}{z}dy & \textit{if} \quad x<0 \\ -\int_{x}^{+\infty}\int_{y}^{+\infty} \frac{dM_1(z)}{z}dy & \textit{if} \quad x>0\,. \end{cases}$$

Suppose $\phi(t) = i\mu t - \sigma^2t^2 + \int_R (e^{ixt} - 1 - ixt/(1+x^2))dM_1(x)$ *where* $\mu \in R$, $\sigma^2 \geqq 0$. *Then*

$$t^{-1}\int_0^t \phi(u)du = it((\mu/2) + \int_R \frac{x^3}{(1+x^2)^2}dM_2(x)) - (\sigma^2t^2/3) + \int_R \left(e^{ixt} - 1 - \frac{ixt}{1+x^2}\right)dM_2(x)\,.$$

Proof. Let $T>0$ be fixed and define $K(u,x) = e^{iux} - 1 - iux/(1+x^2)$. Then $K(u,x) = O(x^2)$ as $x\to0$ uniformly for $|u| \leqq T$. Let $\eta>0$. Then

$$t^{-1}\int_0^t du \lim_{\varepsilon\to 0+}\int_\varepsilon^{+\infty} K(u, x)dM_1(x) = t^{-1}\int_0^t du O\Big(\int_{0+}^{\eta} x^2 dM_1(x)\Big)$$
$$+ t^{-1}\int_\eta^\infty \int_0^t K(u, x)du dM_1(x) = O\Big(\int_0^\eta x^2 dM_1(x)\Big) + \int_\eta^{+\infty} L(t, x)\frac{dM_1(x)}{x}$$

where

$$L(t, x) = \frac{e^{itx}-1}{it} - x - \frac{itx^2}{2(1+x^2)}\,.$$

Letting $\eta \to 0^+$, we have that

$$t^{-1}\int_0^t\int_{0+}^{+\infty} K(u, x)dM_1(x)du = \int_{0+}^{+\infty} L(t, x)\frac{dM_1(x)}{x}\,.$$

A similar statement for the negative axis shows that

$$\text{(4)}\qquad \begin{aligned} t^{-1}\int_0^t \phi(u)du &= (i\mu t/2) - (\sigma^2 t^2/3) \\ &+ \int_R \Big(\frac{e^{itx}-1}{it} - x - \frac{itx^2}{2(1+x^2)}\Big)\frac{dM_1(x)}{x}\,. \end{aligned}$$

Now apply integration by parts to the integral in (4), to conclude that

$$\begin{aligned} t^{-1}\int_0^t \phi(u)du &= (i\mu t/2) - (\sigma^2 t^2/3) + \lim_{\varepsilon\to 0+}\Big[-L(t, x)\int_x^{+\infty} z^{-1}dM_1(z)\,\Big|_{x=\varepsilon}^{+\infty} \\ &+ \int_\varepsilon^{+\infty}\frac{\partial L(t, x)}{\partial x}\int_x^{+\infty} z^{-1}dM_1(z)dz + L(t, x)\int_{-\infty}^x z^{-1}dM_1(z)\,\Big|_{x=-\infty}^{-\varepsilon} \\ &+ \int_{-\infty}^{-\varepsilon}\frac{\partial L(t, x)}{\partial x}\int_{-\infty}^x z^{-1}dM_1(z)dx \\ &= (i\mu t/2) - (\sigma^2 t^2/3) + \int_R K(t, x)dM_2(x) \\ &+ it\int_R \frac{x^3}{(1+x^2)^2}dM_2(x)\,. \end{aligned}$$

The last equality follows by observing that $L(t, x)/x^3$ is bounded for $|t| \leqq T$ as $x \to 0$ and using Lemma 1. This completes the proof of Lemma 2.

THEOREM 2. *A necessary and sufficient condition for* $\phi(t)$ *to be the logarithm of an infinitely divisible* ch. f *whose Lévy spectral function is convex on* $(-\infty, 0)$ *and concave on* $(0, +\infty)$ *is that* $\phi(t)$ *may be written* $\phi(t) = t^{-1}\int_0^t \psi(u)du$, *where* ψ *is the logarithm of a certain infinitely divisible* ch.f .

Proof. Suppose $\phi(t) = t^{-1}\int_0^t \psi(u)du$ where ψ and ϕ are as in the statement of the theorem and let M_1 and M_2 be the Lévy spectral functions of ψ and ϕ respectively. Since the Lévy representation is unique, Lemma 2 shows that M_1 and M_2 are related by (3). Clearly M_2 is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$ and so the sufficiency of the condition holds.

