ESSENTIAL SPECTRUM Γ(β) OF A DUAL ACTION ON A VON NEUMANN ALGEBRA

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For a dual action β of a locally compact group G on a von Neumann algebra N we define the essential spectrum $\Gamma(\beta)$ as the intersection of all spectrum sp β^p of the restriction β^p of β to N_p when p runs over all nonzero projections in N^β. $\Gamma(\beta)$ is then an algebraic invariant for a covariant dual system $\{N, \beta\}$. $\Gamma(\beta)$ is a closed subgroup of G (Theorem 3.7). We introduce three kinds of concept for β such as integrable, regular and dominant (§§4, 5). The former concepts are weaker than the dominance. If β is regular, $\Gamma(\beta)$ coincides with the kernel of the action $\hat{\beta}$ on the center of the crossed dual product $N \otimes_\beta^d G$ (Theorem 6.1). If β is regular, $\Gamma(\beta)$ is normal and $\Gamma(\beta) = \Gamma(\tilde{\beta})$. If β is ergodic on the center $Z(N)$ and $\Gamma(\tilde{\beta}) = G$, then $N \otimes_\beta^d G$ is a factor and vice versa (Theorem 6.4). If β is regular, $\Gamma(\beta) = G$ is equivalent to $Z(N^\beta) \subset Z(N)$ (Proposition 6.3). If β is integrable on a factor N and if $\Gamma(\beta) = G$, then there is a lattice isomorphism between the closed subgroups of G and the von Neumann subalgebras of N containing N^β (Theorem 8.4). Moreover, by $N \otimes_\beta^d (H\backslash G)$ we mean the von Neumann algebra generated by $\beta(N)$ and $1 \otimes (L^\times(G) \cap \lambda'(H)')$, where H is a closed subgroup of G and λ' is the right regular representation of G. $N \otimes_\beta^d (H\backslash G)$ coincides with the set of $x \in N \otimes_\beta^d G$ such that $\hat{\beta}_t(x) = x$ for all $t \in H$ (Theorem 7.2).

0. Introduction. In our previous paper [**17, 16, 21**] we have generalized Takesaki's duality to a general locally compact group in terms of a dual action and a crossed dual product as the following:

$$(M \otimes_\alpha G) \otimes_{\tilde{\alpha}}^d G \sim M \otimes B(L^2(G))$$

$$(N \otimes_\beta^d G) \otimes_{\tilde{\beta}} G \sim N \otimes B(L^2(G)).$$

In this paper we continue our study on dual actions and Takesaki's duality obtained in the above from the view point of covariant systems $\{M, \alpha\}$ and covariant dual systems $\{N, \beta\}$. Then we naturally raise some questions:

  a. What is an invariant of equivalent covariant dual systems?

  b. When does Takesaki's duality hold as a covariant (dual) system?
Using the spectrum of β given in [**17**, §5], we can define the essential spectrum $\Gamma(\beta)$ of β in §3 by the same manner as S set. Then $\Gamma(\beta)$ is a closed subgroup of G and an algebraic invariant of dual actions on a

given von Neumann algebra. Therefore we can classify the dual actions into the family of closed subgroups of G.

c. In case of $\Gamma(\beta) = G$, what can we say about β, N^β and $N \otimes_\beta^d G$? For these questions Connes and Takesaki have obtained several interesting results in their paper [6] from the stand point of nonabelian cohomology in properly infinite von Neumann algebras. Following these ideas, we try to reproduce their Chapter III in terms of covariant (dual) systems. Our contents consist of following nine sections:

0. Introduction
1. Preliminary
2. Takesaki's duality
3. Essential spectrum $\Gamma(\beta)$
4. Integrable actions and dual actions
5. Regular actions and dual actions
6. Ergodic actions and dual actions
7. Subgroups and subalgebras
8. Galois correspondence
9. Appendix

More precisely, in §4 we shall recall two equivalent conditions given by Connes and Takesaki for an action to be integrable. The dual version of one of the conditions is utilized to define the integrability of a dual action. However, we are still unclear, whether the both dual versions of the above two conditions are equivalent or not.

In §5, for the sake of convenience, we call an action or a dual action to be regular when it is dual to some dual action or some action. The regularity is then stronger than integrability. Their characterizations are already given by Landstad and others, [15, 17, 16, 21]. Furthermore, we call a (dual) action to be dominant if it is regular and of infinite multiplicity. The dominance in our paper is a generalization of the one given by Connes and Takesaki to a non separable case. The utility of regularity and dominance will become clearer in our later sections as we analyze the equivalence class of covariant (dual) systems.

In §6, $\Gamma(\hat{\alpha})$ turns out to be the set of all $t \in G$ such that α_t is trivial on the center of M. Therefore, if $\beta \equiv \hat{\alpha}$, $\Gamma(\beta) = H$ is characterized by the fact that H is the largest closed subgroup satisfying that the center of $N \otimes_\beta^d G$ commutes with $1 \otimes \lambda'(H)$, where λ' is the left regular representation of G. Further, if β is ergodic on the center of N, then N^β is a factor, in addition, if $\Gamma(\beta) = G$, then $N \otimes_\beta^d G$ is a factor.

Let $M \otimes_\alpha H$ be the von Neumann algebra generated by $\alpha(M)$ and $1 \otimes \lambda(H)$, and let $N \otimes_\beta^d (H \backslash G)$ be the von Neumann algebra generated by $\beta(N)$ and $\mathscr{L}^\infty(H \backslash G)$, where $\mathscr{L}^\infty(H \backslash G) \equiv L^\infty(G) \cap \lambda'(H)'$. In §7 we shall characterize these von Neumann subalgebras, namely, $M \otimes_\alpha H$ is the set of y in $M \otimes_\alpha G$ with $\hat{\alpha}(y) \subset N \otimes \lambda(H)''$ and $N \otimes_\beta^d (H \backslash G)$ is the set of x in $N \otimes_\beta^d G$ with $\hat{\beta}_t(x) = x$ for all $t \in H$.

In §8 we shall give a Galois type correspondence between the closed subgroups of G and the von Neumann subalgebras of a factor N including N^β under the assumption that G is separable, β is integrable and $\Gamma(\beta) = G$. Let β^d be a dual action of G on $N \otimes_\beta^d G$ given by (1.8). If $N \otimes_\beta^d G$ is a factor and if L is a β^d invariant von Neumann subalgebra of $N \otimes_\beta^d G$ including $\beta(N)$, then L is of the form $N \otimes_\beta^d (H \backslash G)$ for some closed subgroup H of G. Conversely, $N \otimes_\beta^d (H \backslash G)$ is β^d invariant.

In the Appendix we shall give a sufficient condition for an action of a locally compact abelian group to be regular.

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1. Preliminary. Let G be a locally compact group, dt the Haar measure, $t \mapsto \lambda(t)$ the right regular representation of G on $L^2(G)$ and $R(G)$ the von Neumann algebra generated by $\lambda(G)$. Let J be the antiunitary involution on $L^2(G)$ defined by $(J^c \xi)(t) \equiv \Delta(t)^{1/2} \xi(t^{-1})$ for $\xi \in L^2(G)$ and $\lambda'(r) \equiv J^c \lambda(r) J^c$, where Δ is the modular function. Then λ' is the left regular representation:

$$(\lambda'(r)\xi)(t) = \Delta(r)^{1/2} \xi(r^{-1}t).$$

Let M and N be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. γ and δ are isomorphisms $R(G) \to R(G) \otimes R(G)$ and $L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ satisfying

$$\gamma\lambda(r) = \lambda(r) \otimes \lambda(r) \quad \text{and} \quad (\delta f)(s,t) = f(st)$$

for $f \in L^\infty(G)$. An *action* α of G on M is an isomorphism of M into $M \otimes L^\infty(G)$ satisfying

(1.1) $$\alpha \otimes \iota \circ \alpha = \iota \otimes \delta \circ \alpha,$$

where ι denotes the identity automorphism and the association of the tensor product $\otimes$ is stronger than that of the composition $\circ$. By [17, Theorem 2.1] an action α is induced from an action $t \mapsto \alpha_t$ of G on M in the usual sense by $(\alpha(x)\xi)(t) = \alpha_t(x)\xi(t)$ for $\xi \in \mathcal{H} \otimes L^2(G)$. As $\alpha(\alpha_t(x)) = Ad\, 1 \otimes \lambda(t)(\alpha(x))$ for all t, $\{M, \alpha\}$ is a *covariant system*. A *dual action* β of G on N is defined as an isomorphism of N into $N \otimes R(G)$ satisfying

(1.2) $$\beta \otimes \iota \circ \beta = \iota \otimes \gamma \circ \beta.$$

Such a pair $\{N, \beta\}$ is called a *covariant dual system*.

The *crossed product* $M \otimes_\alpha G$ of M by G with respect to α is the von Neumann algebra generated by $\alpha(M)$ and $1 \otimes R(G)$. The *crossed dual product* $N \otimes_\beta^d G$ of N and G with respect to β is defined as the von Neumann algebra generated by $\beta(N)$ and $1 \otimes L^\times(G)$.

Now we define a unitary W on $L^2(G) \otimes L^2(G)$ by

$$(W\xi)(s, t) \equiv \xi(s, ts).$$

Then $W^* = Ad J \otimes 1(W)$, where $(J\xi)(t) \equiv \Delta(t)^{1/2} \xi(t^{-1})$ for $\xi \in L^2(G)$. We set

$$W' \equiv Ad\, 1 \otimes J(W), \qquad V \equiv \sigma(W) \quad \text{and} \quad V' \equiv \sigma(W'),$$

where σ denotes the symmetric isomorphism

$$x \otimes y \mapsto y \otimes x.$$

These unitaries satisfy $(W'\xi)(s, t) = \Delta(s)^{1/2}\xi(s, s^{-1}t)$, $(V\xi)(s, t) = \xi(st, t)$ and $(V'\xi)(s, t) = \Delta(t)^{1/2}\xi(t^{-1}s, t)$. Therefore (1.1) and (1.2) are of the forms

$$(1.3) \qquad \begin{aligned} &\alpha \otimes \iota \circ \alpha(x) = Ad\, 1 \otimes V(\alpha(x) \otimes 1_G) \\ &\beta \otimes \iota \circ \beta(y) = Ad\, 1 \otimes W^*(\beta(y) \otimes 1_G). \end{aligned}$$

On the other hand, since

$$\begin{aligned} Ad\, 1 \otimes W^*(\alpha(x) \otimes 1_G) &= \alpha(x) \otimes 1_G, \\ Ad\, 1 \otimes W^*(\lambda_1(r) \otimes 1_G) &= \lambda_1(r) \otimes \lambda(r) \\ Ad\, 1 \otimes V'(\beta(y) \otimes 1_G) &= \beta(y) \otimes 1_G, \\ Ad\, 1 \otimes V'(1 \otimes f \otimes 1_G) &= 1 \otimes \epsilon f, \end{aligned}$$

where $\lambda_1(r) \equiv 1 \otimes \lambda(r)$ and $(\epsilon f)(s, t) \equiv f(t^{-1}s)$, it follows that

$$(1.4) \qquad \begin{aligned} &Ad\, 1 \otimes W^*((M \otimes_\alpha G) \otimes 1_G) \subset (M \otimes_\alpha G) \otimes R(G) \\ &Ad\, 1 \otimes V'((N \otimes_\beta^d G) \otimes 1_G) \subset (N \otimes_\beta^d G) \otimes L^\times(G). \end{aligned}$$

We denote their restrictions by $\hat{\alpha}$ and $\hat{\beta}$:

$$(1.5) \qquad \begin{aligned} \hat{\alpha}(x) &\equiv Ad\, 1 \otimes W^*(x \otimes 1_G) \\ \hat{\beta}(y) &\equiv Ad\, 1 \otimes V'(y \otimes 1_G) \end{aligned}$$

for $x \in M \otimes_\alpha G$ and $y \in N \otimes_\beta^d G$. Since

$$(1.6) \quad \begin{aligned} &W^* \otimes 1_G (\iota \otimes \sigma)(W^* \otimes 1_G) = Ad \, 1_G \otimes W^*(W^* \otimes 1_G) \\ &V' \otimes 1_G (\iota \otimes \sigma)(V' \otimes 1_G) = Ad \, 1_G \otimes V(V' \otimes 1_G), \end{aligned}$$

it is shown in [17, Theorem 2.3] that $\hat{\alpha}$ is a dual action of G on $M \otimes_\alpha G$ and $\hat{\beta}$ is an action of G on $N \otimes_\beta^d G$. $\hat{\alpha}$ and $\hat{\beta}$ are said to be the *dual action dual to* α and the *action dual to* β. We denote the action dual to $\hat{\alpha}$ by $\hat{\hat{\alpha}}$ or $\tilde{\alpha}$ and the dual action dual to $\hat{\beta}$ by $\hat{\hat{\beta}}$ or $\tilde{\beta}$.

We know from (1.3) that

$$(1.7) \quad \begin{aligned} &Ad \, 1 \otimes V(\alpha(M) \otimes 1_G) \subset \alpha \otimes \iota(M \otimes L^\times(G)) \\ &Ad \, 1 \otimes W^*(\beta(N) \otimes 1_G) \subset \beta \otimes \iota(N \otimes R(G)). \end{aligned}$$

Since $[V, \lambda'(r) \otimes f] = 0$ and $[W, f \otimes 1_G] = 0$, it follows that

$$\begin{aligned} &Ad \, 1 \otimes V(1 \otimes R(G) \otimes 1_G) \subset 1 \otimes R(G) \otimes L^\times(G) \\ &Ad \, 1 \otimes W^*(1 \otimes L^\times(G) \otimes 1_G) = 1 \otimes L^\times(G) \otimes 1_G. \end{aligned}$$

Therefore

$$\begin{aligned} &Ad \, 1 \otimes V((M \otimes_\alpha G) \otimes 1_G) \subset (M \otimes_\alpha G) \otimes L^\times(G) \\ &Ad \, 1 \otimes W^*((N \otimes_\beta^d G) \otimes 1_G) \subset (N \otimes_\beta^d G) \otimes R(G), \end{aligned}$$

which is compared with (1.4). If we define α^d and β^d by

$$(1.8) \quad \begin{aligned} &\alpha^d(x) \equiv Ad \, 1 \otimes V(x \otimes 1_G) \\ &\beta^d(y) \equiv Ad \, 1 \otimes W^*(y \otimes 1_G) \end{aligned}$$

for $x \in M \otimes_\alpha G$ and $y \in N \otimes_\beta^d G$, then α^d is an action of G on $M \otimes_\alpha G$ and β^d is a dual action of G on $N \otimes_\beta^d G$. Indeed, (1.3) for α^d and β^d is proved by

$$(1.9) \quad V \otimes 1_G (\iota \otimes \sigma)(V \otimes 1_G) = Ad \, 1_G \otimes V(V \otimes 1_G)$$

and (1.6) for W.

On the other hand, since $[V, 1 \otimes f] = 0$ and $[W, 1 \otimes \lambda'(r)] = 0$, it follows from (1.7) that

$$\begin{aligned} &Ad \, 1 \otimes V^*(\alpha(M)' \otimes 1_G) \subset \alpha(M)' \otimes L^\times(G) \\ &Ad \, 1 \otimes W(\beta(N)' \otimes 1_G) \subset \beta(N)' \otimes R(G). \end{aligned}$$

If we define α' and β' by

$$(1.10) \quad \begin{aligned} &\alpha'(x) \equiv Ad \, 1 \otimes V^*(x \otimes 1_G) \\ &\beta'(y) \equiv Ad \, 1 \otimes W(y \otimes 1_G) \end{aligned}$$

for $x \in \alpha(M)'$ and $y \in \beta(N)'$, then we have the following theorem due to Landstad, [16, the proof of Lemma 1].

THEOREM 1.1. β' is a dual action of G on $\beta(N)'$ satisfying
(i) $(\beta(N)')^{\beta'} = (N \otimes_\beta^d G)'$; and
(ii)   there is an action α of G on $(N \otimes_\beta^d G)'$ satisfying

$$\{\beta(N)', \beta'\} \sim \{(N \otimes_\beta^d G)' \otimes_\alpha G, \hat\alpha\},$$

where $(\beta(N)')^{\beta'} \equiv \{y \in \beta(N)' : \beta'(y) = y \otimes 1_G\}$.

As this theorem is important from the technical point of view, we shall give a dual version, although it is unnecessary for our later use.

An isomorphism α of M into $M \otimes L^\infty(G)$ satisfying $\alpha \otimes \iota \circ \alpha = \iota \otimes \delta^c \circ \alpha$ $(\delta^c f \equiv \text{Ad } V'^*(f \otimes 1_G))$ is called an action of G on M with respect to $R(G)'$, [17, Appendix].

THEOREM 1.2. α' is an action of G on $\alpha(M)'$ with respect to $R(G)'$ satisfying
(i) $(\alpha(M)')^{\alpha'} = (M \otimes_\alpha G)'$; and
(ii)   there is a dual action β^c of G on $(M \otimes_\alpha G)'$ with respect to $R(G)'$ such that

$$\{\alpha(M)', \alpha'\} \sim \{(M \otimes_\alpha G)' \otimes_{\beta^c}^d G, \hat\beta^c\}.$$

Proof.   Since

$$(V^* \otimes 1_G)\iota \otimes \sigma(V^* \otimes 1_G) = \text{Ad } 1_G \otimes V'^*(V^* \otimes 1_G),$$

we have

$\alpha' \otimes \iota \circ \alpha'(x)$

$$= \text{Ad } 1 \otimes V^* \otimes 1_G \circ \iota \otimes \iota \otimes \sigma \circ \text{Ad } 1 \otimes V^* \otimes 1_G(x \otimes 1_G \otimes 1_G)$$

$$= \text{Ad } 1 \otimes 1_G \otimes V'^* \circ \text{Ad } 1 \otimes V^* \otimes 1_G(x \otimes 1_G \otimes 1_G)$$

$$= \iota \otimes \delta^c \circ \alpha'(x),$$

where $\delta^c(f) \equiv \text{Ad } V'^*(f \otimes 1_G)$ for $f \in L^\infty(G)$.   Therefore α' is an action of G on $\alpha(M)'$ with respect to $R(G)'$.

