SQUARE INTEGRABLE PRIMARY REPRESENTATIONS

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If π is a unitary representation of a locally compact group G, a weight ϕ on the von Neumann algebra $R(\pi)$ generated by π is called semi-invariant if it is transformed into scalar multiples of itself by the action of G. If π is primary we show such objects are essentially unique if one specifies the scaling factor (Schur's lemma). We then study square integrable primary representations and show that a number of possible different definitions are equivalent. We show that any such representation π has a semi-invariant weight scaling by the modular function, and this object is seen to be the proper generalization of the formal degree of π. We formulate and prove generalized Schur orthogonality relations for π. Finally we specialize to the semi-finite case and identify the formal degree as an operator affiliated to $R(\pi)$. For irreducible π the results reduce to those of Duflo and the author, and Phillips.

(1) Let G be a separable (second countable) locally compact group with center Z, and let π be an irreducible unitary representation of G. One then says that π is square integrable (in the wide sense) if there are non-zero vectors x, y in the Hilbert space H_π so that the absolute value of the matrix coefficient $|(x, \pi(g)y)|$ is a square integrable function on the quotient group G/Z. That $|(x, \pi(g)y)|$ depends only on the Z coset of g for an irreducible (or more generally, a primary) representation π is clear since $\pi(gz) = \lambda_\pi(z)\pi(g)$ for some continuous homomorphism λ_π of Z into the circle group T. We say that π is associated to λ_π. The representation π is square integrable (in the strict sense) if $(x, \pi(g)y)$ is square integrable on G for some non-zero x and y. If Z is compact the two notions coincide, and in general all questions of substance concerning such representations can be reduced to the strict sense case. To see this, note that λ_π is a homomorphism of Z into T and therefore corresponding to the group extension of the group G/Z by Z, we can construct a corresponding group extension of G/Z by T, say G^*. There is a bijective correspondence between all primary representations π of G associated to λ_π and all primary representations of G^* associated to the homomorphism of T into itself, given by the identity map, so that in addition square integrability in the wide sense is preserved. But for G^* wide sense square integrability is the same as

strict sense square integrability.   Therefore for simplicity of statement
we will talk about strict sense square integrability.

For unimodular groups G and irreducible representations π, the
phenomenon of square integrability has been well understood for many
years; the standard results are that if one matrix coefficient is square
integrable, all are, and this condition is equivalent to π being a summand
of the regular representation.   A key fact is that the matrix coefficients
satisfy orthogonality relations which have the form

$$\int (x, \pi(g)u)\overline{(y, \pi(g)v)}dg = \frac{1}{d_\pi}(x, y)\overline{(u, v)}$$

where d_π is a positive number called the formal degree.   It of course
depends on the normalization of Haar measure and if dg is multiplied by
$\lambda > 0$, d_π has to be multiplied by λ^{-1}.   The reader is referred to [5] for a
more general formulation of these results in the context of Hilbert
algebras.   Recently M. Duflo and the author [2] found appropriate
generalizations of these results when G is no longer required to be
unimodular (cf. also [4]).   The key observation was to realize that the
formal degree d_π had to be replaced by a positive self-adjoint operator
D_π which is semi-invariant for the group in that $\pi(g)D_\pi\pi(g)^{-1} = \Delta(g)D_\pi$
where Δ is the modular function of G.   Then square integrability means
that some matrix coefficient is square integrable and this is equivalent to
being a summand of the regular representation; but it is no longer true
that all matrix coefficients are square integrable; more precisely,
$(x, \pi(g)u)$ is square integrable with respect to left Haar measure on G iff
u is in the domain of $D_\pi^{-1/2}$ and then we have orthogonality relations of
the form

$$\int (x, \pi(g)u)\overline{(y, \pi(g)v)}d\mu_l(g) = (x, y)(D_\pi^{-1/2}u, D_\pi^{-1/2}v)$$

where μ_l is left Haar measure (Theorem 3 of [2]).

It is our object here to investigate the same kinds of questions for
factor representations π.   These questions were inspired by work of
Rosenberg [6], and we profited from conversations with him.   Of course
if π is type I, all questions are trivially reduced to the case π irreducible,
but if π is type II, the situation is not entirely transparent, and if π is type
III, in which case G is forced to be nonunimodular, a new phenomenon
occurs which bears further study.   In particular there is no way to define
a formal degree operator and in fact we shall see that a formal degree
operator exists if and only if the von Neumann algebra $R(\pi)$ generated
by π is semi-finite.   Thus our first task is to find the appropriate notion

of the formal degree of a representation π which will reduce to the correct thing in case $R(\pi)$ is semi-finite. We shall find this extension by employing the notion of semi-invariant weights on von Neumann algebras.

