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EXPONENTIAL REPRESENTATION OF SOLUTIONS TO AN ABSTRACT SEMI-LINEAR DIFFERENTIAL EQUATION

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It is shown that the solutions to the abstract differential equation $u' = -(A + B)u, u(0) = x \in X$, where X is a Banach space, $-A$ is a linear analytic semigroup generator, and B is Lipschitz continuous from the domain of a fractional power of A to X, have the exponential representation $u(t) = \lim_{n\to\infty}(I + t/n\,(A + B))^{-n}x$.

1. Introduction. Let X be a Banach space with norm $\|\ \|$. We are concerned with the abstract semi-linear differential equation in X

$$du(t)/dt = -(A + B)u(t), \quad t > 0, \quad u(0) = x \in X, \tag{1.1}$$

where $-A$ is the generator of an analytic semigroup of linear operators in X and B is Lipschitz continuous from the domain of a fractional power of A to X. The objective of this paper is to obtain the exponential representation of the solutions to (1.1) in the form

$$u(t) = \lim_{n\to\infty}(I + t/n(A + B))^{-n}x. \tag{1.2}$$

Exponential representations of the form (1.2) are very well known for the case that A and B satisfy accretive type conditions (see, e.g., [1] and [8]). In the accretive case the nonlinear resolvent $(I + t/n(A + B))^{-1}$ is Lipschitz continuous with

$$|(I + t/n(A + B))^{-1}|_{\text{Lip}} \leqq (1 - t\gamma/n)^{-1}, \quad t \geqq 0, \quad n \quad \text{sufficiently large},$$

where γ is some real constant. In our case the main difficulty in establishing (1.2) is that the nonlinear resolvent satisfies a more general condition of the form

$$|(I + t/n(A + B))^{-n}|_{\text{Lip}} \leqq M(1 - t\gamma/n)^{-n}, \quad t \geqq 0, \quad n \quad \text{sufficiently large},$$

where M and γ are real constants and $M > 1$.

We make the following assumption on A:

(1.3) there exists $\omega < 0$ such that $A = A_0 - \omega I$, where A_0 is a closed densely defined linear operator from X to X, the resolvent set of A_0 contains the sector $S_\Phi = \{\lambda : \lambda \neq 0, \ \pi/2 - \Phi < \arg\lambda < 3\pi/2 + \Phi\}$, Φ some constant in $(0, \pi/2)$, and $|(\lambda I - A_0)^{-1}| \leqq M/|\lambda|$ for $\lambda \in S_\Phi$, where M is a given constant $\geqq 1$.

As a consequence of (1.3) we have (see [3]):

(1.4) $-A$ is the infinitesimal generator of an analytic semigroup of linear operators $T(t)$, $t \geqq 0$ in X, $|T(t)| \leqq Me^{\omega t}$ for $t \geqq 0$, and $|(I + t/nA)^{-n}| \leqq M(1 - \omega t/n)^{-n}$ for $t \geqq 0$ and n a positive integer;

(1.5) if $0 < \alpha < 1$ then the fractional power $A^{-\alpha}$ is defined as a bounded linear operator in X by $A^{-\alpha} = (1/\Gamma(\alpha))\int_0^\infty e^{-sA}s^{\alpha-1}ds$, and $D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha \overset{\text{def}}{=} \|A^\alpha x\|$ for $x \in D(A^\alpha)$;

(1.6) if $0 < \alpha < 1$, then there exists $C > 0$ such that $|A^\alpha T(t)| \leqq Ce^{\omega t}t^{-\alpha}$ for $t > 0$.

We make the following assumption on B:

(1.7) there exists $\alpha \in (0,1)$ such that B is an everywhere defined operator from $D(A^\alpha)$ to X and there exists $L > 0$ such that $\|Bx - By\| \leqq L\|x - y\|_\alpha$ for $x, y \in D(A^\alpha)$.