(i) Put $N \equiv (\alpha(M)')^{\alpha'}$.   Since M is standard, we have a weakly continuous unitary representation u of G on $\mathcal{H}$ implementing α, [1, 9]. $(M \otimes_\alpha G)'$ is then generated by $M' \otimes 1_G$ and $U(1 \otimes R(G)')U^*$, where U is a unitary on $\mathcal{H} \otimes L^2(G)$ defined by $(U\xi)(s) \equiv u(s)\xi(s)$, [7, 10, 16].   As $(M \otimes_\alpha G)' \subset N$ is clear, it suffices to show the converse inclusion.   For this we suppose that $x \in N$.   Then $x \in \alpha(M)'$ and

$Ad\ 1 \otimes V^*(x \otimes 1_G) = x \otimes 1_G.$ Therefore

$$x \otimes 1_G \in (1 \otimes 1_G \otimes R(G))' \cap Ad\ 1 \otimes V^*(1 \otimes 1_G \otimes R(G))'$$
$$= (1 \otimes R(G) \otimes R(G))'$$

and hence

$$x \in \alpha(M)' \cap (1 \otimes R(G))' = (M \otimes_\alpha G)'.$$

(ii)   Since $\alpha(M) = (M \otimes_\alpha G)^{\hat\alpha}$ by   (2.7)   and   since $\alpha(M) = (M \otimes_\alpha G) \cap (1 \otimes L^\infty(G))'$ by Theorem 7.1, it follows that

$$\alpha(M)' = ((M \otimes_\alpha G)' \cup (1 \otimes L^\infty(G)))''.$$

Moreover,

$$\alpha'_t(1 \otimes f) = Ad\ 1 \otimes \lambda(t)^*(1 \otimes f)$$

for $f \in L^\infty(G)$.   By [16: 17, Theorem 8.3; 18] we have the desired result.

Finally, we recall the equivalence used in the above theorems.   Let α^j (resp. β_j) be an action (resp. a dual action) of G on M_j (resp. N_j) for $j = 1, 2$.   We denote

$$\{M_1, \alpha^1\} \sim \{M_2, \alpha^2\}   (\text{resp. } \{N_1, \beta_1\} \sim \{N_2, \beta_2\}),$$

if there exists an isomorphism ρ of M_1 (resp. N_1) onto M_2 (resp. N_2) satisfying

$$\rho \otimes \iota \circ \alpha^1 = \alpha^2 \circ \rho   (\text{resp. } \rho \otimes \iota \circ \beta_1 = \beta_2 \circ \rho).$$

Then it is direct from (1.1) and (1.2) that

$$\{M, \alpha\} \sim \{\alpha(M), \iota \otimes \delta\}   \text{and}   \{N, \beta\} \sim \{\beta(N), \iota \otimes \gamma\}.$$

2.   Takesaki's duality.

In this section we shall give a few supplements to our previous results obtained in [17].

We begin by recalling Takesaki's duality [16, 17, 18].   If π_M is an isomorphism of M into $(M \otimes_\alpha G) \otimes_{\hat\alpha}^d G$ given by

$$\pi_M(x) \equiv Ad\ 1 \otimes V^*(\alpha(x) \otimes 1_G)$$

for x in M, then $(M \otimes_\alpha G) \otimes_{\hat\alpha}^d G$ is generated by $\pi_M(x)$ $(x \in M)$, $1 \otimes \lambda(r) \otimes \lambda(r)$ $(r \in G)$ and $1 \otimes 1_G \otimes f$ $(f \in L^\infty(G))$, [17].   We denote

by ρ an isomorphism of $M \otimes B(L^2(G))$ onto $(M \otimes_\alpha G) \otimes_{\tilde{\alpha}}^d G$ defined by

$$(2.1) \qquad\qquad\qquad Ad\, 1 \otimes V^* \circ \alpha \otimes \iota.$$

Then $\rho(x \otimes 1_G) = \pi_M(x)$, $\rho(1 \otimes \lambda(r)) = 1 \otimes \lambda(r) \otimes \lambda(r)$ and $\rho(1 \otimes f) = 1 \otimes 1_G \otimes f$.

THEOREM 2.1. *If α is an action of G on M, then*

$$\{(M \otimes_\alpha G) \otimes_{\tilde{\alpha}}^d G, \tilde{\alpha}\} \sim \{M \otimes B(L^2(G)), [\alpha]\},$$

where $\qquad [\alpha] \equiv Ad\, 1 \otimes V' \circ \iota \otimes \sigma \circ \alpha \otimes \iota.$ *In particular,* $\qquad [\alpha]_t = \alpha_t \otimes Ad\, \lambda'(t)$ *for $t \in G$.*

LEMMA 2.2. *If ρ is given by (2.1), then*

$$\tilde{\alpha} \circ \rho = \rho \otimes \iota \circ Ad\, 1 \otimes V' \circ \bar{\alpha}$$

on $M \otimes 1_G$, where $\bar{\alpha} \equiv \iota \otimes \sigma \circ \alpha \otimes \iota.$

Proof. Since

$$(2.2) \qquad V^* \otimes 1_G(\iota \otimes \sigma)(V^* \otimes 1_G) = Ad\, 1_G \otimes V'^*(V^* \otimes 1_G),$$

we have

$$(2.3) \quad (1_G \otimes V')(V^* \otimes 1_G) = (V^* \otimes 1_G)(1_G \otimes V')\iota \otimes \sigma(V \otimes 1_G).$$

If $x \in M$, then $\iota \otimes \sigma(x \otimes 1_G \otimes 1_G) = x \otimes 1_G \otimes 1_G$. Therefore

$$\rho \otimes \iota \circ Ad\, 1 \otimes V' \circ \bar{\alpha}(x \otimes 1_G)$$
$$= Ad\, 1 \otimes V^* \otimes 1_G \circ \alpha \otimes \iota \otimes \iota \circ Ad\, 1 \otimes V' \circ \bar{\alpha}(x \otimes 1_G) \quad \text{(By (2.1))}$$
$$= Ad((1 \otimes V^* \otimes 1_G)(1 \otimes 1_G \otimes V')) \circ \alpha \otimes \iota \otimes \iota \circ \bar{\alpha}(x \otimes 1_G)$$
$$= Ad((1 \otimes V^* \otimes 1_G)(1 \otimes 1_G \otimes V')\iota \otimes \iota \otimes \sigma(1 \otimes V \otimes 1_G))$$
$$\quad \circ \alpha \otimes \iota \otimes \iota(x \otimes 1_G \otimes 1_G) \quad \text{(By (1.3))}$$
$$= Ad((1 \otimes 1_G \otimes V')(1 \otimes V^* \otimes 1_G)) \circ \alpha \otimes \iota \otimes \iota(x \otimes 1_G \otimes 1_G)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(By (2.3))}$$
$$= Ad\, 1 \otimes 1_G \otimes V' \circ \rho \otimes \iota(x \otimes 1_G \otimes 1_G) \quad \text{(By (2.1))}$$
$$= \tilde{\alpha} \circ \rho(x \otimes 1_G).$$

This completes the proof.

Proof of Theorem 2.1.   It is immediate from the following calculation:

$$(\rho \otimes \iota)^{-1} \circ \tilde{\alpha} \circ \rho(1 \otimes \lambda(r)) = (\rho \otimes \iota)^{-1} \circ \tilde{\alpha}(1 \otimes \lambda(r) \otimes \lambda(r))$$
$$= (\rho \otimes \iota)^{-1}(1 \otimes \lambda(r) \otimes \lambda(r) \otimes 1_G)$$
$$= 1 \otimes \lambda(r) \otimes 1_G = [\alpha](1 \otimes \lambda(r))$$

and

$$(\rho \otimes \iota)^{-1} \circ \tilde{\alpha} \circ \rho(1 \otimes f) = (\rho \otimes \iota)^{-1} \circ \tilde{\alpha}(1 \otimes 1_G \otimes f)$$
$$= (\rho \otimes \iota)^{-1}(1 \otimes 1_G \otimes \epsilon f)$$
$$= 1 \otimes \epsilon f = [\alpha](1 \otimes f).$$

This completes the proof.

The dual version of the above result is obtained as follows.   If π_N is an isomorphism of N into $(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G$ given by

$$\pi_N(y) \equiv Ad\ 1 \otimes W'(\beta(y) \otimes 1_G)$$

for y in N, then $(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G$ is generated by $\pi_N(y)$ ($y \in N$), $1 \otimes \epsilon f$ ($f \in L^\infty(G)$) and $1 \otimes 1_G \otimes \lambda(r)$ ($r \in G$), [17]. We denote by ρ an isomorphism of $N \otimes B(L^2(G))$ onto $(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G$ defined by

(2.4) $$Ad\ 1 \otimes 1_G \otimes J \circ Ad\ 1 \otimes W \circ \beta \otimes \iota.$$

where J is given by $(J\xi)(t) \equiv \Delta(t)^{1/2}\xi(t^{-1})$ for $\xi \in L^2(G)$. Then $\rho(y \otimes 1_G) = \pi_N(y)$, $\rho(1 \otimes f) = 1 \otimes \epsilon f$ and $\rho(1 \otimes \lambda'(r)) = 1 \otimes 1_G \otimes \lambda(r)$.

THEOREM 2.3.   *If β is a dual action on N, then*

$$\{(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G, \tilde{\beta}\} \sim \{N \otimes B(L^2(G)), [\beta]\},$$

where $[\beta] \equiv Ad\ 1 \otimes W \circ \iota \otimes \sigma \circ \beta \otimes \iota.$

LEMMA 2.4.   *If ρ is defined by* (2.4), *then*

$$\tilde{\beta} \circ \rho = \rho \otimes \iota \circ Ad\ 1 \otimes W \circ \bar{\beta}$$

on $N \otimes 1_G$, where $\bar{\beta} \equiv \iota \otimes \sigma \circ \beta \otimes \iota.$

Proof.   Since

(2.5) $$W \otimes 1_G(\iota \otimes \sigma)(W \otimes 1_G) = Ad\ 1_G \otimes W^*(W \otimes 1_G),$$

we have

$$(2.6) \quad (W \otimes 1_G)(1_G \otimes W)\iota \otimes \sigma(W^* \otimes 1_G) = (1_G \otimes W)(W \otimes 1_G).$$

If $y \in N$, then $\iota \otimes \sigma(y \otimes 1_G \otimes 1_G) = y \otimes 1_G \otimes 1_G$. Therefore

$$Ad \ 1 \otimes 1_G \otimes J \otimes 1_G \circ \rho \otimes \iota \circ Ad \ 1 \otimes W \circ \bar{\beta}(y \otimes 1_G)$$

$$= Ad \ 1 \otimes W \otimes 1_G \circ \beta \otimes \iota \otimes \iota \circ Ad \ 1 \otimes W \circ \bar{\beta}(y \otimes 1_G) \quad \text{(By (2.4))}$$

$$= Ad((1 \otimes W \otimes 1_G)(1 \otimes 1_G \otimes W)) \circ \beta \otimes \iota \otimes \iota \circ \bar{\beta}(y \otimes 1_G)$$

$$= Ad((1 \otimes W \otimes 1_G)(1 \otimes 1_G \otimes W))$$

$$\circ \iota \otimes \iota \otimes \sigma \circ \beta \otimes \iota \otimes \iota \circ \beta \otimes \iota(y \otimes 1_G)$$

$$= Ad((1 \otimes W \otimes 1_G)(1 \otimes 1_G \otimes W)\iota \otimes \iota \otimes \sigma(1 \otimes W^* \otimes 1_G))$$

$$\circ \beta \otimes \iota \otimes \iota(y \otimes 1_G \otimes 1_G) \quad \text{(By (1.3))}$$

$$= Ad((1 \otimes 1_G \otimes W)(1 \otimes W \otimes 1_G)) \circ \beta \otimes \iota \otimes \iota(y \otimes 1_G \otimes 1_G)$$

$$\text{(By (2.6))}$$

$$= Ad((1 \otimes 1_G \otimes W)(1 \otimes 1_G \otimes J \otimes 1_G)) \circ \rho \otimes \iota(y \otimes 1_G \otimes 1_G)$$

$$\text{(By (2.4))}$$

$$= Ad((1 \otimes 1_G \otimes J \otimes 1_G)(1 \otimes 1_G \otimes W^*) \circ \rho \otimes \iota(y \otimes 1_G \otimes 1_G)$$

$$= Ad(1 \otimes 1_G \otimes J \otimes 1_G) \circ \bar{\beta} \circ \rho(y \otimes 1_G).$$

This completes the proof.

Proof of Theorem 2.3.   It is immediate from the

$$(\rho \otimes \iota)^{-1} \circ \bar{\beta} \circ \rho(1 \otimes f) = (\rho \otimes \iota)^{-1} \circ \bar{\beta}(1 \otimes \epsilon f)$$

$$= (\rho \otimes \iota)^{-1}(1 \otimes \epsilon f \otimes 1_G) = 1 \otimes f \otimes 1_G = [\beta](1 \otimes f)$$

and

$$(\rho \otimes \iota)^{-1} \circ \bar{\beta} \circ \rho(1 \otimes \lambda'(r)) = (\rho \otimes \iota)^{-1} \circ \bar{\beta}(1 \otimes 1_G \otimes \lambda(r))$$

$$= (\rho \otimes \iota)^{-1}(1 \otimes 1_G \otimes \lambda(r) \otimes \lambda(r))$$

$$= 1 \otimes \lambda'(r) \otimes \lambda(r) = [\beta](1 \otimes \lambda'(r)).$$

This completes the proof.

By the use of operator valued weights E_α and E_β we have shown in [16; 17, Proposition 6.1 and 6.4; 18] that

$$(2.7) \qquad (M \otimes_\alpha G)^{\hat{\alpha}} = \alpha(M) \quad \text{and} \quad (N \otimes_\beta^d G)^{\hat{\beta}} = \beta(N).$$

Combining (2.7) and Takesaki's duality in the above, we have the following theorem.

THEOREM 2.5. (i) *If α is an action of G on M, then*

$$(2.8) \qquad M \otimes_\alpha G = (M \otimes B(L^2(G)))^{[\alpha]}.$$

(ii) *If β is a dual action of G on N, then*

$$(2.9) \qquad N \otimes_\beta^d G = (N \otimes B(L^2(G)))^{[\beta]}.$$

Proof. (i) Let ρ be the isomorphism of $M \otimes B(L^2(G))$ onto $(M \otimes_\alpha G) \otimes_{\hat{\alpha}}^d G$ given by (2.1). Then, by Theorem 2.1 and (2.7), we have

$$\rho((M \otimes B(L^2(G)))^{[\alpha]}) = \hat{\alpha}(M \otimes_\alpha G).$$

Since

$$\rho^{-1} \circ \hat{\alpha}(\alpha(x)) = (\alpha \otimes \iota)^{-1} \circ Ad\, 1 \otimes V(\alpha(x) \otimes 1_G) = \alpha(x)$$

by (1.3) and

$$\rho^{-1} \circ \hat{\alpha}(1 \otimes \lambda(r)) = \rho^{-1}(1 \otimes \lambda(r) \otimes \lambda(r)) = 1 \otimes \lambda(r)$$

as before, we have (2.8).

(ii) Let ρ be the isomorphism of $N \otimes B(L^2(G))$ onto $(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G$ given by (2.4). Then, by Theorem 2.3 and (2.7), we have

$$\rho((N \otimes B(L^2(G)))^{[\beta]}) = \hat{\beta}(N \otimes_\beta^d G).$$

Since

$$\rho^{-1} \circ \hat{\beta}(\beta(y)) = (\beta \otimes \iota)^{-1} \circ Ad((1 \otimes W^*)(1 \otimes 1_G \otimes J))(\beta(y) \otimes 1_G)$$
$$= \beta(y)$$

by (1.3) and

$$\rho^{-1} \circ \hat{\beta}(1 \otimes f) = \rho^{-1}(1 \otimes \epsilon f) = 1 \otimes f$$

as before, we have (2.9).

Hereafter we shall use the following notations:

$$\bar{M} \equiv M \otimes B(L^2(G)), \qquad \bar{\alpha} \equiv \iota \otimes \sigma \circ \alpha \otimes \iota, \qquad [\alpha] \equiv Ad\, 1 \otimes V' \circ \bar{\alpha}$$

$$\bar{N} \equiv N \otimes B(L^2(G)), \qquad \bar{\beta} \equiv \iota \otimes \sigma \circ \beta \otimes \iota, \qquad [\beta] \equiv Ad\, 1 \otimes W \circ \bar{\beta}$$

without any reference.

3. Essential spectrum $\Gamma(\beta)$.

In this section we shall define an invariant $\Gamma(\beta)$ of a dual action β in analogy with $\Gamma(\alpha)$ defined for an action α by Connes, [5]. The argument will proceed similarly as that for α.

For a dual action β of G on N we define β_ϕ and Φ_ω for $\phi \in R(G)_*$ and $\omega \in N_*$ by

$$\langle \beta_\phi(y), \omega \rangle = \langle \beta(y), \omega \otimes \phi \rangle = \langle \Phi_\omega(y), \phi \rangle.$$

Let m_y and m denote the set of all $\phi \in R(G)_*$ with $\beta_\phi(y) = 0$ and $\beta_\phi = 0$, respectively. The spectrum $sp_\beta(y)$ of y with respect to β and the spectrum $sp(\beta)$ are given by

$$sp_\beta(y) \equiv \bigcap_{\phi \in m_y} \Gamma(\phi) \quad \text{and} \quad sp(\beta) \equiv \bigcap_{\phi \in m} \Gamma(\phi),$$

where $\Gamma(\phi)$ denotes the set of all $t \in G$ with $\langle \lambda(t), \phi \rangle = 0$. For a closed subset E of G we denote by $N^\beta(E)$ the set of all y in N satisfying $sp_\beta(y) \subset E$. Then $N^\beta(\{e\}) = N^\beta$.

