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THE FINITE WEIL–PETERSSON DIAMETER OF RIEMANN SPACE

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Let T_g be the Teichmüller space and R_g the Riemann space of compact Riemann surfaces of genus g with $g \geqq 2$. The space R_g can be realized as the quotient of T_g by a properly discontinuous group M_g, the modular group. Various metrics have been defined for T_g which are compatible with the standard topology for T_g and induce quotient metrics for R_g. Several authors have considered the Weil–Petersson metric for T_g. A length estimate derived in a previous paper is summarized; combining this with the Ahlfors Schwarz lemma, an estimate of N. Halpern and L. Keen, and an additional argument shows that the Weil–Petersson quotient metric for R_g has finite diameter. A corollary is an estimate relating the Poincaré length of the shortest closed geodesic of a compact Riemann surface to the Poincaré diameter of the surface.

For background material the reader is referred to the articles of L. Ahlfors [**1**] and L. Bers [**3**] and to the article of L. Bers [**5**] for a survey of related topics. T. C. Chu [**7, 8**] and H. Masur [**12**] have obtained results related to ours. The author would like to thank Professor G. Kiremidjian for his assistance.

1. The case of an annulus. Let $A = \{z \mid 1 < |z| < \rho\}$ be an annulus in the plane. Let $M(A)$ be the space of Beltrami differentials of A endowed with the L^∞ metric; let $Q(A)$ be the space of integrable holomorphic quadratic differentials of A. An element of $M(A)$ is a tensor of type $(-1, 1)$ with measurable coefficient.

DEFINITION 1.1. For $\Phi \in Q(A)$ set

$$\|\Phi\|_A = \left(\int |\Phi|^2 \lambda_A^{-2}\right)^{1/2}$$

where λ_A is the Poincaré metric of A. For $\mu \in M(A)$ set

$$\|\mu\|_A = \sup_{\Phi \in Q(A)} |[\mu, \Phi]| / \|\Phi\|_A$$

where $[\mu, \Phi] = \int_A \mu \Phi$.

The metric λ_A is known to be given by the following expression

$$(\pi/\log\rho)\csc(\pi\log|z|/\log\rho)|dz/z|.$$

We consider a particular deformation of the annulus A. For $t \geqq 1$ let $A_t = \{z_t \mid 1 < |z_t| < \rho^t\}$ then the map

$$z \mapsto z|z|^{t-1} = z_t(z) \tag{1.1}$$

is quasiconformal with Beltrami differential

$$(t - 1/t + 1)(z/|z|)^2\,\overline{dz}/dz.$$

By considering solutions $\omega(z)$ of the Beltrami differential equation $\omega_{\bar z} = \mu\omega_z$ where μ is a Beltrami differential it is seen that the curve of Riemann surfaces A_t is represented by the curve

$$(t - 1/t + 1)(z/|z|)^2\overline{dz}/dz \subset M(A), \qquad t \geqq 1.$$

As described in our previous paper **[16]** $(1/2t)(z_t/|z_t|)^2\,\overline{dz_t}/dz_t$ is the tangent to this curve at A_t expressed as an element of $M(A_t)$, $t \geqq 1$. By Definition 1.1

$$\begin{aligned}&\|(1/2t)(z_t/|z_t|)^2\overline{dz_t}/dz_t\|_{A_t}\\ &= \sup_{\Phi\in Q(A_t)} \left|\int_{A_t} (1/2t)(z_t/|z_t|)^2\,\overline{dz_t}/dz_t\,\Phi\right| \Big/ \left(\int_{A_t} |\Phi|^2\lambda_{A_t}^{-2}\right)^{1/2}.\end{aligned} \tag{1.2}$$

It is clear that the extremal Φ is given by $(dz_t/z_t)^2$. The value of the quotient in (1.2) is now equal to

$$(2\pi^3/t^3\log\rho)^{1/2}. \tag{1.3}$$

Thus the length of the curve A_t, $t \geqq 1$ is given by the convergent integral

$$\int_1^\infty (2\pi^3/t^3\log\rho)^{1/2}\,dt. \tag{1.4}$$

For a compact Riemann surface R of genus g, $g \geqq 2$ one can identify the cotangent space at the point R of Teichmüller space with the regular quadratic differentials $Q(R)$ of R and the tangent space at R with the Beltrami differentials $M(R)$ modulo those which are infinitesimally trivial, **[1]**. In this instance the Weil–Petersson metric and cometric are given by Definition 1.1 on replacing A by R, **[15]**.