(2)   Let R be a von Neumann algebra; and ϕ a weight on R. All weights considered will be normal and semi-finite. Now suppose that $R = R(\pi)$ is the von Neumann algebra generated by a primary representation π of a locally compact separable group G and let λ be a continuous homomorphism of G into the positive real numbers.

DEFINITION. Let ϕ be a nonzero normal semi-finite weight on $R(\pi)$; we say that ϕ is semi-invariant of degree λ if $\phi(\pi(g)x\pi(g)^{-1}) = \lambda(g)\phi(x)$ for $x \in R(\pi)^+$.

One's intuition suggests that if such an object exists at all for a given λ, that it should be unique up to scalar multiples since $R(\pi)$ is a factor. This is in fact the case but we first need the following fact.

PROPOSITION 2.1.   *A semi-invariant nonzero weight ϕ is necessarily faithful.*

Proof. Let $N_\phi = \{x : \phi(x^*x) = 0\}$. Then N_ϕ is a σ-weak closed left ideal and if $x \in N_\phi$, $\phi((x\pi(g))^*x\pi(g)) = \lambda(g)^{-1}\phi(x^*x) = 0$ so $x\pi(g) \in N_\phi$. Since the $\pi(g)$ generate $R(\pi)$, and N_ϕ is σ-weak closed, it is then a two-sided ideal, hence 0 or $R(\pi)$. Since $\phi \neq 0$, it is zero, hence ϕ is faithful.

THEOREM 1.   *If ϕ and ψ are two nonzero semi-finite normal weights on $R(\pi)$ semi-invariant of degree λ, then $\psi = a\phi$ for some $a > 0$.*

Proof. We employ the Radon–Nikodym derivative $(D\psi : D\phi)_t = u_t^{\psi,\phi}$ of ψ with respect to ϕ as defined in [1]. If u is a unitary operator in $R(\pi)$ it follows from the chain rule satisfied by Radon–Nikodym derivatives that

$$(D\psi^u : D\phi^u)_t = u^*(D\psi : D\phi)_t u$$

where $\psi^u(x) = \psi(uxu^*)$. Moreover, since the modular automorphism group corresponding to $\lambda\phi$ is the same as that corresponding to ϕ, it follows again from the properties of Radon–Nikodym derivatives that for $\lambda > 0$, $(D\lambda\phi : D\phi)_t = \exp(it(\log\lambda))1$. By using the chain rule again we conclude that $(D\lambda\psi : D\lambda\phi)_t = (D\psi : D\phi)_t$. We now put $u = \pi(g)$ and

conclude that

$$\pi(g)^*(D\psi: D\phi)_t \pi(g) = (D\psi^{\pi(g)}: D\phi^{\pi(g)})$$
$$= (D\lambda(g)\psi: D\lambda(g)\phi)_t = (D\psi: D\phi)_t.$$

Consequently $(D\psi: D\phi)_t$ commutes with $\pi(g)$ and since the operators $\pi(g)$ generate the factor $R(\pi)$, $(D\psi: D\phi)_t$ is central and hence a multiple of the identity. By the cocycle property of these functions, it follows now that $(D\psi: D\phi)_t$ is a one-parameter unitary group and so $(D\psi: D\phi)_t = \exp(+i(\log a)t)1$ for some $a > 0$. But we have already noted that $(Da\phi: D\phi)_t = \exp(+i\log(a)t)1$ and hence by the unicity part of Theorem 1.2.4 of [1] it follows that $\psi = a\phi$.

We shall see that such an object will be the proper one to play the role of the formal degree in the general setting. A more restricted (as it turns out) notion is as follows.

DEFINITION. An (unbounded) positive operator affiliated to the algebra $R(\pi)$ is said to be semi-invariant of degree λ if $\pi(g)^{-1}D\pi(g) = \lambda(g)D$ where λ is as usual a homomorphism of G into R^+.

The relation between these two notions is made clear by the following result which also makes precise our statement in the first section that π can have an operator as its formal degree (if and) only if π is semi-finite.