Now, we shall give some properties of spectrum $sp_\beta(y)$ and the related ones.

LEMMA 3.1. *For any ϕ and ψ in $R(G)_*$ and for closed subsets E and F of G the following eight statements hold:*

(i) $sp_\beta(\beta_\phi(y)) \subset sp_\beta(y) \backslash \Gamma(\phi)^\circ$, *where $\Gamma(\phi)^\circ$ denotes the interior of $\Gamma(\phi)$;*

(ii) $y \in N^\beta(E)$ *if and only if $E\mathcal{U} \subset \Gamma(\phi)$ implies $\beta_\phi(y) = 0$ for all neighbourhood $\mathcal{U}$ of e and for all $\phi \in R(G)_*$;*

(iii) $N^\beta(E)$ *is a weakly closed vector subspace of N;*

(iv) $sp_\beta(x^*) = sp_\beta(x)^{-1}$;

(v) *if D is weakly total in N, then $sp(\beta)$ is the closure of union of all $sp_\beta(y)$ with $y \in D$;*

(vi) $t \in sp(\beta)$ *if and only if $N^\beta(\mathcal{U}) \neq \{0\}$ for all compact neighbourhoods $\mathcal{U}$ of t;*

(vii) *if $\lambda_* \phi = \lambda_* \psi$ on some neighbourhood of $sp_\beta(y)$, then $\beta_\phi(y) = \beta_\psi(y)$; and*

(viii) *if E or F is compact, $N^\beta(E)N^\beta(F) \subset N^\beta(E \cdot F)$.*

Proof. (i) Put $z \equiv \beta_\phi(y)$. If $\beta_\psi(y) = 0$, then $\beta_\psi(z) = \beta_\phi(\beta_\psi(y)) =$

0. Indeed, $\langle \lambda(t), \phi\psi \rangle = \langle \lambda(t), \phi \rangle \langle \lambda(t), \psi \rangle = \langle \lambda(t), \psi\phi \rangle$ and $\beta_\psi \beta_\phi = \beta_{\phi\psi} = \beta_\phi \beta_\psi$ by the property of a dual action. Therefore $m_y \subset m_z$ and hence $sp_\beta(z) \subset sp_\beta(y)$.

If t is in $\Gamma(\phi)^\circ$, there exists a ψ in $R(G)_*$ such that $\langle \lambda(t), \psi \rangle \neq 0$ and $\phi\psi = 0$. Since $\phi\psi = 0$ implies $\beta_\psi(z) = 0$, it follows that $t \notin sp_\beta(z)$. Therefore $sp_\beta(z) \subset G \setminus \Gamma(\phi)^\circ$.

(ii) Suppose that $y \in N^\beta(E)$. For any neighbourhood $\mathcal{U}$ of e and ϕ in $R(G)_*$, if $E\mathcal{U} \subset \Gamma(\phi)$, then $E \subset \Gamma(\phi)^\circ$ and hence $E \setminus \Gamma(\phi)^\circ = \varnothing$. Since $sp_\beta(\beta_\phi(y)) = \varnothing$ by (i), we have $\beta_\phi(y) = 0$.

Conversely, if $t \notin E$, there exist a neighbourhood $\mathcal{U}$ of e and a ϕ in $R(G)_*$ such that $E\mathcal{U} \subset \Gamma(\phi)$ and $\langle \lambda(t), \phi \rangle \neq 0$. From assumption it follows that $\beta_\phi(y) = 0$. Since $\langle \lambda(t), \phi \rangle \neq 0$, we have $t \notin sp_\beta(y)$. Therefore $sp_\beta(y) \subset E$, namely, $y \in N^\beta(E)$.

(iii) If $y_\kappa \in N^\beta(E)$ and y_κ converges weakly to y, then

$$\langle \beta_\phi(y), \omega \rangle = \langle \beta(y), \omega \otimes \phi \rangle = \lim \langle \beta(y_k), \omega \otimes \phi \rangle$$

for all $\omega \in N_*$. Therefore $E\mathcal{U} \subset \Gamma(\phi)$ implies $\beta_\phi(y) = 0$, and hence $y \in N^\beta(E)$ by (ii).

(iv) Let m_t denote the ideal $\{\lambda(t)\}^\perp$ of $R(G)_*$. If $t \in sp_\beta(x^*)$, then $m_{x^*} \subset m_t$. Since $(m_x)^* = m_{x^*}$ and $(m_t)^* = m_{t^{-1}}$, we have $m_x \subset m_{t^{-1}}$ and hence $t^{-1} \in sp_\beta(x)$. Therefore $sp_\beta(x^*) \subset sp_\beta(x)^{-1}$. Changing the role of x and x^*, we have $sp_\beta(x) \subset sp_\beta(x^*)^{-1}$. Thus $sp_\beta(x^*) = sp_\beta(x)^{-1}$.

(v) Let E be the closure of union of all $sp_\beta(y)$ with $y \in D$. Since $E \subset sp(\beta)$ is clear, it suffices to show the converse inclusion. For this we suppose tht $sp(\beta) \setminus E$ is nonempty and t is its element. For any compact neighbourhood $\mathcal{U}$ of e with $t\mathcal{U} \cap E = \varnothing$ there exists a $\phi \in R(G)_*$ such that $\langle \lambda(t), \phi \rangle \neq 0$ and $G \setminus \Gamma(\phi)^\circ \subset t\mathcal{U}$. Since

$$sp_\beta(y) \setminus \Gamma(\phi)^\circ \subset E \cap t\mathcal{U} = \varnothing,$$

we have $sp_\beta(\beta_\phi(y)) = \varnothing$ by (i) and so $\beta_\phi(y) = 0$ for all $y \in D$. Since D is weakly total in N, $\beta_\phi = 0$. Since $\langle \lambda(t), \phi \rangle \neq 0$, we have $t \notin sp(\beta)$, which is a contradiction.

(vi) Suppose that $N^\beta(\mathcal{U}) \neq \{0\}$ for all compact neighbourhoods $\mathcal{U}$ of t. Then $\mathcal{U}$ has a nonempty intersection with some $sp_\beta(y)$ with $y \in N$. Thus t belongs to the closure of union of $sp_\beta(y)$ with $y \in N$ and hence to $sp(\beta)$ by (v).

Conversely, suppose $t \in sp(\beta)$. For any compact neighbourhood $\mathcal{U}$ of t, there exists a $\phi \in R(G)_*$ such that $\langle \lambda(t), \phi \rangle \neq 0$ and $G \setminus \Gamma(\phi)^\circ \subset \mathcal{U}$. Since $t \in sp(\beta)$, $\beta_\phi(y) \neq 0$ for some $y \in N$. Since $sp_\beta(\beta_\phi(y)) \subset \mathcal{U}$ by (i), it follows that $\beta_\phi(y) \in N^\beta(\mathcal{U})$.

(vii) Since $sp_\beta(y) \subset \Gamma(\phi - \psi)^\circ$, $sp_\beta(\beta_{\phi-\psi}(y)) = \varnothing$ by (i). Thus $\beta_\phi(y) - \beta_\psi(y) = \beta_{\phi-\psi}(y) = 0$.

(viii) We may assume that F is compact by (iv). Let $x \in N^\beta(E)$ and $y \in N^\beta(F)$. If $\omega \in N_*$ and $\psi \in R(G)_*$, then

$$\langle \Phi_\omega(xy), \psi \rangle = \langle \beta(x)\beta(y), \omega \otimes \psi \rangle$$

$$= \lim \int \langle \beta(x), \beta_{\lambda(r)^*\phi}(y)\omega \otimes \lambda(r)\psi \rangle \, dr$$

$$= \lim \int \langle \Phi_{\beta_{\lambda(r)^*\phi}(y)\omega}(x)\lambda(r), \psi \rangle \, dr,$$

where the second equality is due to [**17**, Lemma 4.3]. Therefore

$$\Phi_\omega(xy) = \lim \int z_r\lambda(r) \, dr. \qquad (z_r \equiv \Phi_{\beta_{\lambda(r)^*\phi}(y)\omega}(x))$$

Since $sp_\gamma(z_r\lambda(r)) = sp_\gamma(z_r)r$ and since $sp_\gamma(z_r) \subset sp_\beta(x)$ by [**17**, Proposition 5.4], it follows that $sp_\gamma(z_r\lambda(r)) \subset sp_\beta(x)r$. If $\mathcal{U}$ is a symmetric compact neighbourhood of e, then $r \mapsto \beta_{\lambda(r)^*\phi}(y)$ vanishes on $G\backslash\mathcal{U}F$ for all ϕ with car $\lambda_*\phi \subset \mathcal{U}$. Therefore (iii) implies $sp_\gamma(\Phi_\omega(xy)) \subset sp_\beta(x)\mathcal{U}F$, which implies (viii) by [**17**, Proposition 5.4].

For any projection p in N^β we denote by β^p the restriction of β defined by

$$\beta^p(x_p) \equiv \beta(x)_{p \otimes 1_G} \qquad x \in N,$$

where x_p is the restriction of px to $p\mathcal{H}$. Then β^p is an isomorphism of N_p into $N_p \otimes R(G)$ satisfying $\beta^p \otimes \iota \circ \beta^p = \iota \otimes \gamma \circ \beta^p$, that is, β^p is a dual action of G on N_p.

DEFINITION 3.2. $\Gamma(\beta)$ is the intersection of all $sp\,\beta^p$, where p are nonzero projections in N^β.

LEMMA 3.3.   *If p is a projection in N^β and q the central carrier of p in N^β, then*
   (i)   *if $N^\beta(E) \cap N_q \neq \{0\}$, then $N^\beta(E) \cap N_p \neq \{0\}$;*
   (ii) $sp(\beta^p) = sp(\beta^q)$; *and*
   (iii) $\Gamma(\beta^p) = \Gamma(\beta^q)$.

Proof. (i) Let $(N^\beta)^u$ be the set of all unitaries in N^β. Since $q = \sup\{upu^*: u \in (N^\beta)^u\}$, if $x \in N^\beta(E) \cap N_q$ and $x \neq 0$, there exist two unitaries u and v in N^β such that $upu^*xvpv^* \neq 0$. If we put $y \equiv pu^*xvp$, then $y \in N_p$, $y \neq 0$ and $y \in N^\beta(E)$ by (viii) of Lemma 3.1.
   (ii)   It is clear from (vi) of Lemma 3.1 and (i).

(iii)   Since $Z(N_p^\beta)$ is isomorphic to $Z(N_q^\beta)$ by the correspondence $e_p \mapsto e_q$ for $e \in Z(N^\beta)$, the central carrier of e_p in N_q^β coincides with e_q. Since $sp \beta^{e_p} = sp \beta^{e_q}$ by (ii), $\Gamma(\beta^p) = \Gamma(\beta^q)$.

When u is a weakly continuous α twisted unitary representation of G in M, i.e., $u(st) \equiv u(s)\alpha_s(u(t))$. Then the unitary U in $M \otimes L^\infty(G)$ defined by $(U\xi)(r) = u(r)\xi(r)$ for $\xi \in \mathcal{H} \otimes L^2(G)$ satisfies

(3.1) $$U \otimes 1_G (\alpha \otimes \iota)U = \iota \otimes \delta(u).$$

PROPOSITION 3.4.   *If a unitary U in $M \otimes L^\infty(G)$ satisfies (3.1), there exists a weakly continuous α twisted unitary representation u of G in M such that $(U\xi)(r) = u(r)\xi(r)$ locally almost everywhere. (We can use the same letter for both U and u.)*

Proof.   For g in $L^1(G)$ we define $U(g)$ in M by

(3.2) $$\langle U(g), \omega \rangle = \langle U, \omega \otimes g \rangle$$

for all ω in M_*.   If $f \in L^1(G)$, then

$$\langle U\alpha(U(g)), \omega \otimes f \rangle = \langle U(g), \alpha_*((\omega \otimes f)U) \rangle$$
$$= \langle U, \alpha_*((\omega \otimes f)U) \otimes g \rangle \quad \text{(By (3.2))}$$
$$= \langle U \otimes 1_G (\alpha \otimes \iota)U, \omega \otimes f \otimes g \rangle$$
(3.3)
$$= \langle \iota \otimes \delta(U), \omega \otimes f \otimes g \rangle \quad \text{(By (3.1))}$$
$$= \langle U, \omega \otimes f * g \rangle$$
$$= \int g(t)\langle U, \omega \otimes Ad\lambda(t)^*(f) \rangle dt$$
$$= \int g(t)\langle Ad\lambda_1(t)(U), \omega \otimes f \rangle dt.$$

Therefore

$$U^* \int g(t)Ad\lambda_1(t)(U)dt \; (= \alpha(U(g)))$$

belongs to $\alpha(M)$.   If we make $g(t)dt$ converge to the Dirac measure at s, then it converges weakly to $U^*Ad\lambda_1(s)(U)$, which belongs to $\alpha(M)$.   Define a unitary $u(s)$ in M by

(3.4) $$u(s) \equiv \alpha^{-1}(U^*Ad\lambda_1(s)(U)).$$

Then $s \mapsto u(s)$ is weakly continuous.   Combining (3.3) and (3.4), we have

$$U(g) = \int g(t)u(t)dt.$$

It follows from (3.2) that $(U\xi)(r) = u(r)\xi(r)$ locally almost everywhere.   Besides, since $\{M, \alpha\} \sim \{\alpha(M), \iota \otimes \delta\}$ and $\alpha \circ \alpha_t \circ \alpha^{-1} = (\iota \otimes \delta)_t = Ad\lambda_1(t)$, it follows from (3.4) that $u(s)\alpha_s(u(t)) = u(st)$.

Let α^1 and α^2 be two actions of G on M.   We denote by $\alpha^1 \sim \alpha^2$ if there exists a unitary u in $M \otimes L^\infty(G)$ satisfying (3.1) for α^1 and $\alpha^2 = Adu \circ \alpha^1$.

DEFINITION 3.5.   Let β_1 and β_2 be dual actions of G on N.   Denote by $\beta_1 \sim \beta_2$ if there exists a unitary v in $N \otimes R(G)$ such that

$$v \otimes 1_G(\beta_1 \otimes \iota)v = \iota \otimes \gamma(v) \quad \text{and} \quad \beta_2 = Adv \circ \beta_1.$$

It should be noted that, when u is a weakly continuous α twisted unitary representation of G in N, $u(t) \in M^\alpha$ for all t if and only if $\bar{\alpha}(u) = u \otimes 1_G$.

LEMMA 3.6.   *Let $\{e_{ij} : i, j = 1, 2\}$ be matrix units of a type I_2 factor F_2.   If $\beta_1 \sim \beta_2$, there exists a dual action β of G on $N \otimes F_2$ satisfying $\beta(y \otimes e_{kk}) = \bar{\beta}_k(y \otimes e_{kk})$ for $k = 1, 2$, where $\bar{\beta}_k \equiv \iota \otimes \sigma \circ \beta_k \otimes \iota$ on $N \otimes F_2$.*

Proof.   Since $\beta_1 \sim \beta_2$, there exists a unitary v in $N \otimes R(G)$ such that $\beta_2 = Adv \circ \beta_1$ and

(3.5) $v \otimes 1_G(\beta_1 \otimes \iota)v = \iota_N \otimes \gamma(v).$

If we set $w \equiv \iota_N \otimes \sigma(1 \otimes 1_G \otimes e_{11} + v \otimes e_{22})$, then w belongs to $N \otimes F_2 \otimes R(G)$.   Define a mapping β of $N \otimes F_2$ into $N \otimes F_2 \otimes R(G)$ by

$$\beta(\Sigma y_{ij} \otimes e_{ij}) = \overline{\beta_1}(y_{11} \otimes e_{11}) + \overline{\beta_1}(y_{12} \otimes e_{12})\iota_N \otimes \sigma(v^* \otimes e_{22})$$

$$+ \iota_N \otimes \sigma(v \otimes e_{22})\overline{\beta_1}(y_{21} \otimes e_{21}) + Ad(\iota_N \otimes \sigma(v \otimes e_{22}))(\overline{\beta_1}(y_{22} \otimes e_{22})).$$

Then $\beta = Adw \circ \overline{\beta_1}$ and hence β is an isomorphism.   Moreover, since

$$\sigma \otimes \iota \circ \iota \otimes \sigma(1_G \otimes 1_G \otimes e_{11}) = \iota_{F_2} \otimes \gamma \circ \sigma(1_G \otimes e_{11})$$

$$\iota_N \otimes \sigma \otimes \iota \circ \iota_N \otimes \iota \otimes \sigma \circ \iota_N \otimes \gamma \otimes \iota_{F_2}(v \otimes e_{22})$$

$$= \iota_N \otimes \iota_{F_2} \otimes \gamma \circ \iota_N \otimes \sigma(v \otimes e_{22})$$

it follows from (3.5) that

$$w \otimes 1_G \overline{(\beta_1 \otimes \iota)} w$$

$$= \iota_N \otimes \sigma \otimes \iota(1 \otimes 1_G \otimes e_{11} \otimes 1_G + v \otimes e_{22} \otimes 1_G)$$

$$\cdot \overline{(\beta_1} \otimes \iota_{F_2} \circ \iota_N \otimes \sigma)(1 \otimes 1_G \otimes e_{11} + v \otimes e_{22})$$

$$= \iota_N \otimes \sigma \otimes \iota \circ \iota_N \otimes \iota \otimes \sigma(1 \otimes 1_G \otimes 1_G \otimes e_{11}$$

$$+ v \otimes 1_G \otimes e_{22}(\beta_1 \otimes \iota \otimes \iota_{F_2})(v \otimes e_{22}))$$

$$= \iota_N \otimes \sigma \otimes \iota \circ \iota_N \otimes \iota \otimes \sigma(1 \otimes 1_G \otimes 1_G \otimes e_{11}$$

$$+ \iota_N \otimes \gamma \otimes \iota_{F_2}(v \otimes e_{22}))$$

$$= \iota_N \otimes \iota_{F_2} \otimes \gamma \circ \iota_N \otimes \sigma(1 \otimes 1_G \otimes e_{11} + v \otimes e_{22})$$

$$= (\iota_N \otimes \iota_{F_2} \otimes \gamma)w.$$

Since $\overline{\beta_1}$ is a dual action of G on $N \otimes F_2$, β is also a dual action of G on $N \otimes F_2$.