2. Finite diameter of Riemann space. The Riemann space R_g of genus g, $g \geqq 2$ is the space of conformal equivalence classes of similarly oriented compact Riemann surfaces of genus g, [**14**]. A natural projection π_g of T_g to R_g exists; this projection can be given by the action of a properly discontinuous group M_g, the modular group, [**6**]. S. Kravetz showed that every metric $d(\,,\,)$ for T_g compatible with the topology of T_g induces a quotient metric $\tilde{d}(\,,\,)$ for R_g defined as

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf_{\substack{\pi_g(x)=\tilde{x} \\ \pi_g(y)=\tilde{y}}} d(x, y)$$

for $x, y \in T_g$ and $\tilde{x}, \tilde{y} \in R_g$, [**11**].

DEFINITION 2.1. For $\tilde{x}, \tilde{y} \in R_g$ let

$$\omega(\tilde{x}, \tilde{y}) = \inf_{\substack{\pi_g(x)=\tilde{x} \\ \pi_g(y)=\tilde{y}}} d_{w-p}(x, y)$$

where $d_{w-p}(\,,\,)$ is the Weil–Petersson metric for T_g.

Let $H = \{z \mid \operatorname{Im} z > 0\}$ denote the upper half plane and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ the Laplacian. The following definition and theorem are due to L. Ahlfors, [**2**].

DEFINITION 2.2. A metric $\rho \mid dz \mid$, $\rho \geqq 0$ is said to be ultrahyperbolic in H if it has the following properties:

(i) ρ is upper semicontinuous;

(ii) at every $z_0 \in H$ with $\rho(z_0) > 0$ there exists a ρ_0 defined and of class C^2 in a neighborhood V of z_0 such that $\Delta \log \rho_0 \geqq \rho_0^2$ and $\rho \geqq \rho_0$ in V while $\rho(z_0) = \rho_0(z_0)$.

The Poincaré metric of H is $\mid dz \mid / y$.

THEOREM 2.3. *Let $\rho \mid dz \mid$ be an ultrahyperbolic metric for H. Then $\rho \mid dz \mid \leqq \mid dz \mid / y$.*

The following theorem is due to L. Bers, [**4**] and D. Mumford, [**13**].

THEOREM 2.4. *For $c > 0$, let $K_c \subset R_g$, $g \geqq 2$ consist of those Riemann surfaces R for which each closed Poincaré geodesic has length at least c. Then K_c is a compact set.*

THEOREM 2.5. *R_g has finite diameter for the $\omega(\,,\,)$ metric.*

Proof. Consider the following regions in H $C(l, \theta_0) = \{z \mid \operatorname{Im} z >$