Under the assumptions (1.3) and (1.7) and the assumption that $x \in D(A^\alpha)$ the equation (1.1) can be integrated to yield the equivalent integral equation

$$u(t) = T(t)A^\alpha x - \int_0^t A^\alpha T(t-s)BA^{-\alpha}u(s)ds, \quad t \geqq 0. \tag{1.8}$$

(see, e.g., [5], Chapter 3). The equation (1.8) is a singular Volterra integral equation and has been subject of extensive study. The case in which $B: D(A^\alpha) \to X$ is locally Lipschitz continuous is treated in [5], the case in which A^{-1} is compact and $B: D(A^\alpha) \to X$ is Holder continuous is treated in [3], and the case in which B is accretive and $B: D(A^\alpha) \to X$ is locally continuous is treated in [7]. In our treatment of (1.8) the following lemma (which is similar to Gromwall's lemma) will play a fundamental role (see also [3], Theorem 7.1.1):

LEMMA 1.1. *Let $w: [0, t_0] \to [0, \infty)$ be continuous, let $a \geqq 0$, $b \geqq 0$,*

$\omega \in R$, $0 < \alpha < 1$, and let

$$w(t) \leqq ae^{\omega t} + b\int_0^t e^{\omega(t-s)}(t-s)^{-\alpha}w(s)ds, \qquad 0 \leqq t \leqq t_0. \tag{1.9}$$

Then, for all real γ such that $\gamma > \omega$ and $b\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1} < 1$, we have that

$$w(t) \leqq a(1 - b\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1})^{-1}e^{\gamma t}, \qquad 0 \leqq t \leqq t_0. \tag{1.10}$$

Proof. We will use the gamma function formula

$$\Gamma(z) = \beta^z \int_0^\infty e^{-\beta s}s^{z-1}ds \quad \text{for} \quad z > 0, \quad \beta > 0 \tag{1.11}$$

(see [**9**], p. 265). Let γ be as above, and let $S = \sup_{0 \leqq t \leqq t_0} e^{-\gamma t}w(t)$. For $0 \leqq t \leqq t_0$ (1.9) implies

$$\begin{aligned} e^{-\gamma t}w(t) &\leqq ae^{(\omega-\gamma)t} + b\int_0^t e^{(\omega-\gamma)(t-s)}(t-s)^{-\alpha}e^{-\gamma s}w(s)ds \\ &\leqq a + bS\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1}. \end{aligned}$$

Thus, $S \leqq a + bS\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1}$, which implies (1.10).

2. Existence of solutions. It is well known that under the assumptions (1.3) and (1.7) there exixts a solution to (1.8) for each $x \in D(A^\alpha)$ (see, e.g., [**5**], Theorem 3.3.3). We will prove this fact below, however, since our proof will be instructive for the techniques we use to prove (1.2).

PROPOSITION 2.1. *Let* (1.3) *and* (1.7) *hold and let $x \in D(A^\alpha)$. There exists a unique continuous function $u\colon [0,\infty) \to D(A^\alpha)$ satisfying*

$$u(t) = T(t)x - \int_0^t T(t-s)Bu(s)\,ds, \quad t \geqq 0. \tag{2.1}$$

Proof. Let γ be real such that

$$\gamma > \omega \quad \text{and} \quad CL\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1} < 1 \tag{2.2}$$

and let $t_0 > 0$. Let Y be the Banach space of all continuous functions u from $[0, t_0]$ to X with norm $\|u\|_Y = \sup_{0 \leqq t \leqq t_0} e^{-\gamma t}\|u(t)\|$. Define the mapping $F\colon Y \to Y$ by

$$(Fu)(t) = T(t)A^{\alpha}x - \int_0^t A^{\alpha}T(t-s)BA^{-\alpha}u(s)\,ds, \quad 0 \leqq t \leqq t_0.$$

We will use (1.11) to show that F maps Y into Y and F is a contraction on Y.

First, we see that $(Fu)(t)$ is continuous in t for each $u \in Y$, since for $0 \leqq t_1 \leqq t_2 \leqq t_0$

$$\begin{aligned}\|(Fu)(t_1) - (Fu)(t_2)\| &\leqq \left\| \int_{t_1}^{t_2} A^{\alpha}T(s)BA^{-\alpha}u(t_2 - s)\,ds \right\| \\ &\quad + \left\| \int_0^{t_1} A^{\alpha}T(s)\,(BA^{-\alpha}u(t_1 - s) - BA^{-\alpha}u(t_2 - s))\,ds \right\| \\ &\quad + \|(T(t_1) - T(t_2))A^{\alpha}x\| \\ &\leqq \int_{t_1}^{t_2} Ce^{\omega s}s^{-\alpha} \sup_{t_1 \leqq s \leqq t_2} \|BA^{-\alpha}u(t_2 - s)\|\,ds \\ &\quad + \int_0^{t_1} CLe^{\omega s}s^{-\alpha}\|u(t_1 - s) - u(t_2 - s)\|\,ds + \|(T(t_1) - T(t_2))A^{\alpha}x\|.\end{aligned}$$