When G is abelian, if $p, q \in N^\beta$ and $p \sim q$ in N, then $\Gamma(\beta^p) = \Gamma(\beta^q)$. However, if G is nonabelian, this is not necessarily assured.

Let β_1 and β_2 be dual actions of G on N such that $\beta_2 = Ad v \circ \beta_1$ for some $v \in N \otimes R(G)$ with $v \otimes 1_G(\beta_1 \otimes \iota)v = \iota \otimes \gamma(v)$. Then we have a dual action β of G on $N \otimes F_2$ satisfying

$$1 \otimes e_{jj} \in (N \otimes F_2)^\beta \quad \text{and} \quad 1 \otimes e_{11} \sim 1 \otimes e_{22}$$

as in Lemma 3.6. Then $\Gamma(\beta^{1 \otimes e_{jj}}) = \Gamma(\beta_j)$ and hence $\Gamma(\beta_2)sp_\beta(1 \otimes e_{21}) = sp_\beta(1 \otimes e_{21})\Gamma(\beta_1)$.

· The condition (ii) in the following theorem is a consequence of Theorems 5.2 and 6.1, which will be proved in §§5 and 6.

THEOREM 3.7. (i) $\Gamma(\beta)$ *is a closed subgroup of* G.

(ii) *If* β *is dual to some action (or if* β *is regular),* $\Gamma(\beta)$ *is normal and* $\Gamma(\beta) = \Gamma(\tilde{\beta})$.

Proof. (i) Since $\Gamma(\beta)$ is clearly closed, it suffices to prove the group property. Since $\Gamma(\beta)^{-1} = \Gamma(\beta)$, by (iv) of Lemma 3.1, we have only to

show $\Gamma(\beta)sp\beta^p \subset sp\beta^p$ for all projections p in N^β. Since $sp\beta^p$ is closed, we have only to show that, for any compact neighbourhood $\mathcal{U}$ of e, $sp\beta^p \cap st\mathcal{U}\beta \neq \varnothing$ for any $s \in \Gamma(\beta)$ and $t \in sp\beta^p$. Choose a symmetric neighbourhood $\mathcal{V}$ of e with $t^{-1}\mathcal{V}t\mathcal{V} \subset \mathcal{U}$. Since $t \in sp\beta^p$, there exists an element x in N such that $sp_\beta(x) \subset t\mathcal{V}$ and $x = pxp \neq 0$. Let q be the carrier projection of x^*. Since $s \in \Gamma(\beta)$ and $0 < q \leq p$, $s \in sp\beta^q$ and hence there exists an element y in N such that $sp_\beta(y) \subset s\mathcal{V}$ and $y = qyq \neq 0$. Since $yx \in N$, $yx = pyxp \neq 0$ and $sp_\beta(yx) \subset st\mathcal{U}$ by (viii) of Lemma 3.1, it follows that $sp\beta^p \cap st\mathcal{U} \neq \varnothing$.

(ii) When β is regular, $\Gamma(\beta)$ coincides with the set of $t \in G$ with $\hat{\beta}_t = \iota$ on $Z(N\otimes_\beta^d G)$ by Theorem 6.1. Since $Z(N\otimes_\beta^d G)$ is globally invariant under $\hat{\beta}_s$ for all s, $\Gamma(\beta)$ is normal.

If β is regular, then $\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}, [\beta]\}$ and so $\Gamma(\bar{\beta}) = \Gamma([\beta]) = \Gamma(\bar{\beta})$. Since $\Gamma(\beta) = \Gamma(\bar{\beta})$ by Theorem 5.2, $\Gamma(\beta) = \Gamma(\tilde{\beta})$.

4. Integrable actions and dual actions.

Here and hereafter, we denote the center of a given von Neumann algebra A by $Z(A)$.

This section is prepared only for Theorem 8.4. First we recall a result on a weight on $M\otimes_\alpha G$, [17, Section 6]. Let Δ be a semi-finite faithful normal weight on $L^\infty(G)$ defined by

$$\langle f, \Delta \rangle = \int f(t)\Delta(t)dt$$

for all f in $K(G)$. Here, we need no fair of notational confusion for Δ. Let $\mathfrak{n}_\alpha$ be the set of all x in M satisfying

$$|\langle \alpha(x^*x), \omega \otimes \Delta \rangle| \leq \lambda_x \|\omega\| \qquad (\lambda_x > 0)$$

for all ω in M_*. Then there exists a faithful normal M^α valued weight E_α on M, whose domain is $\mathfrak{n}_\alpha^*\mathfrak{n}_\alpha$.

In case of compact G it is straightforward that
(a) Δ is bounded (it may be assumed a state);
(b) $E_\alpha^2 = E_\alpha$ and $\|E_\alpha\| = 1$ (if $\langle 1, \Delta \rangle = 1$);
(c) $M^\alpha \subset \mathfrak{n}_\alpha$; and
(d) $\mathfrak{n}_\alpha$ is σ-weakly dense in M.

However, for a noncompact G, the assertions (a), (b) and (c) do not hold. Following Connes and Takesaki [6] we call α to be integrable, if E_α is semi-finite, namely, α satisfies (d). It is shown in [17] that the action $\hat{\beta}$ dual to some β is always integrable. Since $(\mathfrak{n}_\alpha)_p \subset \mathfrak{n}^{\alpha^p}$ for $p \in M^\alpha$, if α is integrable, so is α^p.

THEOREM 4.1. (Connes and Takesaki). *Assume that G is separable. If M^α is properly infinite, the following three conditions are equivalent*:

(i) α *is integrable*;

(ii) *for each non zero projection f in M^α there is an x in $\bar{M}$ satisfying $qxq \neq 0$ $(q \equiv f \otimes 1_G)$ and*

(4.1) $$(1 \otimes V')\bar{\alpha}(qxq) = qxq \otimes 1_G;$$

(iii) $\{\bar{M}, \bar{\alpha}\} \sim \{\bar{M}_p, [\alpha]^p\}$ *for some projection p in $\bar{M}^{[\alpha]}$.*

Proof. (i) $\Rightarrow$ (ii) Since α is integrable by assumption (i), there exists a nonzero z in $\mathfrak{n}_\alpha$ with $z = fzf$. For any g in $K(G)$ we set

(4.2) $$(x\xi)(s) \equiv \Delta(s)^{1/2}\alpha_s(z)\eta, \qquad \eta \equiv \int g(t)\xi(t)dt$$

for ξ in $\mathcal{H} \otimes L^2(G)$. Then $qxq = (f \otimes 1_G)x(f \otimes 1_G) = x$. Since $z \neq 0$, $x \neq 0$. Since

$$\|x\xi\|^2 = \int \|\alpha_s(z)\eta\|^2\Delta(s)ds = \langle E_\alpha(z^*z), \omega_\eta \rangle$$

$$\leq \|E_\alpha(z^*z)\|\|\eta\|^2 \leq \|E_\alpha(z^*z)\|\|g\|_2^2\|\xi\|_2^2,$$

x is bounded on $\mathcal{H} \otimes L^2(G)$. If we replace ξ by $(x' \otimes 1_G)\xi$ with $x' \in M'$ in (4.2), then $[x, x' \otimes 1_G] = 0$ and so $x \in \bar{M}$. Since

$$((1 \otimes V')\bar{\alpha}(x)\xi)(s, t) = \Delta(t)^{1/2}(\bar{\alpha}(x)\xi)(t^{-1}s, t)$$

$$= \Delta(t)^{1/2}\bar{\alpha}_t(x)\xi(t^{-1}s, t)$$

$$= \Delta(t)^{1/2}\Delta(t^{-1}s)^{1/2}\alpha_t(\alpha_{t^{-1}s}(z))\int g(r)\xi(r, t)dr \quad \text{(By (4.2))}$$

$$= \Delta(s)^{1/2}\alpha_s(z)\int g(r)\xi(r, t)dr$$

$$= ((x \otimes 1_G)\xi)(s, t),$$

we have qxq $(= x)$ satisfies the equality in (ii).

(ii) $\Rightarrow$ (iii) It suffices to show that the set I_α of x in $\bar{M}$ with $(1 \otimes V')\bar{\alpha}(x) = x \otimes 1_G$ contains an isometry v. Indeed, $vv^* \in \bar{M}^{[\alpha]}$ and $Ad v \otimes 1_G \circ \bar{\alpha} = [\alpha] \circ Ad v$.

The case where M^α is σ-infinite. Since

$$1 \otimes V' \otimes 1_G(\bar{\alpha} \otimes \iota)(1 \otimes V') = \iota_{\bar{M}} \otimes \delta(1 \otimes V'),$$

we can define an action $\langle \alpha \rangle$ of G on $\bar{M} \otimes F_2$ by

$$\langle \alpha \rangle (\Sigma \, x_{ij} \otimes e_{ij}) = \overline{[\alpha]}(x_{11} \otimes e_{11})$$

$$+ \overline{[\alpha]}(x_{12} \otimes e_{12})\iota_{\bar{M}} \otimes \sigma(1 \otimes V' \otimes e_{22})$$

$$+ \iota_{\bar{M}} \otimes \sigma(1 \otimes V'^* \otimes e_{22})\overline{[\alpha]}(x_{21} \otimes e_{21})$$

$$+ \iota_{\bar{M}} \otimes \sigma \circ \bar{\alpha} \otimes \iota_{F_2}(x_{22} \otimes e_{22}),$$

where $\overline{[\alpha]} \equiv \iota_{\bar{M}} \otimes \sigma \circ [\alpha] \otimes \iota_{F_2}$. Then $\Sigma \, x_{ij} \otimes e_{ij} \in (\bar{M} \otimes F_2)^{\langle \alpha \rangle}$ if and only if

$$x_{11} \in \bar{M}^{[\alpha]}, \quad x_{12} \in I_\alpha, \quad x_{21} \in I_\alpha^* \quad \text{and} \quad x_{22} \in \bar{M}^{\bar{\alpha}}.$$

Therefore the central carrier of $1_{\bar{M}} \otimes e_{22}$ in $(\bar{M} \otimes F_2)^{\langle \alpha \rangle}$ is majorized by the central carrier of $1_{\bar{M}} \otimes e_{11}$ by (ii). Since M^α is properly infinite and G is separable by assumption, $1_{\bar{M}} \otimes e_{11}$ is properly infinite and $1_{\bar{M}} \otimes e_{22}$ is σ-infinite in $(\bar{M} \otimes F_2)^{\langle \alpha \rangle}$. Therefore $1_{\bar{M}} \otimes e_{22} \prec 1_{\bar{M}} \otimes e_{11}$ in $(\bar{M} \otimes F_2)^{\langle \alpha \rangle}$ from the above discussion. Thus there exists a partial isometry $v \otimes e_{12}$ in $(\bar{M} \otimes F_2)^{\langle \alpha \rangle}$ whose initial and final projections are $1_{\bar{M}} \otimes e_{22}$ and the one majorized by $1_{\bar{M}} \otimes e_{11}$, respectively. Then v is an isometry in I_α.

The general case. There exists a partition $\{e_\iota : \iota \in I\}$ in $Z(M^\alpha)$ satisfying either

(a) e_ι is σ-finite in M^α; or

(b) e_ι is divided into uncountable set $\{f_\kappa : \kappa \in K\}$ of mutually equivalent, properly infinite and σ-infinite projections in M^α. Since M^α is properly infinite, e_ι with (a) is also properly infinite in M^α. Thus we can apply the above σ-finite case of M^α to $\{M_{e_\iota}, \alpha^{e_\iota}\}$ or $\{M_{f_\kappa}, \alpha^{f_\kappa}\}$.

(iii) $\Rightarrow$ (i) Since $\bar{\alpha}$ is dual to $\hat{\alpha}$, $[\alpha]$ is integrable. If $p \in \bar{M}^{[\alpha]}$ is nonzero, $[\alpha]^p$ is also integrable on $\bar{M}_p$. Therefore $\bar{\alpha}$ is integrable on $\bar{M}$ by our assumption (iii). Let q be a projection in $\bar{M}$ of the form $1 \otimes e$ for some minimal e in $B(L^2(G))$. Since $q \in \bar{M}^{\bar{\alpha}}$, $\bar{\alpha}^q$ is integrable on $\bar{M}_q$. Since $\{M, \alpha\}$ is equivalent to $\{\bar{M}_q, \bar{\alpha}^q\}$, α is integrable on M.

Now we consider the dual version. Let ω_e be the semi-finite faithful normal weight on $R(G)$ defined by

$$\langle \lambda(\Delta^{1/2}(\tilde{g} * g)), \omega_e \rangle = \tilde{g} * g(e), \quad \tilde{g}(t) \equiv \overline{g(t^{-1})}$$

for g in $K(G)$ and $\mathfrak{n}_\beta$ the set of all y in N satisfying

$$|\langle \beta(y^* y), \omega \otimes \omega_e \rangle| \leq \lambda_y \|\omega\|$$

for all ω in N_*.   There exists a faithful normal N^β valued weight E_β on N with domain $\mathfrak{n}_\beta^* \mathfrak{n}_\beta$.   When G is discrete,

    (a) ω_e is bounded (it may be assumed a state);

    (b) $E_\beta^2 = E_\beta$ and $\|E_\beta\| = 1$ (if $\langle 1, \omega_e \rangle = 1$);

    (c) $N^\beta \subset \mathfrak{n}_\beta$; and

    (d) $\mathfrak{n}_\beta$ is σ-weakly dense in N.

However, for a nondiscrete G, (a), (b) and (c) do not hold, and moreover,

    (e) $(N^\beta(\{s\}) \cap \mathfrak{n}_\beta)\backslash\{0\} = \varnothing$ for all s.

DEFINITION 4.2. β is *integrable* if for any nonzero projection f in N^β there is an element y in $\bar{N}$ satisfying $qyq \neq 0$ $(q \equiv f \otimes 1_G)$ and

$$(4.3) \qquad\qquad (1 \otimes W)\bar{\beta}(qyq) = qyq \otimes 1_G.$$

THEOREM 4.3.   *Assume that G is separable and N^β is properly infinite.   If β is integrable, then*

$$\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}_p, [\beta]^p\}$$

for some projection p in $\bar{N}^{[\beta]}$.

Proof.   We denote by I_β the set of all y in $\bar{N}$ satisfying

$$(4.4) \qquad\qquad (1 \otimes W)\bar{\beta}(y) = y \otimes 1_G.$$

Since $\bar{\beta}(y^*y) = y^*y \otimes 1_G$, $q_0 \equiv \sup\{s(y^*y): \; y \in I_\beta\}$ belongs to $\bar{N}^\beta$, where $s(y*y)$ is the carrier of $y*y$.   Since $I_\beta u = I_\beta$ for u in $\bar{N}^\beta$, $q_0 \in Z(\bar{N}^\beta)$.   Since β is integrable, $q_0 = 1$.   Since

$$1 \otimes W \otimes 1_G (\bar{\beta} \otimes \iota)(1 \otimes W) = \iota_N \otimes \gamma(1 \otimes W),$$

we can define a dual action $\langle \beta \rangle$ of G on $\bar{N} \otimes F_2$ by

$$\langle \beta \rangle(\Sigma \, y_{ij} \otimes e_{ij}) = \overline{[\beta]}(y_{11} \otimes e_{11})$$

$$+ \overline{[\beta]}(y_{12} \otimes e_{12})\iota_{\bar{N}} \otimes \sigma(1 \otimes W \otimes e_{22})$$

$$+ \iota_{\bar{N}} \otimes \sigma(1 \otimes W^* \otimes e_{22})\overline{[\beta]}(y_{21} \otimes e_{21})$$

$$+ \iota_{\bar{N}} \otimes \sigma \circ \bar{\beta} \otimes \iota(y_{22} \otimes e_{22})$$

as shown in Lemma 3.6.   Then $\Sigma \, y_{ij} \otimes e_{ij} \in (\bar{N} \otimes F_2)^{\langle \beta \rangle}$ if and only if

$$y_{11} \in \bar{N}^{[\beta]}, \quad y_{12} \in I_\beta, \quad y_{21} \in I_\beta^* \quad \text{and} \quad y_{22} \in \bar{N}^{\bar{\beta}}.$$

Since G is separable and N^β is properly infinite, $1_{\bar{N}} \otimes e_{22} < 1_{\bar{N}} \otimes e_{11}$ in $(\bar{N} \otimes F_2)^{(\beta)}$ from the above. Therefore there exists an isometry w in I_β.

Consequently, (4.4) implies

$$Ad\, w \otimes 1_G \circ \bar{\beta} = [\beta] \circ Ad\, w.$$

Since $[\beta](ww^*) = ww^* \otimes 1_G$ by (4.3), $p \equiv ww^*$ belongs to $\bar{N}^{[\beta]}$ and $\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}_p, [\beta]^p\}$.