$0, 1 < |z| < \exp l, \quad \theta_0 < \arg z < \pi - \theta_0\}$ and $\theta_1 < \theta_2$ $C(l, \theta_1, \theta_2) = C(l, \theta_1) - C(l, \theta_2)$. The Poincaré area of $C(l, \theta_0)$ (resp. $C(l, \theta_1, \theta_2)$) is $2l \cot \theta_0$ (resp. $2l(\cot \theta_1 - \cot \theta_2)$). The self map of H $z \mapsto z \exp l$ identifies the boundaries of $C(l, \theta_0)$ such that the quotient $A(l, \theta_0) = C(l, \theta_0)/\{z \mapsto z \exp l\}$ is conformally an annulus. Let $\tilde{C}(l, \theta_1, \theta_2)$ denote $C(l, \theta_1, \theta_2)$ with the boundaries $\bar{C}(l, \theta_1, \theta_2) \cap \{z \mid \arg z = \theta_2\}$ and $\bar{C}(l, \theta_1, \theta_2) \cap \{z \mid \arg z = \pi - \theta_2\}$ identified by the map $z \mapsto z \exp i(\pi - 2\theta_2)$; the quotient $A(l, \theta_1, \theta_2) = \tilde{C}(l, \theta_1, \theta_2)/\{z \mapsto z \exp l\}$ is conformally an annulus. Let $\alpha(\theta)$ (resp. $\beta(\theta)$) denote the projection to $A(l, \theta_0)$ (resp. $A(l, \theta_1, \theta_2)$) of the curve $z = r \exp i\theta$, $1 \leqq r \leqq \exp l$ provided $\theta_0 \leqq \theta \leqq \pi - \theta_0$ (resp. $\theta_1 \leqq \theta \leqq \theta_2$). A quotient metric for $A(l, \theta_0)$ (resp. $A(l, \theta_1, \theta_2)$) is obtained from the restriction to $C(l, \theta_0)$ (resp. $C(l, \theta_1, \theta_2)$) of the line element $|dz|/y$. The distance between the boundaries of $A(l, \theta_0)$ (resp. $A(l, \theta_1, \theta_2)$) in the quotient metric will be referred to as the width of $A(l, \theta_0)$ (resp. $A(l, \theta_1, \theta_2)$). Since each curve $z = r \exp i\theta \subset H$ $0 < \theta < \pi$ is a Poincaré geodesic it follows that the width of $A(l, \theta_0)$ is given by the integral $\int_{\theta_0}^{\pi - \theta_0} r d\theta / r \sin \theta = 2 \ln(\cot \theta_0 + \csc \theta_0)$. The induced quotient metric for $A(l, \theta_1, \theta_2)$ is not differentiable on the curve $\beta(\theta_2)$; nevertheless, it is straightforward that the width of $A(l, \theta_1, \theta_2)$ is $2 \ln(\cot \theta + \csc \theta)|_{\theta_2}^{\theta_1}$. The curve $\beta(\theta_2)$ has length $\int_1^{\exp l} dr / r \sin \theta_2 = l \csc \theta_2$.

The following lemmas of N. Halpern **[9]** and L. Keen **[10]** are essential to our argument.

LEMMA 2.6. *Let R be a compact Riemann surface. For every $c_1 > 0$ there exists a $c_2 > 0$ such that for γ a simple closed Poincaré geodesic of length l at most c_1, the region $A(l, \theta_l)$, $\theta_l = \cot^{-1}(c_2/2l)$, can be isometrically imbedded into R with $\alpha(\pi/2)$ realizing γ.*

Observe that $2l \cot \theta_l$ represents the area of $A(l, \theta_l)$.

LEMMA 2.7. *Let R be a compact Riemann surface of genus g, $g \geqq 2$. There exists a constant $c_3 > 0$ such that there are at most $3g - 3$ simple closed Poincaré geodesics of length at most c_3.*

Proof of Lemma 2.7. By Lemma 2.6 one can choose $c_3 < c_1$ such that the width of $A(l, \theta_1)$ for $l \leqq c_3$ is at least c_3. The conclusion now follows since there are at most $3g - 3$ mutually disjoint, homotopically nontrivial, simple closed curves on R which are mutually not freely homotopic.

Let $\Phi_l = \cot^{-1}(c_2/4l)$ and consider the domain $A(l, \theta_l, \Phi_l)$. The width of $A(l, \theta_l, \Phi_l)$ is $2 \ln(\cot \theta + \csc \theta)|_{\Phi_l}^{\theta_l}$ which is bounded from below for $l \leqq c_3$ provided there exists a constant $c > 0$ such that

$$(\cot \theta_l + \csc \theta_l)/(\cot \Phi_l + \csc \Phi_l) \geqq c \qquad \text{for } l \leqq c_3.$$

For c_3 sufficiently small $\csc \Phi_l \leqq 2 \cot \Phi_l$ thus

$$(\cot \theta_l + \csc \theta_l)/(\cot \Phi_l + \csc \Phi_l) \geqq \cot \theta_l / 3 \cot \Phi_l \geqq 2/3. \tag{2.1}$$

The length of $\beta(\Phi_l)$ is

$$l \csc(\cot^{-1}(c_2/4l)) \geqq l \cot(\cot^{-1}(c_2/4l)) = c_2/4. \tag{2.2}$$

For an annulus $A = \{z \mid 1 < |z| < r\}$ we make the following definition.