Next, we see that F is a contraction, since for $u, v \in Y$, $0 \leqq t \leqq t_0$

$$\begin{aligned}&e^{-\gamma t}\|(Fu)(t) - (Fv)(t)\| \\ &\quad \leqq CL \int_0^t e^{(\omega-\gamma)(t-s)}(t-s)^{-\alpha}e^{-\gamma s}\|u(s) - v(s)\|\,ds \\ &\quad \leqq (CL\Gamma(1-\alpha)(\gamma-\omega)^{\alpha-1})\|u - v\|_Y.\end{aligned}$$

By the contraction mapping theorem there exists a unique $u_0 \in Y$ such that $Fu_0 = u_0$. Define $u(t) = A^{-\alpha}u_0(t)$ for $0 \leqq t \leqq t_0$ and obviously $u(t)$ satisfies (2.1) uniquely. Furthermore, $u(t) \in D(A^{\alpha})$ for $0 \leqq t \leqq t_0$ and $u(t)$ is continuous from $[0, t_0]$ to $D(A^{\alpha})$ (here we have used the continuity of $A^{-\alpha}$). Since t_0 is arbitrarily large, the proof is finished.

DEFINITION 2.1. Define the family of operators $U(t), t \geqq 0$ in $D(A^{\alpha})$ by $U(t)x = u(t)$, where u is the unique solution of (2.1) for a given $x \in D(A^{\alpha})$. Then, $U(t), t \geqq 0$ is a strongly continuous semigroup of nonlinear operators in $D(A^{\alpha})$. In fact, we have the following:

PROPOSITION 2.2. *Let* (1.3) *and* (1.7) *hold and for each* $t \geqq 0$ *let* $U(t)$: $D(A^{\alpha}) \to D(A^{\alpha})$ *be defined as above. Then,* $U(t), t \geqq 0$ *satisfies the following*:

(2.3) $U(0) = I$ and $U(t)X$: $[0,\infty) \to D(A^{\alpha})$ is continuous in t for each fixed $x \in D(A^{\alpha})$;

(2.4) $U(t_1 + t_2)x = U(t_1)U(t_2)x$ for $t_1, t_2 \geqq 0$, $x \in D(A^\alpha)$;

(2.5) $\| U(t)x - U(t)y \|_\alpha \leqq M(1 - CL\Gamma(1-\alpha)(\gamma - \omega)^{\alpha-1})^{-1} e^{\gamma t} \| x - y \|_\alpha$ for $t \geqq 0, x, y \in D(A^\alpha)$, and γ satisfying (2.2).

Proof. (2.3) follows immediately from Proposition 2.1. (2.4) follows from the uniqueness of solutions to (2.1) (see [7], Proposition 3.6). (2.5) follows immediately from Lemma 1.1, since

$$e^{-\gamma t} \| U(t)x - U(t)y \|_\alpha \leqq Me^{(\omega-\gamma)t} \| x - y \|_\alpha + \int_0^t Ce^{(\omega-\gamma)(t-s)}(t-s)^{-\alpha} Le^{-\gamma s} \| U(s)x - U(s)y \|_\alpha ds.$$

COROLLARY 2.1. *If* $L < (-\omega)^{1-\alpha}/C\Gamma(1-\alpha)$, *then there exists a unique* $x_0 \in D(A^\alpha)$ *such that* $\lim_{t\to\infty} \| U(t)x - x_0 \|_\alpha = 0$ *for all* $x \in D(A^\alpha)$.

Proof. Choose $\gamma < 0$ satisfying (2.2) and by (2.5) we have that $U(t)$ is a strict contradiction for t sufficiently large, say $t \geqq t_1$. Let x_t be the unique fixed point of $U(t)$ for each $t \geqq t_1$. If $s, t \geqq t_1$, then $U(t)x_s = U(t)U(s)x_s = U(t+s)x_s = U(s)U(t)x_s$, which implies $x_s = U(t)x_s$, and which in turn implies $x_s = x_t$. The conclusion follows immediately using (2.5).