5. Regular actions and dual actions.

In this section we shall discuss the relation between a (dual) action dual to some one and a dominant (dual) action. The former is initially characterized by Landstad and the latter concept is initially given by Connes and Takesaki. For our convenience we shall introduce the following terminology for α and β.

DEFINITION 5.1. An action α (resp. a dual action β) of G on M (resp. N) is *regular* if there exists a covariant dual system $\{N, \beta\}$ (resp. covariant system $\{M, \alpha\}$) satisfying $\{M, \alpha\} \sim \{N \otimes_\beta^d G, \hat{\beta}\}$ (resp. $\{N, \beta\} \sim \{M \otimes_\alpha G, \hat{\alpha}\}$).

Therefore, β is regular if and only if there exists a weakly continuous unitary representation u of G in N such that $\beta(u(t)) = u(t) \otimes \lambda(t)$ for all t; and α is regular if and only if there exists an isomorphism π of $L^\infty(G)$ into M such that $\alpha_t \circ \pi = \pi \circ Ad\lambda'(t)$ for all t, [15; 16; 17, Theorems 8.1 and 8.3; 21]. It is immediate from our definition that $\hat{\alpha}$, $\hat{\beta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $[\alpha]$ and $[\beta]$ are regular.

The ergodicity of α (resp. β) on $Z(M)$ (resp. $Z(N)$) is defined by $M^\alpha \cap Z(M) = \mathbf{C}1$ (resp. $N^\beta \cap Z(N) = \mathbf{C}1$).

THEOREM 5.2. (i) $Z(M \otimes_\alpha G) \subset \alpha(M)$ (resp. $Z(N \otimes_\beta^d G) \subset \beta(N)$) if and only if $Z(\tilde{M} \otimes_{\tilde{\alpha}} G) \subset \tilde{\alpha}(\tilde{M})$ (resp. $Z(\tilde{N} \otimes_{\tilde{\beta}}^d G) \subset \tilde{\beta}(\tilde{N})$).

(ii) α (resp. β) is ergodic on $Z(M)$ (resp. $Z(N)$) if and only if $\tilde{\alpha}$ (resp. $\tilde{\beta}$) is ergodic on $Z(\tilde{M})$ (resp. $Z(\tilde{N})$).

(iii) $\mathrm{Ker}\, \alpha \upharpoonright Z(M) = \mathrm{Ker}\, \tilde{\alpha} \upharpoonright Z(\tilde{M})$, where $\mathrm{Ker}\, \alpha \upharpoonright Z(M)$ is the set of all t in G with $\alpha_t = \iota$ on $Z(M)$.

(iv) $\Gamma(\beta) = \Gamma(\bar{\beta})$.

Proof. Our proof owes to Takesaki's duality:

(5.1) $\{\tilde{M}, \tilde{\alpha}\} \sim \{\bar{M}, [\alpha]\}$ and $\{\tilde{N}, \tilde{\beta}\} \sim \{\bar{N}, [\beta]\}$.

(i) *The case of* α. Let $N \equiv M \otimes_\alpha G$ and $\beta \equiv \hat{\alpha}$. According to (5.1) we have only to show the equivalence of $Z(N) \subset N^\beta$ and $Z(\bar{N}) \subset$

$\bar{N}^{[\beta]}$. Since $Z(\bar{N}) = Z(N) \otimes 1_G$, we have only to show that $y \in N^\beta$ if and only if $y \otimes 1_G \in \bar{N}^{[\beta]}$ for all $y \in Z(N)$. However, this is clear from the following equality

$$Ad \, 1 \otimes W \circ \iota \otimes \sigma \circ \beta \otimes \iota (y \otimes 1_G) = y \otimes 1_G \otimes 1_G.$$

The case of β. Let $M \equiv N \otimes_\beta^d G$ and $\alpha \equiv \hat{\beta}$. According to (5.1) we have only to show the equivalence of $Z(M) \subset M^\alpha$ and $Z(\bar{M}) \subset \bar{M}^{[\alpha]}$. Since $Z(\bar{M}) = Z(M) \otimes 1_G$, we have only to show that $x \in M^\alpha$ if and only if $x \otimes 1_G \in \bar{M}^{[\alpha]}$ for all $x \in Z(M)$. This is clear from

$$Ad \, 1 \otimes V' \circ \iota \otimes \sigma \circ \alpha \otimes \iota (x \otimes 1_G) = x \otimes 1_G \otimes 1_G.$$

(ii) *The case of α.* Since $\alpha(x) = x \otimes 1_G$ if and only if $[\alpha](x \otimes 1_G) = x \otimes 1_G \otimes 1_G$, it follows that α is ergodic on $Z(M)$ if and only if $[\alpha]$ is ergodic on $Z(M) \otimes 1_G$. Thus we have (ii) for α by (5.1).

The case of β. Since $\beta(y) = y \otimes 1_G$ if and only if $[\beta](y \otimes 1_G) = y \otimes 1_G \otimes 1_G$, both of the ergodicity of β on $Z(N)$ and $[\beta]$ on $Z(N) \otimes 1_G$ are equivalent. Thus we have (ii) for β by (5.1).

(iii) It is immediate from (5.1) that

$$\text{Ker} \, \alpha \upharpoonright Z(M) = \text{Ker} \, [\alpha] \upharpoonright Z(\bar{M}) = \text{Ker} \, \tilde{\alpha} \upharpoonright Z(\tilde{M}).$$

(iv) If $y \in N$, $z \in B(L^2(G))$, $\omega_1 \in N_*$, $\omega_2 \in B(L^2(G))_*$ and $\phi \in R(G)_*$, then

$$\langle \bar{\beta}_\phi(y \otimes z), \omega_1 \otimes \omega_2 \rangle = \langle \beta_\phi(y) \otimes z, \omega_1 \otimes \omega_2 \rangle$$

and hence $\beta_\phi = 0$ is equivalent to $\bar{\beta}_\phi = 0$. Since $Z(\bar{N}^\beta) = Z(N^\beta) \otimes 1_G$, it follows that $sp \, \beta^p = sp \, \bar{\beta}^{p \otimes 1_G}$ for $p \in Z(N^\beta)$. Consequently, $\Gamma(\beta) = \Gamma(\bar{\beta})$.

When M is properly infinite and M contains a partition $\{e_\iota : \iota \in I\}$ of the identity such that $e_\iota \sim e_\kappa$ in M for all ι, κ and $\dim L^2(G) \leqq \text{Card} \, I$, Takesaki's duality tells us that $(M \otimes_\alpha G) \otimes_{\tilde{\alpha}}^d G$ is isomorphic to M. Here we raise a question, when is $\{(M \otimes_\alpha G) \otimes_{\tilde{\alpha}}^d G, \tilde{\tilde{\alpha}}\}$ equivalent to $\{M, \alpha\}$?, namely, when are they isomorphic as a covariant system?

We shall begin with the following lemma.

LEMMA 5.3. *If M^α (resp. N^β) contains a partition $\{e_\iota : \iota \in I\}$ of the identity satisfying*
  (i) *$e_\iota \sim 1$ in M^α (resp. N^β) for all $\iota \in I$; and*
  (ii) *$\dim H \leqq \text{Card} \, I$,*
then $\{M, \alpha\}$ (resp. $\{N, \beta\}$) is spatially isomorphic to

(5.2) $\{M \otimes B(H), \iota \otimes \sigma \circ \alpha \otimes \iota\}$ (*resp.* $\{N \otimes B(H), \iota \otimes \sigma \circ \beta \otimes \iota\}$).

Proof. From our assumptions (i) and (ii) we can obtain a partition $\{f_\iota : \iota \in I_H\}$ of the identity such that $\mathrm{Card}\, I_H = \dim H$ and $f_\iota \sim 1$ in M^α (resp. N^β) for all $\iota \in I_H$. Let v_ι be an isometry in M^α (resp. N^β) with $v_\iota v_\iota^* = f_\iota$. Let ρ be the spatial isomorphism of M (resp. N) onto $M \otimes B(H)$ (resp. $N \otimes B(H)$) such that

$$\rho(x) = \sum_{\iota, \kappa} v_\iota^* x v_\kappa \otimes e_{\iota\kappa}$$

for x in M (resp. N), where $\{e_{\iota\kappa} : \iota, \kappa \in I_H\}$ is the set of matrix units of $B(H)$. Then

$$\rho(\alpha_t(x)) = \sum v_\iota^* \alpha_t(x) v_\kappa \otimes e_{\iota\kappa}$$

$$= \sum \alpha_t(v_\iota^* x v_\kappa) \otimes e_{\iota\kappa} = (\alpha_t \otimes \iota)(\rho(x))$$

$$\left(\text{resp. } (\rho \otimes \iota)\beta(x) = (\iota \otimes \sigma \circ (\iota \otimes \sigma \circ \rho \otimes \iota))\beta(x) \right.$$

$$\left. = \iota \otimes \sigma \Big(\sum \beta(v_\iota^* x v_k) \otimes e_{\iota\kappa} \Big) = (\iota \otimes \sigma \circ \beta \otimes \iota)\rho(x), \right.$$

where the second equality follows from the following reason. If $sp_\beta(x)$ is compact, then, by [**17**, Lemma 4.3],

$$(v_\iota^* \otimes 1_G)\beta(x)(v_\kappa \otimes 1_G) = \lim_\phi \int \beta_{\lambda(r)^* \phi}(v_\iota^* x v_k) \otimes \lambda(r) dr \Big).$$

An action α (resp. a dual action β) is said to be *of infinite multiplicity* if it satisfies the conditions (i) and (ii) for a Hilbert space $H \equiv L^2(G)$ and an infinite G in Lemma 5.3.

DEFINITION 5.4.   An action α (resp. a dual action β) of G on M (resp. N) is said to be *dominant* if
(i) α (resp. β) is of infinite multiplicity, and
(ii) $\{\bar{M}, \bar{\alpha}\} \sim \{\bar{M}, [\alpha]\}$ (resp. $\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}, [\beta]\}$).
For a dominant α (resp. β) it holds that

$$\{M, \alpha\} \sim \{\tilde{M}, \tilde{\alpha}\}   (\text{resp. } \{N, \beta\} \sim \{\tilde{N}, \tilde{\beta}\})$$

by Lemma 5.3.

REMARK 5.5.   If G is infinite and α (resp. β) is regular, $\bar{\alpha}$ (resp. $\bar{\beta}$) is dominant.

THEOREM 5.6.   *If α (resp. β) is regular, then*

$$\{\bar{M}, \bar{\alpha}\} \sim \{\bar{M}, [\alpha]\}   (resp. \{\bar{N}, \bar{\beta}\} \sim \{\bar{N}, [\beta]\}).$$

Proof.   The case of $\{M, \alpha\}$.   If α is regular, there exists a covariant dual system $\{N, \beta\}$ satisfying

$$(5.3) \qquad \{M, \alpha\} \sim \{N \otimes_\beta^d G, \hat{\beta}\}.$$

Let $\mathcal{H}$ denote the underlying Hilbert space of N.   Since

$$(5.4) \qquad \begin{aligned} \hat{\beta}_t(\beta(y)) &= \beta(y) = Ad\,\lambda_1'(t)(\beta(y)) \\ \hat{\beta}_t(1_{\mathcal{H}} \otimes f) &= 1_{\mathcal{H}} \otimes f_{t^{-1}} = Ad\,\lambda_1'(t)(1_{\mathcal{H}} \otimes f) \end{aligned}$$

for $\lambda_1'(t) \equiv 1_{\mathcal{H}} \otimes \lambda'(t)$, we have $\hat{\beta}_t = Ad\,\lambda_1'(t)$ on $N \otimes_\beta^d G$.   Let w be the unitary on $\mathcal{H} \otimes L^2(G) \otimes L^2(G)$ defined by

$$(1_{\mathcal{H}} \otimes W^*)(1_{\mathcal{H}} \otimes 1_G \otimes J).$$

Since $Ad\,W(\lambda'(r) \otimes 1_G) = \lambda'(r) \otimes \lambda(r)$, it follows that

$$w^* Ad\,\lambda_1'(r) \otimes 1_G(w) = 1_{\mathcal{H}} \otimes 1_G \otimes \lambda'(r).$$

Therefore, by (5.4)

$$(Ad\,w)^{-1} \circ \hat{\beta}_t \otimes \iota \circ Ad\,w = \hat{\beta}_t \otimes Ad\,\lambda'(t)$$

on $(N \otimes_\beta^d G) \otimes B(L^2(G))$, which completes the proof for α.

The case of $\{N, \beta\}$.   If β is regular, there exists a weakly continuous unitary representation u of G in N satisfying $\beta(u(t)) = u(t) \otimes \lambda(t)$.   We denote by the same letter u the unitary in $N \otimes L^\infty(G)$ defined   by $(u\xi)(r) = u(r)\xi(r)$ for $\xi \in \mathcal{H} \otimes L^2(G)$.   Then $(1 \otimes W)\bar{\beta}(u^*) = u^* \otimes 1_G$.   Indeed, if $\xi \in \mathcal{H} \otimes L^2(G) \otimes L^2(G)$, then

$$\begin{aligned} ((u \otimes 1_G)(1 \otimes W)\xi)(s, t) &= u(s)\xi(s, ts) \\ &= u(s)((1 \otimes U)\xi)(ts, s) \\ &= \beta(u(s)) \otimes 1_G((1 \otimes U)\xi)(t, s) \\ &= ((\beta \otimes \iota(u))(1 \otimes U)\xi)(t, s) \\ &= ((1 \otimes U)(\beta \otimes \iota(u))(1 \otimes U)\xi(s, t) \\ &= (\bar{\beta}(u)\xi)(s, t), \end{aligned}$$

where U is a unitary on $L^2(G) \otimes L^2(G)$ defined by $(U\eta)(s, t) \equiv \eta(t, s)$ for all η in $L^2(G) \otimes L^2(G)$.   Therefore

$$Ad\, u \otimes 1 \circ Ad\, 1 \otimes W \circ \bar{\beta} = \bar{\beta} \circ Ad\, u$$

and hence $\{\bar{N}, [\beta]\} \sim \{\bar{N}, \bar{\beta}\}$.

It should be noted that the proof of Theorem 5.6 for β does not use the homomorphism property of u.

COROLLARY 5.7. α (resp. β) is dominant if and only if it is regular and of infinite multiplicity.

6.   Ergodic actions and dual actions. This section is devoted to further investigation of [6, §3 of Chapter III].   The following theorem generalizes [6, Theorem III, 3.2], whose proof will go along the same line of argument.   By Lemma 3.3

$$(6.1) \qquad\qquad \Gamma(\beta) = \cap \{sp\, \beta^p : p \in Z(N^\beta),\, p \neq 0\}.$$

For a given action α we denote by $\operatorname{Ker} \alpha \upharpoonright Z(M)$ the set of all t in G satisfying $\alpha_t = \iota$ on $Z(M)$.

THEOREM 6.1.   (i) $\Gamma(\beta) = \operatorname{Ker} \hat{\beta} \upharpoonright Z(N \otimes^d_\beta G)$ if β is regular.
(ii) $\Gamma(\hat{\alpha}) = \operatorname{Ker} \alpha \upharpoonright Z(M)$.

Proof.   (i) If β is regular, there exists an isomorphism ρ of $N^\beta \otimes_\alpha G$ onto N satisfying $\beta \circ \rho = \rho \otimes \iota \circ \hat{\alpha}$, where α is an action of G on N^β.   Then Takesaki's duality gives us that

$$\{N \otimes^d_\beta G, \hat{\beta}\} \sim \{N^\beta \otimes B(L^2(G)), [\alpha]\}.$$

Therefore

$$(6.2) \quad \operatorname{Ker} \hat{\beta} \upharpoonright Z(N \otimes^d_\beta G) = \operatorname{Ker}[\alpha] \upharpoonright Z(N^\beta) \otimes 1_G = \operatorname{Ker} \alpha \upharpoonright Z(N^\beta).$$

Since $\{N^\beta, \alpha\}$ is covariant and $\rho \circ \alpha(N^\beta) = N^\beta$ by (2.7), we have

$$\alpha^\circ_t \equiv (\rho \circ \alpha) \circ \alpha_t \circ (\rho \circ \alpha)^{-1} \in \operatorname{Aut} N^\beta.$$

Here we set $u(t) \equiv \rho(1 \otimes \lambda(t))$.   Since $\beta(u(t)) = u(t) \otimes \lambda(t)$, $N^\beta(\{t\}) = N^\beta u(t)$ by [17, Proposition 5.2].   If p is a nonzero projection in $Z(N^\beta)$, then $\alpha^\circ_t(p) = u(t)pu(t)^*$ and

$$(6.3) \qquad\qquad pN^\beta(\{t\})p = pN^\beta u(t)p = pN^\beta \alpha^\circ_t(p)u(t).$$

Consequently, $s \in \operatorname{Ker} \hat{\beta} \upharpoonright Z(N \otimes_\beta^d G)$ if and only if $\alpha_s = \iota$ on $Z(N^\beta)$ by (6.2), if and only if $\alpha_s^\circ = \iota$ on $Z(N^\beta)$, if and only if $p\alpha_s^\circ(p) \neq 0$ for all nonzero projection p in $Z(N^\beta)$, if and only if $s \in \Gamma(\beta)$ by (6.1) and (6.3).

(ii) Combining (i) and Theorem 5.2, we have

$$\Gamma(\hat{\alpha}) = \operatorname{Ker} \tilde{\alpha} \upharpoonright Z(\tilde{M}) = \operatorname{Ker} \alpha \upharpoonright Z(M).$$

Here we denote the von Neumann algebra generated by $\beta(N)$ and $1 \otimes (L^\infty(G) \cap \lambda'(H)')$ by $N \otimes_\beta^d (H \backslash G)$, whose characterization will be discussed in §7.