DEFINITION 2.8. The extremal length $E(A)$ of A is given by $E(A) = 2\pi/\log r$.

Now the extremal length of $A(l, \theta_l, \Phi_l)$ is $E(A(l, \theta_l, \Phi_l)) = l/2(\Phi_l - \theta_l) = l/2(\cot^{-1}(c_2/4l) - \cot^{-1}(c_2/2l))$ where by l'Hopital's rule

$$\lim_{l \to 0} l/2(\cot^{-1}(c_2/4l) - \cot^{-1}(c_2/2l)) = c_2/4. \tag{2.3}$$

It is now clear that c', $0 < c' < c_3$ can be chosen such that for $l \leqq c'$

$$2 \ln(\cot \theta + \csc \theta)\big|_{\Phi_l}^{\theta_l} \geqq c' \tag{2.4}$$

$$l \csc \Phi_l \geqq c' \tag{2.5}$$

and

$$l/2(\Phi_l - \theta_l) \leqq c_2. \tag{2.6}$$

These inequalities will now be used to estimate the diameter of R_g. The region $K_{c'} \subset R_g$ is compact and thus has finite ω diameter. Let a Riemann surface R represent a point in T_g such that $\pi_g(R) \not\in K_{c'}$ with $\gamma_1, \cdots, \gamma_n$ the geodesics of R of length less than c'. The object is to "fatten" R in a neighborhood of each of $\gamma_1, \cdots, \gamma_n$ thereby obtaining a surface in $K_{c'}$. By Lemma 2.6 a region $A(l, \theta_l)$ can be considered as a coordinate neighborhood of γ_1 where l is the length of γ_1. A new surface R^* can be formed by removing the part of $A(l, \theta_l)$ corresponding to $A(l, \Phi_l)$ and identifying the boundaries by the map $z \mapsto z \exp i(\pi - 2\Phi_l)$. Thus $A(l, \theta_l, \Phi_l)$ represents a coordinate patch in a neighborhood of the gluing and the original coordinates are chosen otherwise. In a neighborhood of the gluing $\lambda_R|_{R^*}$, the Poincaré metric of R restricted to R^*, is defined in terms of the coordinate patch $A(l, \theta_l, \Phi_l)$; for coordinate patches disjoint from the gluing $\lambda_R|_{R^*} = \lambda_R$. Assuming that $\lambda_R|_{R^*}$ is ultrahyperbolic Theorem 2.3 implies that $\lambda_R|_{R^*} \leqq \lambda_{R^*}$ where λ_{R^*} is the Poincaré metric of R^*. To show that $\lambda_R|_{R^*}$ is ultrahyperbolic it suffices to consider the metric in a neighborhood of the gluing. Define the metric $\tilde{\lambda}(z)|dz|$ on $\tilde{C}(l, \theta_l, \Phi_l)$ by setting $\tilde{\lambda}(z)|dz| = |dz|/\operatorname{Im} z$ for $1 < |z| < \exp l$, $\theta_l < \arg z < \Phi_l$ and $\tilde{\lambda}(z)|dz| = |dz|/\operatorname{Im}(z \exp i(2\Phi_l - \pi))$ for $1 < |z| < \exp l$, $\pi - \Phi_l < \arg z < \pi - \theta_l$; that $\tilde{\lambda}(z)|dz|$ satisfies (ii) of Definition 2.2 relative to the quotient metric of $\tilde{C}(l, \theta_l, \Phi_l)$ is clear. The objective is to show that R^* is "fat" in