DEFINITION 2.2. Define the infinitesimal generator of $U(t), t \geqq 0$ to be the nonlinear operator $F: D(A^\alpha) \to D(A^\alpha)$ given by

$$D(F) = \left\{ x \in D(A^\alpha): \lim_{t\to 0} (U(t)x - x)/t \text{ exists in } D(A^\alpha) \right\}$$

$$Fx = \lim_{t\to 0} (U(t)x - x)/t \text{ (where the limit is taken in } D(A^\alpha)).$$

PROPOSITION 2.3. $D(F) = \{x \in D(A): (A+B)x \in D(A^\alpha)\}$ *and* $Fx = -(A+B)x$ *for all* $x \in D(F)$.

Proof. First, let $x \in D(F)$. Then, $X - \lim_{t\to 0}(U(t)x - x)/t = Fx$, since the α-norm dominates the X-norm. Since $X - \lim_{t\to 0} \int_0^t T(t-s)BA^{-\alpha}A^\alpha U(s)x ds/t = Bx$ (for any $x \in D(A^\alpha)$) and since $(T(t)x - x)/t = (U(t)x - x)/t + \int_0^t T(t-s)BU(s)x ds/t$, we see that $x \in D(A)$ and $X - \lim_{t\to 0}(U(t)x - x)/t = -(A+B)x$. But then $\alpha - \lim_{t\to 0}(U(t)x - x)/t = -(A+B)x$ as well, and therefore $(A+B)x \in D(A^\alpha)$ and $Fx = -(A+B)x$.

Now let $x \in D(A)$ and let $(A+B)x \in D(A^\alpha)$. Since $(A^\alpha + A^{\alpha-1}B)x = A^{-1}A^\alpha(A+B)x$, we have that

(2.6) $(A^\alpha + A^{\alpha-1}B)x \in D(A)$ and $A(A^\alpha + A^{\alpha-1}B)x = A^\alpha(A+B)x$.

Since $(T(t)-I)A^{-1}Bx = -\int_0^t T(s)Bx\,ds$ (see p. 481, [6]), we have that

(2.7) $(T(t)-I)A^{\alpha-1}Bx = A^\alpha(T(t)-I)A^{-1}Bx = -\int_0^t A^\alpha T(s)Bx\,ds.$

Next, we show

(2.8) there is a constant C_x depending only on x such that $\| U(t)x - x \|_\alpha \leqq tC_x$ for $0 \leqq t \leqq 1$.

To prove (2.8) let $0 < t_0 \leqq 1$ and define $w: [0, t_0] \to [0, \infty)$ by $w(t) = \| U(t)x - x \|_\alpha$, $0 \leqq t \leqq t_0$. Using (2.6) and (2.7) we have for $0 \leqq t \leqq t_0$

$$\begin{aligned} w(t) &= \| (T(t)-I)(A^\alpha + A^{\alpha-1}B)x - (T(t)-I)A^{\alpha-1}Bx \\ &\qquad - \int_0^t A^\alpha T(t-s)BU(s)x\,ds \| \\ &= \left\| \int_0^t T(s)A(A^\alpha + A^{\alpha-1}B)x + \int_0^t A^\alpha T(t-s)[Bx - BU(s)x]\,ds \right\| \\ &\leqq Mt_0 e^{|\omega|t} \| A^\alpha(A+B)x \| + \int_0^t Ce^{|\omega|(t-s)}(t-s)^{-\alpha} L w(s)\,ds. \end{aligned}$$

Then, (2.8) follows from Lemma 1.1 with $a = Mt_0 \| (A+B)x \|_\alpha$, $b = CL$, and $\omega = |\omega|$. To complete the proof it remains to show that

(2.9) $$\lim_{t \to 0} \| (U(t)x - x)/t + (A+B)x \|_\alpha = 0.$$

For $0 < t \leqq 1$

$$\begin{aligned} &\left\| (T(t)x - x)/t - \int_0^t T(t-s)BU(s)x\,ds/t + (A+B)x \right\|_\alpha \\ &\quad = \left\| ((T(t)-I)/t)(A^\alpha + A^{\alpha-1}B)x + A^\alpha(A+B)x \right. \\ &\qquad \left. + \int_0^t A^\alpha T(t-s)[Bx - BU(s)x]\,ds/t \right\|. \end{aligned}$$

Then, (2.9) follows using (2.6), (2.8), and the estimate

$$\left\| \int_0^t A^{\alpha} T(t-s)[Bx - BU(s)x]\, ds/t \right\|$$
$$\leqq CL \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha} C_x ds$$
$$\leqq \text{const } t^{1-\alpha}.$$