COROLLARY 6.2.   *Assume β is regular.   For a closed subgroup H of G, $\Gamma(\beta) = H$ if and only if H is the largest subgroup satisfying $Z(N \otimes_\beta^d G) \subset N \otimes_\beta^d (H \backslash G)$.   In particular, $\Gamma(\beta) = G$ if and only if $Z(N \otimes_\beta^d G) \subset \beta(N)$.*

Proof. $\Gamma(\beta) = H$ if and only if H is the largest subgroup satisfying the condition that $\hat{\beta}_t = \iota$ on $Z(N \otimes_\beta^d G)$ for all $t \in H$ by Theorem 6.1.   The condition is equivalent to

$$Z(N \otimes_\beta^d G) \subset \{x \in N \otimes_\beta^d G : \hat{\beta}_t(x) = x, t \in H\} = N \otimes_\beta^d (H \backslash G)$$

by Theorem 7.2 in §7.

Here, if we combine Theorem 5.2 and Corollary 6.2, we have that $\Gamma(\tilde{\beta}) = G$ if and only if $Z(N \otimes_\beta^d G) \subset \beta(N)$.

From our previous result [14, Proposition 3.1] we have the following proposition.

PROPOSITION 6.3.   *If α (resp. β) is regular, the following two conditions are equivalent:*
    (i) $Z(M \otimes_\alpha G) \subset \alpha(M)$ *(resp.* $\Gamma(\beta) = G$); *and*
    (ii) $Z(M^\alpha) \subset Z(M)$ *(resp.* $Z(N^\beta) \subset Z(N)$).
If α (resp. β) in the above is ergodic on $Z(M)$ (resp. $Z(N)$), then M^α (resp. N^β) is a factor.

Proof. *The case of $\{M, \alpha\}$.* If α is regular, there exists a dual action β of G on M^α such that

(6.4) $\{M, \alpha\} \sim \{N \otimes_\beta^d G, \hat{\beta}\}$,

where $N \equiv M^\alpha$.

(i) $\Rightarrow$ (ii) Since $Z(M \otimes_\alpha G) \subset \alpha(M)$ by (i) and $\alpha(M) = (M \otimes_\alpha G)^{\hat{\alpha}}$, it follows from (6.4) that

$$Z(N) \otimes 1_G = Z(N \otimes B(L^2(G))) \subset (N \otimes B(L^2(G)))^{[\beta]}.$$

If $z \in Z(N)$, then $[\beta](z \otimes 1_G) = z \otimes 1_G \otimes 1_G$ and hence

$$\beta \otimes \iota (z \otimes 1_G) = (\iota \otimes \sigma \circ Ad \, 1 \otimes W^* \circ [\beta])(z \otimes 1_G) = z \otimes 1_G \otimes 1_G.$$

Therefore $\beta(z) = z \otimes 1_G$ for $z \in Z(N)$. Since $\beta(z)$ commutes with $\beta(N)$ and $1 \otimes L^\infty(G)$, $\beta(z)$ belongs to $Z(N \otimes_\beta^d G)$, which shows that $\beta(Z(N)) \subset Z(N \otimes_\beta^d G)$. Consequently, (6.4) implies that $Z(M^\alpha) \subset Z(M)$, because $\beta(N) = (N \otimes_\beta^d G)^{\hat\beta}$.

(ii) $\Rightarrow$ (i) We apply Takesaki's duality to (6.4) and we have

(6.5) $\{M \otimes_\alpha G, \hat\alpha\} \sim \{N \otimes B(L^2(G)), [\beta]\}.$

If we can show that

(6.6) $Z(N) \otimes 1_G \subset N \otimes_\beta^d G,$

then $Z(M \otimes_\alpha G) \subset (M \otimes_\alpha G)^{\hat\alpha} = \alpha(M)$ by (6.5). Therefore we want to show (6.6).

Now, since β is a dual action of G on N, we have

(6.7) $\{N, \beta\} \sim \{\beta(N), \iota \otimes \gamma\}$

and $(\iota \otimes \gamma)\beta(y) = Ad \, 1 \otimes W^*(\beta(y) \otimes 1_G)$. According to our assumption $Z(M^\alpha) \subset Z(M)$, we have

$$Z((\iota \otimes \gamma)\beta(N)) \subset Z(\beta(N) \otimes_{\iota \otimes \gamma}^d G) \subset (\beta(N) \otimes_{\iota \otimes \gamma}^d G)'$$

by (6.4) and (6.7). Therefore $(\iota \otimes \gamma)\beta(Z(N))$ commutes with $1 \otimes 1_G \otimes L^\infty(G)$ and hence $\beta(Z(N)) \otimes 1_G$ commutes with $Ad \, 1 \otimes W(1 \otimes 1_G \otimes L^\infty(G))$. Since $Z(\beta(N)) \otimes 1_G$ commutes with $\beta(N)' \otimes 1_G$, we have

$$Z(\beta(N)) \otimes 1_G \subset \{\beta(N)' \otimes 1_G, \, Ad \, 1 \otimes W(1 \otimes 1_G \otimes L^\infty(G))\}',$$

the right hand side of which is $\beta(N) \otimes_{\iota \otimes \gamma}^d G$ by [**16**, Theorem 5]. Therefore, (6.6) is proved by (6.7).

The case of $\{N, \beta\}$. If β is regular, there exists an action α of G on N^β such that

(6.8) $\{N, \beta\} \sim \{M \otimes_\alpha G, \hat\alpha\},$

where $M \equiv N^\beta$.

(i) $\Rightarrow$ (ii) Since $\mathrm{Ker}\,\hat{\beta} \restriction Z(N\otimes^d_\beta G) = \Gamma(\beta) = G$ by Theorem 6.1 and (i), $\hat{\beta}_t = \iota$ on $Z(N\otimes^d_\beta G)$ and hence $[\alpha]_t = \iota$ on $Z(M\otimes B(L^2(G)))$ by (6.8). Therefore $\alpha_t = \iota$ on $Z(M)$ for all t and so $\alpha(z) = z\otimes 1_G$ for $z\in Z(M)$. Since $\alpha(z)\in M\otimes_\alpha G$ and it commutes with $\alpha(M)$ and $1\otimes R(G)$, $\alpha(z)$ belongs to $Z(M\otimes_\alpha G)$, namely, $Z(\alpha(M))\subset Z(M\otimes_\alpha G)$. Consequently, (6.8) implies that $Z(N^\beta)\subset Z(N)$, because $\alpha(M) = (M\otimes_\alpha G)^{\hat\alpha}$.

(ii) $\Rightarrow$ (i) Since β is regular, there is a weakly continuous unitary representation u of G in N such that $\beta(u(t)) = u(t)\otimes\lambda(t)$. Since $Z(N^\beta)\subset Z(N)$ by (ii), if p is a nonzero projection in $Z(N^\beta)$, β^p is a dual action of G on N_p and $t\mapsto u(t)_p$ is a weakly continuous unitary representation of G in N_p satisfying $\beta^p(u(t)_p) = u(t)_p\otimes\lambda(t)$. Therefore $sp\,\beta^p = G$ for all p in $Z(N^\beta)$. Consequently, $\Gamma(\beta) = G$.

Finally, if α (resp. β) is ergodic on $Z(M)$ (resp. $Z(N)$), then M^α (resp. N^β) is a factor from the above.

THEOREM 6.4.   *The following two conditions are equivalent*:
(i) $M\otimes_\alpha G$ *(resp. $N\otimes^d_\beta G$) is a factor*; *and*
(ii) $Z(M\otimes_\alpha G)\subset\alpha(M)$ *(resp. $\Gamma(\tilde\beta) = G$) and α (resp. β) is ergodic on $Z(M)$ (resp. $Z(N)$).*

Proof.   The case of $\{M,\alpha\}$.   (i) $\Rightarrow$ (ii) If $\alpha(z) = z\otimes 1_G$ for $z\in Z(M)$, $\alpha(z)$ commutes with $\alpha(M)$ and $1\otimes R(G)$, and hence it belongs to $Z(M\otimes_\alpha G)$. Since $M\otimes_\alpha G$ is a factor, (ii) follows immediately.

(ii) $\Rightarrow$ (i) We denote the covariant system $\{(M\otimes_\alpha G)\otimes_{\hat\alpha} G, \hat{\hat{\alpha}}\}$ by $\{\tilde M, \tilde\alpha\}$. Since $\hat\alpha(M\otimes_\alpha G) = \tilde M^{\tilde\alpha}$, it suffices to show that $\tilde M^{\tilde\alpha}$ is a factor. Since $\tilde\alpha$ is regular, it suffices to show that
(a) $Z(\tilde M\otimes_{\tilde\alpha} G)\subset\tilde\alpha(\tilde M)$; and
(b) $\tilde\alpha$ is ergodic on $Z(\tilde M)$
by Proposition 6.3. (a) and (b) are immediate from (ii) by Theorem 5.2.

The case of $\{N,\beta\}$.   (i) $\Rightarrow$ (ii) Since $\Gamma(\tilde\beta) = G$ is clear, it suffices to show the ergodicity of β on $Z(N)$. If $\beta(z) = z\otimes 1_G$ for $z\in Z(N)$, then $\beta(z)$ commutes with $\beta(N)$ and $1\otimes L^\infty(G)$, and hence $\beta(z)$ belongs to $Z(N\otimes^d_\beta G)$. Since $N\otimes^d_\beta G$ is a factor by (i), β is ergodic on $Z(N)$.

(ii) $\Rightarrow$ (i) Let $\{\tilde N, \tilde\beta\} \equiv \{(N\otimes^d_\beta G)\otimes_{\hat\beta} G, \hat{\hat\beta}\}$.   Since $\hat\beta(N\otimes^d_\beta G) = \tilde N^{\tilde\beta}$, it suffices to show that $\tilde N^{\tilde\beta}$ is a factor.   Since $\tilde\beta$ is regular, it suffices to note that
(c) $\Gamma(\tilde{\tilde\beta}) = G$; and
(d) $\tilde\beta$ is ergodic on $Z(\tilde N)$
by Proposition 6.3.   (d) is clear from (ii) by Theorem 5.2.

COROLLARY 6.5. *If α (resp. β) is regular and if $Z(M \otimes_\alpha G) \subset \alpha(M)$ (resp. $\Gamma(\beta) = G$), then α^p (resp. β^p) is regular for all $p \in Z(M^\alpha)$ (resp. $Z(N^\beta)$).*

Proof. The case of α. Since α is regular, $\{M, \alpha\}$ is identified with $\{N \otimes_\beta^d G, \hat{\beta}\}$ for some $\{N, \beta\}$. Since $\hat{\beta}_t(\beta(y)) = \beta(y)$ and $\hat{\beta}_t(1 \otimes f) = 1 \otimes f_{t^{-1}}$, it follows that $\hat{\beta}_t = Ad\lambda_1'(t)$ on $N \otimes_\beta^d G$. Since $Z(M^\alpha) \subset Z(M)$ by Proposition 6.3, p belongs to $B(\mathcal{K}) \otimes L^\infty(G)$, where $\mathcal{K}$ is the underlying Hilbert space of N. An action $t \mapsto Ad\lambda_1'(t)$ on $B(\mathcal{K}) \otimes L^\infty(G)$ coincides with $\hat{\beta}_t$ on $Z(M^\alpha)$. Since $\hat{\beta}_t(p) = p$, $Ad\lambda_1'(t)(p) = p$. Since $L^\infty(G) \cap R(G) = \mathbf{C}1_G$, p is of the form $e \otimes 1_G$ for some projection e in $B(\mathcal{K})$. Let π be an isomorphism of $L^\infty(G)$ into M_p defined by $\pi(f) \equiv e \otimes f$ ($\in M_p$). Then π satisfies $\alpha_t \circ \pi = \pi \circ Ad\lambda'(t)$ on $L^\infty(G)$. Therefore α^p is regular on M_p.

The case of β. Since β is regular, there exists a weakly continuous unitary representation u of G in N such that $\beta(u(t)) = u(t) \otimes \lambda(t)$. Since $Z(N^\beta) \subset Z(N)$ by Proposition 6.3, $t \mapsto u(t)_p$ is a weakly continuous unitary representation of G in N_p and hence β^p is a regular dual action of G on N_p.

LEMMA 6.6. *Let p be a projection in M. If*
(i)   *α is regular and $Z(M) \subset M^\alpha$; and*
(ii) *p is properly infinite,*
then $p \otimes 1_G \sim \alpha(p)$ in $M \otimes L^\infty(G)$.

Proof. Since α is regular by (i), $\{M, \alpha\}$ is identified with $\{N \otimes_\beta^d G, \hat{\beta}\}$ for some covariant dual system $\{N, \beta\}$. Then $\alpha(x) = Ad\,1_N \otimes V'(x \otimes 1_G)$ for x in M. Since

$$(6.8) \qquad Ad\,1_N \otimes V'(1 \otimes f) = 1 \otimes f, \qquad (1 \equiv 1_M = 1_N \otimes 1_G)$$

$M \otimes L^\infty(G)$ is globally invariant under $Ad\,1_N \otimes V'$ by Lemma 8.2. Let ρ be the restriction of $Ad\,1_N \otimes V'$ to $M \otimes L^\infty(G)$. Then the condition (i) implies that $\rho = \iota$ on $Z(M) \otimes L^\infty(G)$ by (6.8). Therefore $p \otimes 1_G \sim \rho(p)$ ($= \alpha(p)$) by Suzuki's Theorem [N. Suzuki, Tôhoku Math. J. 7 (1955), 186–191, Theorem 1]*.

THEOREM 6.7. *If*
(i)   *G is separable;*
(ii)   *β is integrable and $\Gamma(\tilde{\beta}) = G$; and*
(iii) *N^β is properly infinite,*
then $\{\tilde{\bar{N}}, \iota \otimes \sigma \circ \bar{\beta} \otimes \iota\} \sim \{\tilde{\bar{N}}, [\bar{\beta}]\}$, where $\bar{\bar{N}} \equiv \bar{N} \otimes B(L^2(G))$.

Proof. According to (i), (ii) and (iii) we have

$$(6.9) \qquad \{\bar{N}, \bar{\beta}\} \sim \{\bar{N}_p, [\beta]^p\}$$

for some p in $\bar{N}^{[\beta]}$ by Theorem 4.3. Since N is properly infinite by (iii), so is p. Let q be the central carrier of p in $\bar{N}^{[\beta]}$. Since

$$(6.10) \qquad Z(\bar{N} \otimes_{[\beta]}^d G) \subset [\beta](\bar{N})$$

by (ii) and Theorem 5.2, it follows from Corollary 6.5 that $[\beta]^q$ is regular. Therefore we can identify $\{\bar{N}_q, [\beta]^q\}$ with $\{M \otimes_\alpha G, \hat{\alpha}\}$ for some covariant system $\{M, \alpha\}$. Since $Z(\bar{N}^{[\beta]}) \subset Z(\bar{N})$ by Proposition 6.3, (6.10) implies that $Z(\bar{N}_q \otimes_{[\beta]^q}^d G)$ is included in $[\beta]^q(\bar{N})$ and hence $Z(\bar{M}) \subset \bar{M}^{[\alpha]}$. Therefore, if $z \in Z(M)$, then $[\alpha](z \otimes 1_G) = z \otimes 1_G \otimes 1_G$ and hence $\alpha(z) = z \otimes 1_G$. Namely, $Z(M) \subset M^\alpha$ or $Z(\alpha(M)) = M^\alpha \otimes 1_G$.

Now we apply Lemma 6.6 to the covariant system $\{\alpha(M), \iota \otimes \delta\}$ and the projection $p \in \alpha(M)$. Then we have a partial isometry v in $\alpha(M) \otimes L^\infty(G)$ satisfying $vv^* = p \otimes 1_G$ and $v^*v = \iota \otimes \delta(p) = \alpha \otimes \iota(p)$. As $1_M \otimes V$ is a function $t \mapsto 1_M \otimes \lambda(t)$ in $(1_M \otimes R(G)) \otimes L^\infty(G)$ and

$$v V_M (v V_M)^* = vv^* = p \otimes 1_G \qquad (V_M \equiv 1_M \otimes V)$$

$$(v V_M)^* v V_M = V_M^* \alpha \otimes \iota(p) V_M = p \otimes 1_G,$$

it follows that $u \equiv (v V_M)_{p \otimes 1_G}$ is a unitary in $\bar{N}_p \otimes L^\infty(G)$ and

$$[\beta]^p(u(t)) = u(t) \otimes \lambda(t).$$

Therefore the note given after Theorem 5.6 gives our desired results.

7. Subgroups and subalgebras.

Throughout this section H, d_H and Δ_H denote a closed subgroup of G, the right invariant Haar measure and the modular function, respectively. We define a subalgebra of $M \otimes_\alpha G$ (resp. $N \otimes_\beta^d G$) associated with H by

$$(7.1) \qquad M \otimes_\alpha H \equiv \{\alpha(M), 1 \otimes \lambda(H)\}''$$

$$(7.2) \qquad (\text{resp. } N \otimes_\beta^d (H \backslash G) \equiv \{\beta(N), 1 \otimes (L^\infty(G) \cap \lambda'(H)')\}'').$$

Utilizing $\hat{\alpha}$ (resp. $\hat{\beta}$), we shall give a correspondence between a subgroup of G and a subalgebra of $M \otimes_\alpha G$ (resp. $N \otimes_\beta^d G$) of the form (7.1) (resp. (7.2)), which generalizes a result due to Takesaki, [23, Theorems 7.2 and

7.3]. For the notational convenience we set

$$\mathscr{L}^\infty(G/H) \equiv L^\infty(G) \cap \lambda(H)' \quad \text{and} \quad \mathscr{L}^\infty(H\backslash G) \equiv L^\infty(G) \cap \lambda'(H)'$$

in what follows.