the free homotopy class of γ_1 and that no new (i.e., other than $\gamma_2, \cdots, \gamma_n$) "pinched" free homotopy classes were introduced. Let $\gamma_0^* \subset R^*$ be a simple closed λ_{R^*} geodesic of length less than c'. If γ_0^* does not intersect the gluing then γ_0^* can also be considered as a curve γ_0 on R. Since $\lambda_R|_{R^*} \leqq \lambda_{R^*}$ the length of γ_0 is also less than c'. If γ_0 is freely homotopic to γ_1 then γ_0 can be lifted to the universal cover H of R with initial point z_0 and end point z_1 such that $|z_0| = 1$ and $|z_1| = \exp l$. By the assumption that γ_0^* is disjoint from the gluing the lift of γ_0 is disjoint from the domain $A(l, \Phi_l)$ and thus by estimate (2.5) has length at least c', a contradiction. By Lemma 2.7 γ_0^* cannot intersect and yet be distinct from the geodesics $\gamma_2, \cdots, \gamma_n$. Thus γ_0 must be freely homotopic to one of $\gamma_2, \cdots, \gamma_n \subset R$ or γ_0^* intersects the gluing. If γ_0^* is contained in $A(l, \theta_l, \Phi_l)$ then it must be freely homotopic to γ_1 a case considered above; otherwise γ_0^* intersects the gluing and the boundaries of $A(l, \theta_l, \Phi_l)$ hence crosses the domain. By estimate (2.4) γ_0^* has length at least c' in terms of the $\lambda_R|_{R^*} \leqq \lambda_{R^*}$ metric, a contradiction. Thus γ_0^* is freely homotopic to one of $\gamma_2, \cdots, \gamma_n$. The deformation corresponding to the replacing of $A(l, \theta_l)$ by $A(l, \theta_l, \Phi_l)$ can be realized in terms of quasiconformal maps. For $A = A(l, \theta_l, \Phi_l) = \{z \mid 1 < |z| < \rho\}$ the domain $A(l, \theta_l)$ corresponds to the deformation of A given by the element $(t - 1/t + 1)(z/|z|)^2 \overline{dz}/dz \in M(A(l, \theta_l, \Phi_l))$ where $t = (\pi - 2\theta_l)/2(\Phi_l - \theta_l)$. We consider $(\tau - 1/\tau + 1)(z/|z|)^2 \overline{dz}/dz$ restricted to $A(l, \theta_l, \Phi_l) \subset R^*$ $1 \leqq \tau \leqq t$ as a curve in $M(R^*)$. The estimate for an annulus given by (1.4) can be now applied upon noting that $\lambda_R|_A \leqq \lambda_A$ and $Q(R)|_A \subset Q(A)$, **[16]**. The Weil–Petersson length of this curve is seen to be bounded in terms of $E(A(l, \theta_l, \Phi_l))^{1/2}$. Estimate (2.6) bounds the latter quantity by the constant $c_2^{1/2}$. Repeating this "fattening" process n times a surface $\tilde{R} \in K_{c'}$ is obtained. By Lemma 2.7 $n \leqq 3g - 3$; the above remarks now yield $\omega(R, \tilde{R}) \leqq (3g - 3)c_2^{1/2}$. The proof is complete.