3. Exponential representation of solutions. Before proving (1.2) we require the following lemmas:

LEMMA 3.3. *Let* $0 < \alpha < 1$. *There exists a constant* K *such that if* $\beta > 0, t > 0$, *and* n *is a positive integer, then*

$$(3.1) \qquad (t/n)^{1-\alpha} \sum_{k=1}^{n} \Gamma(k-\alpha) e^{-t\beta k/n}/\Gamma(k) < K\Gamma(1-\alpha)\beta^{\alpha-1}.$$

Proof. We will use the fact that there exists a constant K such that

$$(3.2) \qquad \Gamma(k-\alpha)/\Gamma(k) < Kk^{-\alpha} \quad \text{for} \quad k = 1, 2, \cdots,$$

which follows immediately from the fact that $\lim_{k\to\infty}\Gamma(k+1-\alpha)/\Gamma(k)k^{1-\alpha} = 1$ (see **[4]**, p. 195). Using (3.2) we have that

$$(t/n)^{1-\alpha} \sum_{k=1}^{n} \Gamma(k-\alpha) e^{-t\beta k/n}/\Gamma(k)$$
$$\leqq t^{1-\alpha} K \sum_{k=1}^{n} (e^{-t\beta k/n}(k/n)^{-\alpha})(1/n).$$

Since $\Sigma_{k=1}^{n}(e^{-t\beta k/n}(k/n)^{-\alpha})(1/n)$ is a lower approximating sum to the integral $\int_0^1 e^{-t\beta x} x^{-\alpha} dx$, (3.1) follows immediately using (1.11).

LEMMA 3.2. *Let* (1.3) *hold and let* $\alpha \in (0,1)$. *If* $x \in X, t \geqq 0$, *and* n *is a positive integer,*

$$(3.3) \qquad \|(I + t/nA)^{-k} x\|_{\alpha} \leqq (n/t)^{\alpha} C\Gamma(k-\alpha)(1-\omega t/n)^{\alpha-k} \|x\|/\Gamma(k)$$
$$\text{for } k = 1, \cdots, n.$$

Proof. We will use the formula

$$(3.4) \qquad (\lambda I + A)^{-k} = \int_0^{\infty} e^{-\lambda s} s^{k-1} T(s) x\, ds/(k-1)!$$

for $\lambda > \omega$, $k = 1, 2, \cdots$ (see **[2]**, p. 623). From (3.4) we obtain

$$\begin{aligned} &\| (I + t/nA)^{-k} \|_\alpha \\ &\quad = \left\| (n/t)^k \int_0^\infty e^{-ns/t} s^{k-1} A^\alpha T(s) x ds/(k-1)! \right\| \\ &\quad \leqq (n/t)^k \int_0^\infty C e^{(\omega - n/t)s} s^{k-1-\alpha} \| x \| ds/\Gamma(k) \end{aligned}$$

and (3.3) follows immediately using (1.11).

LEMMA 3.3. *Let* (1.3) *and* (1.7) *hold. If* $t > 0$ *and* n *is a positive integer sufficiently large, then*

(3.5) $(I + t/nF)^{-1}$ *exists as an everywhere defined mapping from* $(D(A^\alpha)$ *into* $D(A^\alpha)$;

(3.6) *for* $x \in D(A^\alpha)$ *and* $j = 1, 2, \cdots, n$, $(I + t/n)F)^{-j} x = (I + t/nA)^{-j} x$
$$- (t/n) \sum_{k=1}^{j} (I + t/nA)^{-k} B (I + t/nF)^{-(j-k+1)} x.$$

Proof. Let $x \in D(A^\alpha)$, let $t > 0$, let n be a positive integer, and define $G: D(A^\alpha)$ to $D(A^\alpha)$ by

$$G(y) = (I + t/nA)^{-1}(x - t/nBy), \quad y \in D(A^\alpha)$$

(note that G maps into $D(A) \subset D(A^\alpha)$, see **[3]**, p. 159). From (3.3) and (1.11) we obtain for $y_1, y_2 \in D(A^\alpha)$

$$\| G(y_1) - G(y_2) \|_\alpha \leqq C\Gamma(1-\alpha)(1 - \omega t/n)^{\alpha - 1}(t/n)^{1-\alpha} L \| y_1 - y_2 \|_\alpha.$$

For n sufficiently large G is a contraction from $D(A^\alpha)$ to $D(A^\alpha)$ and has a unique fixed point y which satisfies $(I + t/n(A + B))y = x$. (3.5) follows immediately and a simple induction argument proves (3.6).