THEOREM 7.1. *let* $N \equiv M \otimes_\alpha G$ *and* $\beta \equiv \hat{\alpha}$.
(i) $M \otimes_\alpha H = \{y \in N : \beta(y) \in N \otimes \lambda(H)''\}$
   $\qquad = N \cap (1 \otimes \mathscr{L}^\infty(G/H))'$.
(ii) *H coincides with the smallest closed subgroup H' of G satisfying* $\beta(M \otimes_\alpha H) \subset N \otimes \lambda(H')''$.

Proof. (i) Since

$$\beta(\alpha(x)) = \alpha(x) \otimes 1_G \quad \text{and} \quad \beta(\lambda_1(t)) = \lambda_1(t) \otimes \lambda(t),$$

it follows that $\beta(M \otimes_\alpha H) \subset N \otimes \lambda(H)''$.
  Next we shall show that $\beta(y) \in N \otimes \lambda(H)''$ implies $y \in (1 \otimes \mathscr{L}^\infty(G/H))'$ for $y \in N$. For this we have only to show

$$(7.3) \quad \mathscr{L}^\infty(G/H) \otimes 1_G \subset \{Ad\, W(1_G \otimes \lambda(H)') \cup 1_G \otimes B(L^2(G))\}''.$$

Indeed, $y \otimes 1_G$ commutes with $Ad\, 1 \otimes W(N' \otimes \lambda(H)')$ and $1_N \otimes B(L^2(G))$ by assumption and hence y commutes with $1 \otimes \mathscr{L}^\infty(G/H)$ by (7.3). Now, if $f \in \mathscr{L}^\infty(G/H) \cap C(G)$, $\delta^c f \in \mathscr{L}^\infty(G/H) \otimes L^\infty(G)$, where $(\delta^c f)(s, t) \equiv f(ts)$. For any $g, h \in K(G)$ with $\|\Delta g\|_1 = 1$ we set

$$F_{g,h} \equiv \left(\int (\delta^c f_{r^{-1}})(1_G \otimes g_{r^{-1}}) dr \right)(1_G \otimes h).$$

Since $f \in \lambda(H)'$ and $\delta^c f = Ad\, W(1_G \otimes f)$, $F_{g,h}$ belongs to the right hand side of (7.3). If $\xi \in L^2(G) \otimes L^2(G)$, then

$$(F_{g,h}\xi)(s, t) = \left(\int f(r^{-1}s)g(r^{-1}) dr \right) \Delta(t)^{-1}h(t)\xi(s, t).$$

Since $r \mapsto f(r^{-1}s)$ is continuous and bounded, if $g(r^{-1})dr$ converges to the Dirac measure at the unit e, then $F_{g,h}$ converges weakly to $f \otimes (\Delta^{-1}h)$. Therefore $f \otimes (\Delta^{-1}h)$ belongs to the right hand side of (7.3). Making $\Delta^{-1}h$ converge weakly to 1_G, we have the inclusion (7.3) for $\mathscr{L}^\infty(G/H) \cap C(G)$ instead of $\mathscr{L}^\infty(G/H)$. Since $\mathscr{L}^\infty(G/H) \cap C(G)$ is weakly dense in $\mathscr{L}^\infty(G/H)$, we have (7.3).
  Finally we shall show $M \otimes_\alpha H = N \cap (1 \otimes \mathscr{L}^\infty(G/H))'$. Our proof will go along the same line as Takesaki's proof. Suppose that $y \in N \cap (1 \otimes \mathscr{L}^\infty(G/H))'$. We may assume that M is standard. Let J be the

modular conjugation operator of $M \otimes_\alpha G$. Since $J = v(J_M \otimes J_G) = (J_M \otimes J_G)v$ for a certain weakly continuous unitary representation v of G in M, we have

$$J(1 \otimes \mathscr{L}^\infty(G/H))'J = (1 \otimes \mathscr{L}^\infty(H \backslash G))'$$

and hence $JyJ \in N' \cap (1 \otimes \mathscr{L}^\infty(H \backslash G))'$. Here we apply the modified Blattner–Mackey's theorem for induced covariance representations [22, Theorem 4.3]. There exists a natural isomorphism ρ of $(M \otimes_{\alpha \restriction H} H)'$ onto $N' \cap (1 \otimes \mathscr{L}^\infty(H \backslash G))'$ such that

$$\rho(x' \otimes 1_H) = x' \otimes 1_G \quad \text{and} \quad \rho(u(t) \otimes \lambda'_H(t)) = u(t) \otimes \lambda'(t)$$

for $x' \in M'$ and $t \in H$, where u is a strongly continuous unitary representation of G on $\mathscr{H}$ implementing α and where λ'_H is the left regular representation of H on $L^2(H)$. Therefore JyJ belongs to $\rho((M \otimes_{\alpha \restriction H} H)')$, which is generated by $x' \otimes 1_G$ and $u(t) \otimes \lambda'(t)$ for $x' \in M'$ and $t \in H$. Since $v(t)u(t)^* \in M'$ and since

$$J(x' \otimes 1_G)J = \alpha(J_M x' J_M)$$

and

$$J(v(t) \otimes \lambda'(t))J = 1 \otimes \lambda(t),$$

it follows that y belongs to $M \otimes_\alpha H$.

(ii) It is clear from the first equality in (i).

In the above proof we have established a bijective correspondence of a closed subgroup H and a subalgebra $\mathscr{L}^\infty(G/H)$ by the relation

$$\mathscr{L}^\infty(G/H)' = \{x \in B(L^2(G)) : Ad\, W^*(x \otimes 1_G) \in B(L^2(G)) \otimes \lambda(H)''\}.$$

Indeed, since $\mathscr{L}^\infty(G/H)'$ is generated by $L^\infty(G)$ and $\lambda(H)$, it is included in the right hand side. The converse inclusion is direct from (7.3).

The dual version of Theorem 7.1 is the following.

THEOREM 7.2. Let $M \equiv N \otimes_\beta^d G$ and $\alpha \equiv \hat{\beta}$.
(i) $N \otimes_\beta^d (H \backslash G) = \{x \in M : \alpha_t(x) = x, t \in H\}$
$$= M \cap (1 \otimes \lambda'(H))'.$$
(ii) $H = \{t \in G : \alpha_t(x) = x, x \in N \otimes_\beta^d (H \backslash G)\}$.

Proof. (i) Since the action α dual to β is defined by $\alpha(z) = Ad\, 1 \otimes V'(z \otimes 1_G)$ for $z \in N \otimes_\beta^d G$, we have

$$\alpha_t(z)\xi(s,t) = (\alpha(z)\xi)(s,t)$$

$$= ((1 \otimes V')(z \otimes 1_G)(1 \otimes V')^*\xi)(s,t)$$

(7.4)

$$= \lambda_1'(t)z((1 \otimes V')^*\xi)(s,t)$$

$$= \lambda_1'(t)z\lambda_1'(t)^*\xi(s,t),$$

where $\lambda_1'(t) \equiv 1 \otimes \lambda'(t)$. Therefore the second equality in (i) is proved. Moreover, $\alpha_t = Ad\,\lambda_1'(t)$ on $N\otimes_\beta^d G$.

Now we shall show the first equality. Put $\alpha_t^H \equiv \alpha_t$ for all $t \in H$. α^H is then an action of H on $N\otimes_\beta^d G$. Since $N\otimes_\beta^d(H\backslash G)$ is clearly included in $(N\otimes_\beta^d G)^{\alpha^H}$, it remains to show the inclusion relation

(7.5) $N_0 \equiv (N\otimes_\beta^d(H\backslash G))' \subset ((N\otimes_\beta^d G)^{\alpha^H})'.$

According to Theorem 1.1 we have

$$(N_0)^{\beta^H} \subset (\beta(N)')^{\beta'} = (N\otimes_\beta^d G)' \subset ((N\otimes_\beta^d G)^{\alpha^H})'$$

where β^H is defined by (7.6) below. Moreover, $1\otimes\lambda'(H)$ commutes with $(N\otimes_\beta^d G)^{\alpha^H}$. Therefore Lemma 7.3 below implies (7.5).

(ii) Let H_0 be the set of all t such that $\alpha_t(x) = x$ for all $x \in N\otimes_\beta^d(H\backslash G)$. Then H_0 is a closed subgroup of G. Since $H \subset H_0$ and $N\otimes_\beta^d(H\backslash G) \subset N\otimes_\beta^d(H_0\backslash G)$ by (i), we have $H = H_0$.

LEMMA 7.3.   *If β^H is a mapping defined on $(N\otimes_\beta^d(H\backslash G))'$ by*

(7.6) $\beta^H(y) \equiv Ad\,1 \otimes W(y \otimes 1_G),$

then   (i)   *β^H is a dual action of G on $(N\otimes_\beta^d(H\backslash G))'$; and*
         (ii) *$(N\otimes_\beta^d(H\backslash G))' = \{((N\otimes_\beta^d(H\backslash G))')^{\beta^H}, 1\otimes\lambda'(H)\}''.$*

Proof.   (i) Since $[W, 1_G\otimes\lambda'(t)] = 0$ and

$$Ad\,1 \otimes W^*(\beta(N)\otimes 1_G) \subset \beta(N)\otimes R(G)$$

$$Ad\,1 \otimes W^*(1\otimes\mathscr{L}^\infty(H\backslash G)\otimes 1_G) = 1\otimes\mathscr{L}^\infty(H\backslash G)\otimes 1_G,$$

we have $\beta^H(N_0) \subset N_0\otimes R(G)$, where $N_0 \equiv (N\otimes_\beta^d(H\backslash G))'$. Since $W\otimes 1_G$ satisfies (2.5), β^H is a dual action of G on N_0.

(ii) As $N_0 \subset (1\otimes\mathscr{L}^\infty(H\backslash G))'$, $\beta^H(N_0)$ is included in $N_0\otimes\lambda(H)''$. Let ρ be an isomorphism of $\lambda(H)''$ onto $\lambda_H(H)''$ with $\rho(\lambda(t)) = \lambda_H(t)$ and let $\beta_H \equiv \iota\otimes\rho\circ\beta^H$, where λ_H is the right regular representation of H. Then β_H is a dual action of H on N_0. Since $1\otimes\lambda'(H) \subset N_0$ and $\beta_H(1\otimes\lambda'(r)) = 1\otimes\lambda'(r)\otimes\lambda_H(r)$, N_0 is generated

by $(N_0)^{\beta_H}$ and $1 \otimes \lambda'(H)$ by a characterization of a crossed product due to Landstad. Since $(N_0)^{\beta^H} = (N_0)^{\beta_H}$, we complete the proof.

COROLLARY 7.4. *There exists a semi-finite faithful normal operator valued weight E on $N \otimes_\beta^d G$ onto $N \otimes_\beta^d (H \backslash G)$.*

PROPOSITION 7.5. *Let α be an action of G on M and H a closed subgroup of G. Then H is normal if and only if $M \otimes_\alpha H$ is invariant under $Ad\,\lambda_1(t)$ for all t.*

Proof. Since

$$Ad\,\lambda_1(t)(\alpha(x)) = \alpha(\alpha_t(x)), \qquad Ad\,\lambda_1(t)(\lambda_1(s)) = \lambda_1(tst^{-1}),$$

if H is normal, $M \otimes_\alpha H$ is invariant under $Ad\,\lambda_1(t)$. On the other hand, if H is nonnormal, $M \otimes_\alpha H$ is not invariant.

8. Galois correspondence. In this section we shall give a Galois correspondence between closed subgroups of G and globally β invariant von Neumann subalgebras of N containing N^β in Theorem 8.4, which generalizes [**6**, Theorem III.4.3]. If L_0 is an α invariant subalgebra, then $\alpha(L_0) \subset L_0 \otimes L^\infty(G)$, and vice versa. Therefore a von Neumann subalgebra L of N is said to be β *invariant* if $\beta(L) \subset L \otimes R(G)$. In such cases $\alpha \upharpoonright L_0$ and $\beta \upharpoonright L$ are an action of G on L_0 and a dual action of G on L, respectively.

THEOREM 8.1. *Let α be an action of G on M and L a von Neumann subalgebra with $\alpha(M) \subset L \subset M \otimes_\alpha G$. If M is a factor, then the following two conditions are equivalent:*
  (i) *L is $\hat{\alpha}$ invariant; and*
  (ii) *there exists a closed subgroup H of G such that $L = M \otimes_\alpha H$ (or $L = \{y \in M \otimes_\alpha G : \hat{\alpha}(y) \in (M \otimes_\alpha G) \otimes \lambda(H)''\})$.*

Before going into the proof we shall prepare the following lemma, which is implicitly proved in the proof of [**17**, Theorem 3.1].

LEMMA 8.2. $M \otimes L^\infty(G) = \{\alpha(M), 1 \otimes L^\infty(G)\}''$.

Proof. Since the right hand side is included in the left hand side, we want to show the converse inclusion. For this we set $y_{f,g}$ for $f, g \in K(G)$ and $y \in M$ by

$$(8.1) \qquad y_{f,g} \equiv \int (1 \otimes_{t^{-1}} f)\alpha(\alpha_t^{-1}(y))g(t)dt.$$

Since $g \in K(G)$ and $\|(1 \otimes_{t^{-1}} f) \alpha(\alpha_t^{-1}(y))\| \leqq \|f\|_\infty \|y\|$, the right hand side of (8.1) is Bochner integrable and hence $y_{f,g}$ exists in $\{\alpha(M), 1 \otimes L^\infty(G)\}''$. If $\xi, \eta \in K(G, \mathcal{H})$, then

$$(y_{f,g} \xi \mid \eta) = \int \int (f(st^{-1}) \alpha_{st^{-1}}(y) g(t) \xi(s) \mid \eta(s)) dt ds$$

$$= \int \int f(t^{-1}) g(ts) (\alpha_{t^{-1}}(y) \xi(s) \mid \eta(s)) dt ds$$

and $(s, t) \to g(ts)(\alpha_{t^{-1}}(y) \xi(s) \mid \eta(s))$ belongs to $K(G \times G)$. If $f(t^{-1}) dt$ tends to the Dirac measure at the unit e of G, then

$$(y_{f,g} \xi \mid \eta) \to ((y \otimes g) \xi \mid \eta).$$

Since $\|y_{f,g}\| \leqq \|g\|_\infty \|\Delta f\|_1 \|y\|$ and $K(G, \mathcal{H})$ is dense in $\mathcal{H} \otimes L^2(G)$, $y_{f,g}$ converges weakly to $y \otimes g$. Since y and g are arbitrary in M and $K(G)$ respectively, $M \otimes L^\infty(G)$ is included in $\{\alpha(M), 1 \otimes L^\infty(G)\}''$.

Proof of Theorem 8.1. (i) $\Rightarrow$ (ii) We set $N \equiv M \otimes_\alpha G$. Let H be the smallest closed subgroup of G satisfying $\hat{\alpha}(L) \subset N \otimes \lambda(H)''$. Since $\hat{\alpha}(y) \in N \otimes \lambda(H)''$ is equivalent to $sp_{\hat{\alpha}}(y) \subset H$, H coincides with the closed subgroup spanned by $sp_{\hat{\alpha}}(y)$ for all y in L. Therefore, L is included in the set of all $y \in N$ with $\hat{\alpha}(y) \in N \otimes \lambda(H)''$, or in $M \otimes_\alpha H$ by Theorem 7.1. Let ρ be the isomorphism of $M \otimes_\alpha H$ onto $M \otimes_{\alpha^H} H$ satisfying $\rho(\alpha(x)) = \alpha^H(x)$ and $\rho(\lambda(r)) = \lambda_H(r)$. Then

$$(8.2) \qquad \alpha^H(M) \subset \rho(L) \subset M \otimes_{\alpha^H} H.$$

If we denote $(\alpha^H)^\wedge$ by β, then $\Gamma(\beta) = \text{Ker } \alpha^H \upharpoonright Z(M)$ by Theorem 6.1. Since M is a factor by assumption, $\Gamma(\beta) = H$.

Now, we shall show that $\rho(L) \otimes_\beta^d H$ is a factor. Since $\Gamma(\beta) = H$, we have only to prove $\beta \upharpoonright Z(\rho(L))$ is ergodic by Theorem 6.4. For this we suppose that $\beta(z) = z \otimes 1_H$ for z in $Z(\rho(L))$, that is, $z \in Z(\rho(L))^\beta$. (8.2) implies that

$$\alpha^H(M) \subset \rho(L)^\beta \subset (M \otimes_{\alpha^H} H)^\beta = \alpha^H(M).$$

Since M is a factor, so is $\rho(L)^\beta$. Since $Z(\rho(L))^\beta$ is included in $Z(\rho(L)^\beta)$, z is a scalar operator.

On the other hand, by (8.2) we have

$$(8.3) \qquad \alpha^H(M) \otimes_\beta^d H \subset \rho(L) \otimes_\beta^d H \subset (M \otimes_{\alpha^H} H) \otimes_\beta^d H.$$

If we apply Takesaki's duality to (8.3), we have an isomorphism ρ' of

$(M \otimes_{\alpha^H} H) \otimes_{\beta}^d H$ onto $M \otimes B(L^2(H))$ as in the proof of Theorem 2.1 such that

$$M \otimes L^x(H) \subset \rho'(\rho(L) \otimes_{\beta}^d H) \subset M \otimes B(L^2(H)),$$

where the first inclusion is obtained by Lemma 8.2. Since $\rho(L) \otimes_{\beta}^d H$ is a factor as shown in the above, we have $\rho'(\rho(L) \otimes_{\beta}^d H) = M \otimes B(L^2(H))$ by [6, Lemma III.4.2]. Therefore

$$\rho(L) \otimes_{\beta}^d H = (M \otimes_{\alpha^H} H) \otimes_{\beta}^d H,$$

which implies $\rho(L) = M \otimes_{\alpha^H} H$ by (2.7) and hence $L = M \otimes_{\alpha} H$.