THEOREM 2. *Let π be primary and ϕ a semi-invariant weight of degree λ for π. Then there exists a positive operator D affiliated to $R(\pi)$, semi-invariant of degree λ, if and only if $R(\pi)$ is semi-finite. In this case $\phi(x) = \mathrm{Tr}(D^{1/2}xD^{1/2})$ (or $\phi = \mathrm{Tr}(D \cdot)$ in the language of [3]) where Tr is a trace on $R(\pi)$.*

Proof. If there exists such a D, then $u_t = D^{it}$ is a unitary group in $R(\pi)$. Moreover, $\pi(g)u_t\pi(g)^{-1} = \lambda(g)^{-it}u_t$ by semi-invariance; we then have $u_t\pi(g)u_t^* = \pi(g)\lambda(g)^{it}$. Now if σ_t^ϕ is the modular automorphism group associated to ϕ, $\sigma_t^\phi = \sigma_t^{a\phi}$ for any $a > 0$, and by the properties of Radon–Nikodym derivatives, $\pi(g)^*\sigma_t^\phi(\pi(g)) = (D\phi^{\pi(g)}: D\phi)_t = \exp(+it\log(\lambda(g)))1$. After rearranging this, we see that $\sigma_t^\phi(\pi(g)) = \pi(g)\lambda(g)^{it}$, and hence that σ_t^ϕ agrees on a set of generators of $R(\pi)$ with the inner automorphism defined by u_t. Thus σ_t^ϕ is in fact inner and consequently $R(\pi)$ is semi-finite.

In this case $\mathrm{Tr}(D^{1/2} \cdot D^{1/2}) = \mathrm{Tr}(D \cdot)$ in the language of [3] evidently

defines a semi-finite weight which one sees at once is semi-invariant of degree λ, and hence by Theorem 1 is a multiple of ϕ.

Conversely if $R(\pi)$ is semi-finite, let Tr be a trace. Then the Radon–Nikodym derivative $(D\phi: D \text{ Tr})_t$ is a one-parameter group in $R(\pi)$ and is therefore of the form D^{it} with D affiliated to $R(\pi)$. By the same kind of calculations one sees immediately that D is semi-invariant of degree λ and this completes the proof.

(3) If ϕ is a semi-invariant weight on $R(\pi)$ we introduce the space $n_\phi = \{x: \phi(x^*x) < \infty\}$. This is a left ideal and is two-sided invariant under multiplication by $\pi(g)$. It is also σ-weak dense in $R(\pi)$ as this is part of the definition of a semi-finite weight. We let m_ϕ be the set of linear combinations of elements of the form y^*x, $x, y \in n_\phi$. Evidently m_ϕ is also closed under left and right multiplication by $\pi(g)$. It is also well known that there is a unique linear functional on m_ϕ denoted again by ϕ which agrees with the original weight on elements of the form x^*x, $x \in n_\phi$.

Since ϕ is faithful, n_ϕ is a pre-Hilbert space, and we let H_ϕ be its completion and let Δ and J be the modular operator and the unitary conjugation defined by the weight ϕ via Tomita–Takesaki theory. We define the left representation of G by $L(g)a = \pi(g)a$ for $a \in n_\phi$. Clearly $L(g)$ extends to a unitary operator on H_ϕ and L is a unitary representation of G on H_ϕ which generates the von Neumann algebra M (isomorphic to $R(\pi)$) of left multiplications on H_ϕ. If we define $R(g)(a) = \lambda(g)^{-1/2}a\pi(g)^{-1}$ for $a \in n_\phi$ it is easy to see that $R(g)$ extends to a unitary operator on H_ϕ.

Now if $x \in n_\phi \cap n_\phi^*$, it follows that $x\pi(g) \in n_\phi \cap n_\phi^*$ since as we have noted n_ϕ is left and right invariant under $\pi(g)$. Now if $S(x) = x^*$ for such elements, we see that $(L(g)S)(x) = \lambda(g)^{1/2}SR(g)(x)$ and hence $L(g)S = \lambda(g)^{1/2}SR(g)$ where S is the closure of the map $x \to x^*$ on $n_\phi \cap n_\phi^*$. It also follows that $R(g)$ and $L(g)$ send the domain of S into itself, and as $S = J\Delta^{1/2}$ is the polar decomposition of S, we see that $R(g)$ and $L(g)$ send the domain of $\Delta^{1/2}$, which is the domain of S, into itself. Moreover

$$L(g)J\Delta^{1/2} = \lambda(g)^{1/2}J\Delta^{1/2}R(g)$$
$$= JR(g)\lambda(g)^{1/2}R(g)^*\Delta^{1/2}R(g)$$

and by the unicity of the polar decomposition we have $L(g)J = JR(g)$ or $JL(g)J = R(g)$ and $\Delta^{1/2} = \lambda(g)^{1/2}R(g)^*\Delta^{1/2}R(g)$, or equivalently $L(g)\Delta^{1/2}L(g)^{-1} = \lambda(g)^{-1/2}\Delta^{1/2}$.