REMARK 3.1. By virtue of (3.3), (3.6), and Corollary 2, p. 241 of **[9]** we have that $D(F)$ is dense in $D(A^\alpha)$, since

$$\begin{aligned} \| (I + t/nF)^{-1} x - x \|_\alpha \leqq{}& \| (I + t/nA)^{-1} A^\alpha x - A^\alpha x \| \\ &+ (t/n)(\| A^\alpha (I + t/nA)^{-1} (B(I + t/nF)^{-1} x - Bx) \| \\ &+ \| A^\alpha (I + t/nA)^{-1} Bx \|). \end{aligned}$$

LEMMA 3.4. *Let* (1.3) *and* (1.7) *hold, let* $x \in D(A^\alpha)$, $t_0 > 0$, *and* $\epsilon > 0$. *There exists a positive integer* N *such that if* $n \geqq N$, $1 \leqq j \leqq n$, *and* $0 \leqq t \leqq t_0$, *then*

$$\left\| (t/n) \sum_{k=1}^{j} (I + t/nA)^{-k} BU(t(j-k+1)/n)x \right.$$

(3.7)

$$\left. - \int_0^{tj/n} T(s)BU((tj/n) - s)x\,ds \right\|_\alpha < \epsilon.$$

Proof. Let $\{y_1, \cdots, y_m\}$ be a finite set in $(D(A^\alpha)$ *such that if* $0 \leqq s \leqq t_0$, then there exists some integer $i \in [1, m]$ such that $\| BU(s)x - y_i \| < \epsilon$ (here we use the fact that $D(A^\alpha)$ is dense in X and $BA^{-\alpha}A^\alpha U(s)x$ is continuous in s from $[0, t_0]$ to X and hence has compact range). Choose N such that if $n \geqq N$, then both of the following conditions hold:

(3.8) $\left\| (t/n) \sum_{k=1}^{j} ((I + t/nA)^{-k} - T(tk/n))\, y_i \right\|_\alpha < \epsilon$ for all $t \in [0, t_0]$, $j = 1, \cdots, n$, and $i = 1, \cdots, m$ (here we use the fact that for $z \in X$, $\lim_{n\to\infty} \| (I + s/nA)^{-n} z - T(s)z \| = 0$ uniformly for $0 \leqq s \leqq t_0$, see **[6]**, p. 481);

(3.9) $\left\| (t/n) \sum_{k=1}^{j} T(tk/n)\, z_{k,j,n} - \int_0^{tj/n} T(s)BU((tj/n) - s)x\,ds \right\|_\alpha < \epsilon$ for all $t \in [0, t_0]$ and $j = 1, \cdots, n$, where $z_{k,j,n} = BU(t(j-k+1)/n)x$.

Using Lemma 3.1 we can find a constant K_1 such that for all $n = 1, 2, \cdots$, and $0 \leqq t \leqq t_0$

$$(t/n)^{1-\alpha} \sum_{k=1}^{n} C\Gamma(k-\alpha)(1 - \omega t/n)^{\alpha-k}/\Gamma(k) \leqq K_1.$$

Also, there exists a constant K_2 such that for all $n = 1, 2, \cdots$, and $0 \leqq t \leqq t_0$

$$(t/n) \sum_{k=1}^{n} Ce^{\omega kt/n}(tk/n)^{-\alpha} < K_2.$$

Now let $n \geqq N$, let $0 \leqq t \leqq t_0$, let $1 \leqq j \leqq n$, and for each $k = 1, \cdots, j$ choose an integer $i(k) \in [1, m]$ such that $\| z_{k,j,n} - y_{i(k)} \| < \epsilon$. Then, using Lemma 3.2 and (3.8) we have

(3.10) $$\left\| (t/n) \sum_{k=1}^{j} ((I + t/nA)^{-k} - T(tk/n))\, z_{k,j,n} \right\|_\alpha$$

$$\leqq (t/n)^{1-\alpha} \sum_{k=1}^{j} (C\Gamma(k-\alpha)(1 - \omega t/n)^{\alpha-k}/\Gamma(k)) \| z_{k,j,n} - y_{i(k)} \|$$

$$+ \left\| (t/n) \sum_{k=1}^{j} ((I + t/nA)^{-k} - T(tk/n))\, y_{i(k)} \right\|_\alpha$$

$$+ (t/n) \sum_{k=1}^{n} Ce^{tk\omega/n}(tk/n)^{-\alpha} \| y_{i(k)} - z_{k,j,n} \| < K_1\epsilon + \epsilon + K_2\epsilon.$$

Then, (3.7) follows immediately from (3.9) and (3.10).