(ii) $\Rightarrow$ (i) We have only to show that $\hat{\alpha}(L) \subset L \otimes R(G)$. Since

$$\hat{\alpha}(\alpha(x)) = \alpha(x) \otimes 1_G \quad \text{and} \quad \hat{\alpha}(\lambda_1(r)) = \lambda_1(r) \otimes \lambda(r)$$

and $L = M \otimes_{\alpha} H$, we have $\hat{\alpha}(L) \subset L \otimes \lambda(H)'' \subset L \otimes R(G)$.

The dual version of Theorem 8.1 is the following.

THEOREM 8.3. *Let β be a dual action of G on N and L a von Neumann subalgebra with $\beta(N) \subset L \subset N \otimes_{\beta}^d G$. If $N \otimes_{\beta}^d G$ is a factor, then the following two conditions are equivalent:*
  (i)   *L is β^d invariant; and*
  (ii) *there exists a closed subgroup H of G such that $L = N \otimes_{\beta}^d (H \backslash G)$* (*or $L = \{x \in N \otimes_{\beta}^d G : \hat{\beta}_t(x) = x, \ t \in H\}$*).

Proof. We may assume without any loss of generality that N is standard.

(i) $\Rightarrow$ (ii) Let β' be the dual action of G on $\beta(N)'$ defined by (1.10). Theorem 1.1 gives an action α of G on $(N \otimes_{\beta}^d G)'$ satisfying

$$\{\beta(N)', \beta'\} \sim \{(N \otimes_{\beta}^d G)' \otimes_{\alpha} G, \hat{\alpha}\},$$

namely, there exists an isomorphism ρ of $\beta(N)'$ onto $(N \otimes_{\beta}^d G)' \otimes_{\alpha} G$ satisfying $\hat{\alpha} \circ \rho = \rho \otimes \iota \circ \beta'$.

Since L is β^d invariant by (i), $Ad \ 1 \otimes W^*(L \otimes 1_G)$ is included in $L \otimes R(G)$. Since $[W, 1 \otimes \lambda'(r)] = 0$, $Ad \ 1 \otimes W(L' \otimes 1_G)$ is included in $L' \otimes R(G)$. Therefore L' is β' invariant. Moreover, $N \otimes_{\beta}^d G$ is a factor by assumption and

$$(N \otimes_{\beta}^d G)' \subset L' \subset \beta(N)'.$$

Therefore we can apply Theorem 8.1 to $\{(N \otimes_{\beta}^d G)', \alpha\}$ and obtain a

closed subgroup H of G satisfying

$$\rho(L') = (N\otimes^d_\beta G)' \otimes_\alpha H.$$

Let's recall the proof of [17, Theorem 8.1]. Then ρ is of the form $Ad\,1\otimes V'\circ\beta'$ and it satisfies

$$\rho(y) = \alpha(y) \quad \text{and} \quad \rho(\lambda'_1(r)) = 1\otimes 1_G\otimes\lambda(r)$$

for $y \in (N\otimes^d_\beta G)'$ and $r \in G$. Therefore L' is generated by $(N\otimes^d_\beta G)'$ and $1\otimes\lambda'(H)$ and hence

$$L = (N\otimes^d_\beta G)\cap(1\otimes\lambda'(H))' = N\otimes^d_\beta(H\backslash G).$$

(ii) $\Rightarrow$ (i) The commutant of $L \equiv N\otimes^d_\beta(H\backslash G)$ is generated by $(N\otimes^d_\beta G)'$ and $1\otimes\lambda'(H)$. Since $\beta'(y) = y\otimes 1_G$ for $y \in (N\otimes^d_\beta G)'$ by Theorem 1.1 and

$$Ad\,1\otimes W(1\otimes\lambda'(r)\otimes 1_G) = 1\otimes\lambda'(r)\otimes\lambda(r),$$

it follows that L' is β' invariant. Therefore $Ad\,1\otimes W(L'\otimes 1_G)$ is included in $L'\otimes R(G)$. Since $[W, 1\otimes\lambda'(r)] = 0$, L is β^d invariant similarly as before.

THEOREM 8.4. *Assume that G is separable. If*
(i)   *N is a factor, and*
(ii) *β is integrable and $\Gamma(\tilde{\beta}) = G$,*
then there exists a bijective correspondence between the closed subgroups H of G and the β invariant von Neumann subalgebras L of N containing N^β in such a way that

$$L_H = \{y \in N: \beta(y) \in N\otimes\lambda(H)''\}$$
$$H_L = \cap\,\{H': \beta(L)\subset N\otimes\lambda(H')''\},$$

where H' runs over closed subgroups of G.

Proof. Let K be an infinite dimensional Hilbert space with $\dim K \geqq \dim L^2(G)$. We put $\bar{N} \equiv N\otimes B(K)$ and $\bar{\beta} \equiv \iota\otimes\sigma\circ\beta\otimes\iota$. Then $\{\bar{N}, \bar{\beta}\}$ is a covariant dual system. Since K is infinite dimensional, $\bar{N}^{\bar{\beta}} = N^\beta\otimes B(K)$ and $\dim K \geqq \dim L^2(G)$, it follows that $\bar{\beta}$ is of infinite multiplicity. Since β is integrable by assumption, so is $\bar{\beta}$. If $x \in N$, $y \in B(K)$, $z \in B(L^2(G))$ and $f \in L^\infty(G)$, then

$$\pi \circ [\bar{\beta}](x \otimes y \otimes z) = [\beta] \otimes \iota (x \otimes z \otimes y)$$

$$\pi (1_N \otimes 1_K \otimes 1_G \otimes f) = 1_N \otimes 1_G \otimes f \otimes 1_K$$

where $\pi \equiv \iota_N \otimes \iota \otimes \sigma \circ \iota_N \otimes \sigma \otimes \iota$. Therefore

$$\pi ((\bar{N} \otimes B(L^2(G))) \otimes^d_{[\bar{\beta}]} G) = ((N \otimes B(L^2(G))) \otimes^d_{[\beta]} G) \otimes B(K).$$

Since $\Gamma([\beta]) = \Gamma(\tilde{\beta}) = G$ by (ii), it follows from Corollary 6.2 that $\Gamma(\bar{\beta}) = \Gamma([\bar{\beta}]) = G$. Since N^β is properly infinite, Theorem 6.7 implies that $\bar{\beta}$ is dominant. Therefore $\{\bar{N}, \bar{\beta}\}$ is equivalent to $\{M \otimes_\alpha G, \hat{\alpha}\}$, where $M \equiv \bar{N} \otimes^d_{\bar{\beta}} G$ and $\alpha \equiv \hat{\bar{\beta}}$. Since N is a factor by assumption (i), $\bar{N}$ is also a factor and hence $\bar{\beta}$ is ergodic on $Z(\bar{N})$. Since $\Gamma(\bar{\beta}) = G$, M is a factor by Theorem 6.4.

Now, suppose that L is β invariant. Put $\bar{L} \equiv L \otimes B(K)$. Then $\bar{L}$ is $\bar{\beta}$ invariant. Since $\bar{N}^{\bar{\beta}} \subset \bar{L} \subset \bar{N}$, $\alpha(M) \subset \bar{L} \subset M \otimes_\alpha G$. Therefore we can apply Theorem 8.1 to $\{M, \alpha\}$ and obtain

$$\bar{L}_{H_{\bar{L}}} = \bar{L} \quad \text{and} \quad H_{\bar{L}_H} = H \quad \text{(By Theorem 7.1).}$$

Now, since $\beta(L) \subset N \otimes \lambda(H)''$ is equivalent to $\bar{\beta}(\bar{L}) \subset \bar{N} \otimes \lambda(H)''$, we have $H_L = H_{\bar{L}}$. Therefore, if $x \in L_{H_L}$, then $x \otimes 1_K \in \bar{L}_{H_{\bar{L}}} = \bar{L}$ and hence $x \in L$. Since $L \subset L_{H_L}$ is clear, we have $L = L_{H_L}$.

Moreover, since $\bar{L}_H = L_H \otimes B(K) = (L_H)^-$, we have $H = H_{\bar{L}_H} = H_{L_H \otimes B(K)} = H_{L_H}$.

9. Appendix.

1. As shown by Araki and Haagerup [1, 9], each action α of G on a standard M is implemented by a unitary u in $B(\mathcal{H}) \otimes L^\infty(G)$ satisfying

$$u \otimes 1_G (\iota \otimes \sigma)(u \otimes 1_G) = \iota \otimes \delta(u)$$

in such a way $\alpha(x) = Adu(x \otimes 1_G)$ by Proposition 3.4. Then the commutant of $M \otimes_\alpha G$ is given by Digerness, [7, 10, 16] as follows:

$$(M \otimes_\alpha G)' = \{M' \otimes 1_G, u(1 \otimes R(G)')u^*\}''.$$

The dual version of this result is the following:

THEOREM 9.1. *If a dual action β of G on a standard N is implemented by a unitary w in $B(\mathcal{H}) \otimes R(G)$ satisfying*

$$(9.1) \qquad w^* \otimes 1_G (\iota \otimes \sigma)(w^* \otimes 1_G) = \iota \otimes \gamma(w^*)$$

in such a way $\beta(y) = Ad\, w^*(y \otimes 1_G)$, *then*

$$(N \otimes_\beta^d G)' = \{N' \otimes 1_G, w^*(1 \otimes L^\infty(G))w\}''.$$

   Proof.   We set

$$\pi(f) \equiv Ad\, w^*(1 \otimes f)$$

for $f \in L^\infty(G)$. First we shall show $\pi(f) \in (N \otimes_\beta^d G)'$. For this it suffices to show that $[\pi(f), 1 \otimes g] = 0$ for all $g \in L^\infty(G)$. It is clear that $\pi(f) \in \beta(N)'$. Since $\iota \otimes \sigma(w \otimes 1_G)$ belongs to $B(\mathcal{H}) \otimes 1_G \otimes R(G)$ and since

$$(1 \otimes W)(w^* \otimes 1_G)\sigma(w^* \otimes 1_G) = (w^* \otimes 1_G)(1 \otimes W)$$

by (9.1), we have

$$Ad\, 1 \otimes W \circ Ad\, w^* \otimes 1_G(1 \otimes f \otimes 1_G) = Ad\, w^* \otimes 1_G(1 \otimes f \otimes 1_G)$$

and hence $\beta'(\pi(f)) = \pi(f) \otimes 1_G$, where β' is a dual action of G on $\beta(N)'$ defined by (1.10). Then $\pi(f) \in (1 \otimes L^\infty(G))'$ by the argument given after Theorem 7.1.

   Now, as $M \equiv N \otimes_\beta^d G$ is globally invariant under $Ad\, 1 \otimes \lambda'(t)$, we can define an action of G on M' by $\alpha_t \equiv Ad\, 1 \otimes \lambda'(t) \restriction M'$. Since π is an isomorphism of $L^\infty(G)$ into M' satisfying $\alpha_t \circ \pi = \pi \circ Ad\, \lambda'(t)$, M' is generated by $\pi(L^\infty(G))$ and $(M')^\alpha$ by [16; 17, Theorem 8.3]. Besides

$$(M')^\alpha = M' \cap (1 \otimes R(G)')' = \{\beta(N), 1 \otimes L^\infty(G), 1 \otimes R(G)'\}'.$$

It suffices to show the right hand side is $N' \otimes 1_G$.

   Since $\{N, \beta\}$ and $\{\beta(N), \iota \otimes \gamma\}$ are equivalent, it follows that

$$\beta \otimes \iota(\{\beta(N), 1 \otimes L^\infty(G), 1 \otimes R(G)'\}'')$$

$$= \{\iota \otimes \gamma(\beta(N)), 1 \otimes 1_G \otimes L^\infty(G), 1 \otimes 1_G \otimes R(G)'\}'',$$

which is $\beta(N) \otimes B(L^2(G))$ by [16, (38) in the proof of Theorem 3]. Therefore $(M')^\alpha = N' \otimes 1_G$.

   COROLLARY 9.2. $N \otimes B(L^2(G)) = \{\beta(N), 1 \otimes B(L^2(G))\}''$.

   2.   Haga's factorization holds always for a regular action.   Namely, if α is regular, $M \otimes_\alpha G$ is isomorphic to $M^\alpha \otimes B(L^2(G))$ by Theorems 2.5 and 5.6.   However, as for the converse, we have only the following proposition for an abelian G.

PROPOSITION 9.3. *If G is abelian, the following two conditions are equivalent*:

    (i)   *α is regular*; *and*

    (ii)   *there is an isomorphism ρ of $M \otimes_\alpha G$ onto $M^\alpha \otimes B(L^2(G))$ such that $\rho^{-1}(M^\alpha \otimes 1_G)$ is $\hat{\alpha}$ invariant.*

Proof. (i) $\Rightarrow$ (ii). If α is regular, $\{M, \alpha\}$ is equivalent to $\{M^\alpha \otimes_\beta^d G, \hat{\beta}\}$ for some covariant dual system $\{M^\alpha, \beta\}$. Therefore

$$\{\overline{M^\alpha}, [\beta]\} \sim \{(M^\alpha)\check{~}, \bar{\beta}\} \sim \{M \otimes_\alpha G, \hat{\alpha}\}.$$

Since G is abelian, $[\beta](y \otimes 1_G) = y \otimes 1_G \otimes 1_G$ for $y \in M^\alpha$. Thus we have an isomorphism ρ of $M \otimes_\alpha G$ onto $M^\alpha \otimes B(L^2(G))$ such that $\rho^{-1}(M^\alpha \otimes 1_G)$ is $\hat{\alpha}$ invariant.

    (ii) $\Rightarrow$ (i) We set

$$\langle \beta \rangle \equiv \rho \otimes \iota \circ \hat{\alpha} \circ \rho^{-1} \quad \text{and} \quad \underline{\beta} \equiv Ad\, 1 \otimes W^* \circ \langle \beta \rangle.$$

Then we have

(9.2) $\{M \otimes_\alpha G, \hat{\alpha}\} \sim \{M^\alpha \otimes B(L^2(G)), \langle \beta \rangle\}.$

Since $\langle \beta \rangle$ is a dual action of G on $M^\alpha \otimes B(L^2(G))$ and since $Ad\, W^*(f \otimes 1_G) = f \otimes 1_G$, $Ad\, W^*(\lambda(r) \otimes 1_G) = \lambda(r) \otimes \lambda(r)$ and $Ad\, W^*(1_G \otimes \lambda(r))$ belongs to $L^\infty(G) \otimes R(G)$, $\underline{\beta}$ is an isomorphism of $M^\alpha \otimes B(L^2(G))$ into $M^\alpha \otimes B(L^2(G)) \otimes R(G)$. However, G is abelian by assumption, we have

(9.3) $Ad\, W^*(1_G \otimes \lambda(r)) \in 1 \otimes R(G).$

Since $M^\alpha \otimes 1_G$ is $\langle \beta \rangle$ invariant by (ii), it is $\underline{\beta}$ invariant by (9.3). Therefore there is an isomorphism β of M^α into $M^\alpha \otimes R(G)$ such that

(9.4) $\iota \otimes \sigma \circ \beta \otimes \iota = \underline{\beta} \upharpoonright M^\alpha \otimes 1_G.$

    Now, we shall show that $\underline{\beta} \upharpoonright M^\alpha \otimes 1_G$ is a dual action of G on $M^\alpha \otimes 1_G$. If $y \in N$, then

$\underline{\beta} \otimes \iota \circ \underline{\beta}(y \otimes 1_G) = \underline{\beta} \otimes \iota \circ Ad\, 1 \otimes W^* \circ \langle \beta \rangle (y \otimes 1_G)$

$\quad = \iota \otimes \sigma \otimes \iota \circ \beta \otimes \iota \otimes \iota \circ Ad\, 1 \otimes W^* \circ \langle \beta \rangle (y \otimes 1_G)$ (By (9.4))

$\quad = \iota \otimes \sigma \otimes \iota \circ Ad\, 1 \otimes 1_G \otimes W^* \circ \beta \otimes \iota \otimes \iota \circ \langle \beta \rangle (y \otimes 1_G)$

$$= \tau \circ \langle \beta \rangle \otimes \iota \circ \langle \beta \rangle (y \otimes 1_G) \quad \text{(By (9.4))}$$

$$(\tau \equiv \iota \otimes \sigma \otimes \iota \circ Ad\, 1 \otimes 1_G \otimes W^* \circ \iota \otimes \sigma \otimes \iota \circ Ad\, 1 \otimes W^* \otimes 1_G)$$

$$= \tau \circ Ad\, 1 \otimes 1_G \otimes W^* \circ \langle \beta \rangle \otimes \iota (y \otimes 1_G \otimes 1_G)$$

$$= \tau \circ Ad((1 \otimes 1_G \otimes W^*)(1 \otimes W \otimes 1_G)) \circ \beta \otimes \iota (y \otimes 1_G \otimes 1_G)$$

$$= Ad\, 1 \otimes 1_G \otimes W^* \circ \underline{\beta} \otimes \iota (y \otimes 1_G \otimes 1_G),$$

where the last equality follows from

$$W \otimes 1_G (\sigma \otimes \iota) 1_G \otimes W = Ad\, 1_G \otimes W^* (W \otimes 1_G).$$

Consequently, since β is a dual action of G on $M^\alpha \otimes 1_G$ and $\beta \otimes \iota = \iota \otimes \sigma \circ \underline{\beta} \restriction M_\alpha \otimes 1_G$, β is a dual action of G on M^α and $\langle \beta \rangle = Ad\, 1 \otimes W \circ \bar{\beta} = [\beta]$, which implies that

$$M \sim \alpha(M) = (M \otimes_\alpha G)^{\hat{\alpha}} \sim (M^\alpha \otimes B(L^2(G)))^{[\beta]} = M^\alpha \otimes_\beta^d G$$

by (9.2). Therefore α is regular.

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