Conversely suppose a Lévy spectral function M_2 is given and assume further that M_2 is unimodal at 0. Then we can write

$$M_2(x) = \begin{cases} \int_{-\infty}^{x} p(u)du & \text{if } x < 0 \\ -\int_{x}^{+\infty} p(u)du & \text{if } x > 0 \end{cases}$$

where $p \geqq 0$ and is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, +\infty)$. Define $M_1(x) = -\int_{-\infty}^{x} udp(u)$ if $x < 0$ and $M_1(x) = \int_{x}^{+\infty} udp(u)$ if $x > 0$. Then M_1 is also a Lévy spectral function and

$$M_2(x) = \int_{-\infty}^{x}\int_{-\infty}^{y} dp(z)dy = -\int_{-\infty}^{x}\int_{-\infty}^{y} z^{-1}dM_1(z)dy$$

if $x < 0$, and similarly, $M_2(x) = -\int_{x}^{+\infty}\int_{y}^{+\infty} z^{-1}dM_1(z)dy$ if $x > 0$. This shows that M_1 and M_2 are related by (3). So if ϕ has the Lévy triple (μ, σ^2, M_2), define

$$\psi(t) = it\left(2\mu - 2\int_R \frac{x^3}{(1+x^2)^2}dM_2(x)\right) - 3\sigma^2t^2 + \int_R e^{itx} - 1 - \frac{itx}{1+x^2}dM_1(x) .$$

By Lemma 2, $\phi(t) = t^{-1}\int_0^t \psi(u)du$, and hence, the proof of Theorem 2.

Some applications and consequences of Theorem 2 will be given.

(a) Suppose that a Lévy spectral function, M, and a df, G, are related by (2) for some $c \geqq 0$. From (2), it is clear that the (0)-unimodality of G entails that of M. The converse is not true; a counterexample is provided by the function $M(x) = c_1|x|^{-\alpha}$ or $c_2x^{-\alpha}$ according as $x < 0$ or $x > 0$, where $c_1, c_2 > 0$ and $0 < \alpha < 1$.

(b) Medgyessy ([3], Theorem 2.1) proved that if M is symmetric and convex on $(-\infty, 0)$, then the original df is unimodal at 0. Hence, combining our result with Khinchin's theorem on unimodality, one obtains that if $\Phi(t)$ is an infinitely divisible real ch. f and $\ln\Phi(t) = t^{-1}\int_0^t \ln\Psi(u)du$ for some infinitely divisible ch. f Ψ, then $\Phi(t) = t^{-1}\int_0^t \chi(u)du$ for some ch. f $\chi(u)$.

(c) Suppose $\phi(t) = i\mu t - b|t|^{\alpha}(1 + (i\beta t/|t|)\omega(|t|, \alpha))$ corresponds to a stable law of index α. (See [2], p. 136.) In this case

$$\phi(t) = i\gamma t + c\tilde{\phi}(t) \tag{5}$$

where $\gamma \in R$, $c \geqq 0$, and $\tilde{\phi}(t) = t^{-1}\int_0^t \phi(u)du$. Conversely suppose $\phi(t) = \ln \Phi(t)$ for some infinitely divisible ch. f Φ and for some $\gamma \in R$, $c \geqq 0$, (5) holds. Let (μ, σ^2, M) be the Lévy triple of Φ. If $M = 0$, then Φ is a normal ch. f and $c = 3$. Assume M is not identically zero. By Theorem 2, M is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$, and so there exists a nonnegative function $p(x)$ such that p is nondecreasing on $(-\infty, 0)$, nonincreasing on $(0, +\infty)$, and such that

$$M(x) = \begin{cases} \int_{-\infty}^{x} p(u)du & \text{if } x < 0 \\ -\int_{x}^{+\infty} p(u)du & \text{if } x < 0 \end{cases}$$

Since the Lévy representation is unique, if (5) holds, the Lévy spectral functions of ϕ and $c\tilde{\phi}$ agree. Hence M satisfies the identity

$$M(x) = \begin{cases} -c\int_{-\infty}^{x}\int_{-\infty}^{y} z^{-1}dM(z)dy & \text{if } x < 0 \\ -c\int_{x}^{+\infty}\int_{y}^{+\infty} z^{-1}dM(z)dy & \text{if } x > 0 \end{cases}$$

In terms of p, (6) reduces to

$$p(x) = \begin{cases} -c\int_{-\infty}^{x} u^{-1}p(u)du & \text{if } x < 0 \\ \int_{x}^{+\infty} u^{-1}p(u)du & \text{if } x > 0\,. \end{cases}$$

Employing the uniqueness theorem for first order differential equations, it follows that $p(x) = p(-1)|x|^{-c}$ if $x < 0$ or $p(1)x^{-c}$ if $x > 0$. But since $\int_{R\setminus(-1,1)} p(x)dx$ and $\int_{(-1,1)} x^2 p(x)dx$ are both finite, we must have that $1 < c < 3$. This, in turn, forces $\sigma^2 = 0$. Combining this and the form of the Lévy spectral function for stable distributions, we see that (5) characterizes the stable laws.

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UNIVERSITY OF LOUISVILLE
LOUISVILLE KY 40208