THEOREM. *Let* (1.3) *and* (1.7) *hold and let* $x \in D(A^\alpha)$. *Then, uniformly in bounded intervals of* t

$$\lim_{n\to\infty} \| (I + t/nF)^{-n}x - U(t)x \|_\alpha = 0. \tag{3.11}$$

Proof. Let $t_0 > 0$. Let K be a constant as in Lemma 3.1 and observe that

$$(1 - \omega t/n)^{\alpha-k} \leqq 1 \quad \text{for} \quad n \geqq 1, \quad k = 1, \cdots, n, \ 0 \leqq t \leqq t_0. \tag{3.12}$$

Let $\beta > 0$ such that $CLK\Gamma(1-\alpha)\beta^{\alpha-1} < 1/4$. Let $\epsilon > 0$ and choose N such that if $n \geqq N$ then (3.7) holds, as well as the following:

$$\| (I + t/nA)^{-j}x - T(tj/n)x \|_\alpha < \epsilon \text{ for } j = 1, \cdots, n, 0 \leqq t \leqq t_0; \tag{3.13}$$

$e^{t\beta/n} < 2$ for $0 \leqq t \leqq t_0$. (3.14) Now fix $n \geqq N$, $0 < t \leqq t_0$, and define

$$w_j = \| (I + t/nF)^{-j}x - U(tj/n)x \|_\alpha, \ j = 1, \cdots, n, \ S_n(t) = \sup_{1\leqq j\leqq n} e^{-t\beta j/n} w_j.$$

Using (3.12), (3.13), (3.14), and Lemmas 3.1, 3.2, 3.3, and 3.4 we have for $j = 1, \cdots, n$

$$\begin{aligned} w_j &= \Big\| (I + t/nA)^{-j}x + (t/n)\sum_{k=1}^{j} (I + t/nA)^{-k} B(I + t/nF)^{-(j-k+1)}x \\ &\qquad - T(tj/n)x - \int_0^{tj/n} T(s)BU(tj/n - s)x\,ds \Big\|_\alpha \\ &\leqq \| (I + t/nA)^{-j}x - T(tj/n)x \|_\alpha \\ &\quad + \Big\| (t/n)\sum_{k=1}^{j} (I + t/nA)^{-k} (B(I + t/nF)^{-(j-k+1)}x \\ &\qquad - BU(t(j-k+1)/n)x \Big\|_\alpha \\ &\quad + \Big\| (t/n)\sum_{k=1}^{j} (I + t/nA)^{-k} BU(t(j-k+1)/n)x \\ &\qquad - \int_0^{tj/n} T(s)BU((tj/n) - s)x\,ds \Big\|_\alpha \\ &< \epsilon + (t/n)^{1-\alpha}\sum_{k=1}^{j} (C\Gamma(k-\alpha)/\Gamma(k))Lw_{j-k+1} + \epsilon. \end{aligned} \tag{3.15}$$

From (3.15) we obtain for $j = 1, \cdots, n$

$$
\begin{aligned}
& e^{-t\beta j/n} w_j \\
& \quad < 2\epsilon + (t/n)^{1-\alpha} \sum_{k=1}^{j} (CL\Gamma(k-\alpha) e^{-t\beta k/n}/\Gamma(k))(e^{t\beta/n})(e^{-t\beta(j-k+1)/n} w_{j-k+1}) \\
& \quad < 2\epsilon + 2CLK\Gamma(1-\alpha)\beta^{\alpha-1} S_n(t) \\
& \quad < 2\epsilon + S_n(t)/2.
\end{aligned} \tag{3.16}
$$

Then, (3.16) implies $S_n(t) < 2\epsilon + S_n(t)/2$ or, equivalently, $S_n(t) < 4\epsilon$. (3.11) follows immediately and the proof is complete.

In conclusion we remark that our methods can be used to treat the existence and exponential representation of local solutions to (1.1) in the case that B satisfies a local Lipschitz continuity condition from $D(A^\alpha)$ to X. Also, our methods can be used in the numerical study of (1.1), a program which we will carry out elsewhere.

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