The formula $JL(g)J = R(g)$ shows that $R(g)$ is a unitary representation which generates the commutant $M' = JMJ$ of the von Neumann

algebra generated by L.   It is also evidently the contragredient represen-
tation to R.

In the sequel we shall need the following fact about stability of n_ϕ
and m_ϕ.   Let a be a continuous function of compact support on G and
let $A = \int a(s)\pi(s)ds$ be the corresponding operator in $R(\pi)$ where ds
is left Haar measure.

PROPOSITION 3.1.   *If $x \in n_\phi$, Ax and $xA \in n_\phi$; if $y \in m_\phi$, Ay and*
$yA \in m_\phi$ and $\phi(Ay) = \int a(s)\phi(\pi(s)y)ds$, and similarly for $\phi(yA)$.

Proof.   It is immediate that if $x \in n_\phi$ then $Bx \in n_\phi$ for any $B \in$
$R(\pi)$.   Now if A has the special form as in the statement, we may find a
sequence of operators B_n each of which is a finite linear combination of
the operators $\pi(s)$, $s \in G$ such that $|B_n|$ is dominated by the L_1 norm of
a and so that $B_n \to A$ *-strongly.   We simply use "Riemann sums" to
approximate the integral defining A.   Then $(xB_n)^*(xB_n) = B_n^* x^* xB_n$
evidently converges weakly, and hence σ-weakly since $|B_n| \leq K$, to
Ax^*xA.   Since ϕ is σ-weak lower semi-continuous, it follows that

$$\phi((xA)^*xA) = \phi(Ax^*xA) \leqq \lim \phi((xB_n)^*(xB_n)) = \lim |xB_n|^2,$$

norms taken in H_ϕ.   But now if $a(s) = \lambda^{1/2}(s)\Delta(s^{-1})\tilde{a}(s^{-1})$ where Δ is the
usual modular function on G, it follows that when we integrate the right
representation R against the function $\tilde{a}$, $R(\tilde{a})x = \int \tilde{a}(s)R(s)xds$ con-
verges as a Bochner integral in H_ϕ.   On the other hand, it is evident that
xB_n are Riemann sums approximating this integral so that
$xB_n \to R(\tilde{a})x$.   Thus $\lim |xB_n|^2 = |R(\tilde{a})x|^2$ and so $\phi((xA)^*(xA)) \leqq$
$|R(\tilde{a})x|^2$ and $xA \in n_\phi$ as desired.

Moreover, by the same reasoning, for any m

$$\phi((x(A - B_m))^*x(A - B_m)) \leqq \lim_n |x(B_n - B_m)|^2,$$

and since xB_n is Cauchy in H_ϕ it follows that $|x(A - B_m)|^2 \to 0$ in n_ϕ and
hence that $xA = R(\tilde{a})x$.   Thus for any $z \in n_\phi$, $(xA, z) = (R(\tilde{a})x, z)$,
which means using the Bochner integral for $R(\tilde{a})x$ that

$$\phi(z^*xA) = \int \phi(z^*x\pi(s)^{-1})\lambda(s)^{-1/2}\tilde{a}(s)ds$$

$$= \int \phi(z^*x\pi(s))a(s)ds.$$

Now since any element of m_ϕ can be written as a linear combination of elements of the form z^*x, we have established that $yA \in m_\phi$ if $y \in m_\phi$ and that the desired integral formula holds. The result for Ay follows by the same reasoning.

We now introduce the notion of square integrability and establish equivalent criteria. There are several different notions of matrix coefficients and we first examine these possibilities. We first observe that a "geometric" matrix coefficient $(x, \pi(g)y) = (\pi(g)^{-1}x, y)$ of a representation π can be written in the form $f(\pi(g)^{-1})$ where $f(a) = (a(x), y)$ is an element of the predual $R(\pi)_*$ of $R(\pi)$. In fact, it is clear that any $f \in R(\pi)_*$ is the limit in norm of finite linear combinations of f's of the above form. Hence the most general "matrix coefficient" of π is a function of the form $f(\pi(g)^{-1})$ for some f in the predual of $R(\pi)$. Now suppose that ϕ is a semi-invariant weight on $R(\pi)$ and that $x \in m_\phi$. Then $\pi(g)^{-1}x \in m_\phi$ so that $\phi(\pi(g)^{-1}x)$ is well defined and could also with much justification be called a matrix coefficient of π. We denote these matrix coefficients as m_x with ϕ to be understood. It should be noted that the map $z \to \phi(zx)$ may not be everywhere defined and hence not in the predual so that this matrix coefficient may not be of the above form.