3. The Poincaré diameter and length of the shortest closed geodesic. Let R be a compact Riemann surface of genus g, $g \geqq 2$. Let $l(R)$ denote the length of the shortest closed Poincaré geodesic and $d(R)$ the Poincaré diameter of R. The following lemma is a consequence of the considerations of 2.

LEMMA 3.1. *There exist constants $\overline{c_1}$ and $\overline{c_2}$ depending only on the genus such that*

$$\ln(\tilde{c}_1/l(R)) \leqq d(R) \leqq 6g \ln(\tilde{c}_2/l(R)).$$

Proof. Maintaining the constants c_1, c_2, c_3 and c' of § 2 we consider a surface $R \in K_{c'}$. As $K_{c'}$ is compact $l(R)$ and $d(R)$ are bounded above and below hence constants $\tilde{c}_1$ $\tilde{c}_2$ exist to yield

$$\ln(\tilde{c}_1/l(R)) \leqq d(R) \leqq 2\ln(\tilde{c}_2/l(R))$$

for surfaces in $K_{c'}$. Now let $R \notin K_{c'}$ then clearly $d(R)$ is bounded below by one-half the width of $A(l, \theta_l) \subset R$ where $l = l(R)$. Thus

$$\ln(c_2/2l) \leqq \ln(\cot\theta_l + \csc\theta_l) \leqq d(R). \tag{3.1}$$

Setting $\overline{c_2} = \min\{c_2, \tilde{c}_2\}$ the lower bound is established. Assume that $R \notin K_{c'}$ and has only one closed Poincaré geodesic of length less than c'. Forming the surface R^* as in 2. by removing $A(l, \Phi_l)$ from $A(l, \theta_l) \subset R$ where $l = l(R)$ we have that $d(R)$ is bounded by the sum of the width of $A(l, \theta_l)$, $l/2$ and $d(R^*)$. Specifically for two points x, y of R^* we connect them with a λ_{R^*} length minimizing curve $\gamma_{x,y}$. If this curve intersects the gluing a new curve is formed as the union of the shortest segment of $\gamma_{x,y}$ from x to the gluing, a segment along the gluing and the shortest segment of $\gamma_{x,y}$ from the gluing to y. Now taking account of the relation of R to $R^* d(R)$ is seen to be bounded by

$$2\ln(\tilde{c}_2/l(R)) + c' + 2\ln(\tilde{c}_2/l(R^*))$$

where $\tilde{c}_2$ has been appropriately modified. A constant $\overline{c_2}$ can now be chosen to bound this last quantity by $4\ln(\overline{c_2}/l(R))$. In general let S be a surface with exactly n closed Poincaré geodesics of length less than c'. We claim that $d(S) \leqq 2(n+1)\ln(\overline{c_2}/l(S))$ for an appropriate $\overline{c_2}$. Proceeding by induction on n it remains only to consider the induction step. Let $R \notin K_{c'}$ have exactly $n+1$ closed Poincaré geodesics of length less than c'. Forming the surface R^* and arguing as above $d(R)$ is bounded by the sum of the width of $A(l, \theta_l) \subset R$, $l/2$ and $d(R^*)$ where $l = l(R)$. Using the induction hypothesis this is bounded by

$$2\ln(\tilde{c}_2/l(R)) + c' + 2(n+1)\ln(\tilde{c}_2/l(R^*))$$

which in turn is bounded by

$$2(n+2)\ln(\overline{c_2}/l(R)). \tag{3.2}$$

Observing that n is at most $3g-3$ the upper bound is now established.

In contrast to the present lemma the constructive estimate

$$d(R) \leqq (g-1)l(R)/\sinh^2(l(R)/2) \tag{3.3}$$

where

$$l(R)/\sinh^2(l(R)/2) \approx 4/l(R)$$

for $l(R)$ sufficiently small was given by L. Bers, [**4**].

References

1. L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math., **74** (1961), 171–191.
2. ———, *Conformal Invariants*, McGraw-Hill, New York, N.Y., 1973.
3. L. Bers, *Quasiconformal mapping and Teichmüller's theorem*, Analytic Functions by R. Nevanlinna, et al., Princeton University Press, Princeton, N.J., 1960, 89–119.
4. ———, *A remark on Mumford's compactness theorem*, Israel J. Math., 12 (1972), 400–407.
5. ———, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc., **4** (1972), 257–300.
6. ———, *Fiber spaces over Teichmüller spaces*, Acta Math., **130** (1973), 89–126.
7. T. C. Chu, *Noncompleteness of the Weil–Petersson metric*, preprint.
8. ———, Thesis, Columbia University, 1976.
9. N. Halpern, *Collars on Riemann surfaces*, preprint.
10. L. Keen, *Collars on Riemann surfaces*, Ann. of Math. Studies, **79** (1974), 263–268.
11. S. Kravetz, *On the geometry of Teichmüller spaces and the structure of their modular groups*, Ann. Acad. Sci. Fenn., **278** (1959), 1–35.
12. H. Masur, *The extension of the Weil–Petersson metric to the boundary of Teichmüller space*, preprint.
13. D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc., **28** (1971), 289–294.
14. H. E. Rauch, *A transcendental view of the space of algebraic Riemann surfaces*, Bull. Amer. Math. Soc., **71** (1965), 1–39.
15. H. L. Royden, *Invariant metrics on Teichmüller space*, Contributions to Analysis by L. V. Ahlfors, et al., Academic Press, New York, N.Y., 1974, 393–399.
16. S. Wolpert, *Non-completeness of the Weil–Petersson metric for Teichmüller space*, Pacific J. Math., **61** (1975), 573–577.
17. ———, *The Weil–Petersson metric for Teichmüller space and the Jenkins–Strebel differentials*, Thesis, Stanford University, 1976.

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