We fix once and for all a left Haar measure μ_l on G and denote by Δ the modular function. Then μ_r defined by $d\mu_r(x) = \Delta(x)^{-1}d\mu_l(x)$ is a right Haar measure.

DEFINITION. A primary representation π is square integrable if $R(\pi)$ has a semi-invariant weight ϕ of degree Δ so that for some $x \in m_\phi$, $m_x(g) = \phi(\pi(g)^{-1}x)$ is a square integrable function on G with respect to u_l.

We first note the following fact.

PROPOSITION 3.2. If π is square integrable, then the set s_ϕ of $x \in R(\pi)$ such that $m_x \in L^2(\mu_l)$ is σ-strong dense in $R(\pi)$, and $s_\phi \subset m_\phi \subset n_\phi \subset H_\phi$ is dense in the Hilbert space H_ϕ.

Proof. Note that s_ϕ is closed under multiplication by $\pi(g)$ on both sides as $\phi(\pi(g)^{-1}\pi(h)x\pi(k)) = \Delta(k)^{-1}\phi(\pi(h^{-1}gk^{-1})^{-1}x)$ which is square integrable in g if m_x is. It follows that the σ-strong closure of s_ϕ has the same property and hence is a two-sided ideal, which is necessarily $R(\pi)$ as π is primary. The second statement follows in the same way.

We shall subsequently show that if some m_x is square integrable,

then m_x is square integrable for all $x \in m_\phi$. We note of course that this notion of square integrability is invariant under quasi-equivalence as quasi-equivalence of π_1 and π_2 simply means that $R(\pi_1)$ and $R(\pi_2)$ are isomorphic by an isomorphism carrying $\pi_1(g)$ onto $\pi_2(g)$. We shall now show that the various other possible definitions of square integrability all coincide.

THEOREM 3. *For a primary representation π the following are equivalent*:

    (1) π *is square integrable*;

    (2) $(x, \pi(\cdot)y) \in L^2(\mu_l)$ *and is non-zero for some* x, y;

    (3) $\exists f \in R(\pi)_*$, $f(\pi(\cdot)^{-1}) \in L^2(\mu_l)$, $f \neq 0$;

    (4) π *is quasi-equivalent to a summand of the left regular representation*.

Proof. We note that Rosenberg [6] has established the equivalence of (2) and (4). We have already noted that (2) $\Rightarrow$ (3) above. To see that (3) $\Rightarrow$ (4), let $\sigma = \infty\pi$, and then using the standard form for elements in the predual of $R(\pi)$, it is clear that $f(\pi(g)^{-1})$ is an ordinary matrix coefficient of σ. Hence by Rosenberg's result, σ and hence π, is quasi-equivalent to a subrepresentation of the regular representation. Thus (2), (3) and (4) are equivalent and we shall complete the argument by establishing the equivalence of (1) and (4).

Suppose that (4) holds; then as (1) is invariant under quasi-equivalence, we can assume that π is a summand of the left regular representation which is invariant under left and right translations. If M is the von Neumann algebra generated by the left regular representation, we may then identify $R(\pi)$ as the weakly closed two-sided ideal in M determined by the projection onto the subspace $H(\pi)$ of $L^2(\mu_l)$ defined by π.

Now M possesses a distinguished weight ϕ, defined by evaluation at the identity [8]. More precisely, if h is continuous with compact support on G and if $f = h^*h$, then f is continuous and $f(e) = |h|_2^2$, the square of the L^2 norm of h. If $L_f = \pi(f)$ is the corresponding operator of left convolution by f, $L_f \in M$ and one defines $\phi(L_f) = f(e)$. Using Tomita theory, this extends to a normal weight ϕ on M. An easy verification shows that $\phi(\pi(g)L_f\pi(g)^{-1}) = \Delta(g)\phi(L_f)$ and by extension the same is true for the weight on M and so ϕ is semi-invariant of degree Δ and faithful.

We restrict ϕ to $R(\pi)$ and it is again semi-finite and semi-invariant of degree Δ. It remains to find an x so that m_x is square integrable. Now if $f = h^*h$ as above, one verifies by direct calculation that $L_f \in m_\phi$ and $\phi(\pi(s)^{-1}L_f) = f(s)$. Let P be the projection of $L_2(G)$

onto $H(\pi)$; then P is in the center of M and $R(\pi) = PM$. Since $PL_f \le L_f$ and $PL_f = PL \, {}_h^* PL_h$, it follows that PL_f is in m_ϕ and we claim that $\phi(\pi(s)^{-1}PL_f) = (Pf)(s)$, where f is viewed on the right hand side as a vector in $L^2(G)$. In fact, let a be a continuous function with compact support on G and let $A = \int \pi(s)^{-1}a(s)ds \in M$. Then evidently

$$\int a(s)\phi(\pi(s)^{-1}\pi(t)L_f)ds = \phi(A\pi(t)L_f)$$

$$= \int a(s)f(t^{-1}s)ds = (\pi(t)f, \bar{a}),$$

where in the last term a is viewed as an element of $L^2(G)$. Now by the general theory of weights, $AxL_f \in m_\phi$ for $x \in M$ and $x \to \phi(AxL_f)$ is in the predual. Evidently the same is true for $x \to (xf, \bar{a})$ where xf is the result of applying the operator $x \in M$ to the vector $f \in L^2(G)$. The above identity asserts equality of these linear functionals for x of the form $\pi(t)$, $t \in G$, and it follows that they are equal for all $x \in M$ and in particular for $x = P$. Then we have $\phi(APL_f) = (Pf, \bar{a})$, but since $PL_f \in m_\phi$, it follows from Proposition 3.1 that the left side above can be written as an integral over G so that we have

$$\int a(s)\phi(\pi(s)^{-1}PL_f)ds = \int a(s)Pf(s)ds.$$

Since this holds for all a which are continuous and compactly supported, it follows that $\phi(\pi(s)^{-1}PL_f) = Pf(s)$ a.e. and therefore that $\phi(\pi(s)^{-1}PL_f)$ is a square integrable function as desired.

We now turn to the converse $(1) \Rightarrow (4)$, so suppose that ϕ is a semi-invariant weight of degree Δ for $R(\pi)$ where π is a primary representation, and let $n_\phi, m_\phi, H = H_\phi$ (the completion of n_ϕ) $\Delta, J, L(g), R(g)$ be as above and suppose that $m_x(g) = \phi(\pi(g)^{-1}x)$ is in $L_2(\mu_l)$ for some $x \in m_\phi$. Let $x = \Sigma y_i^* z_i$ with $y_i, z_i \in n_\phi$. Now let H^n be the n-fold direct sum of H with itself and let R^n be the n-fold direct sum of R acting on H^n. Then $y = (y_1, \cdots, y_n)$ and $z = (z_1, \cdots, z_n)$ are elements of H^n and the matrix coefficient $(z, R^n(g)y)$ is equal to $\Delta(g)^{-1/2}m_x(g^{-1})$. But since $m_x(\cdot)$ is assumed to be square integrable, it follows now that $(z, R^n(\cdot)y)$ is also square integrable. Then by Rosenberg's result, R^n is quasi-contained in the regular representation and hence so is R. But since R and L are contragredients, and L is quasi-equivalent to π, and as the regular representation is self-contragredient, π is quasi-contained in the regular representation and this is (4).

Now that this result is established we can proceed to establish the orthogonality relations for the "matrix coefficients" m_x. Introduce as above the space $H(\pi)$, the sum of all left invariant subspaces of $L^2(\mu_l)$ which are quasi-equivalent to π.

THEOREM 4. *If π is square integrable, then $m_x \in L^2(\mu_l)$ for all $x \in m_\phi$ and ϕ may (uniquely) be normalized so that $x \to m_x$ extends to unitary operator from H_ϕ onto $H(\pi)$. In particular, we have orthogonality relations*

$$\int m_x(g)\overline{m_y(g)}d\mu_l(g) = \phi(y^*x)$$

for $x, y \in m_\phi$.

Proof. As before, let $s_\phi = \{x : x \in m_\phi, m_x \in L^2(\mu_l)\}$; then easy computations show that $m_{L(h)x}(g) = m_x(h^{-1}g)$ and $m_{R(h)x}(g) = m_x(gh)\Delta(h)^{1/2}$. Thus m is a densely defined linear operator from $s_\phi \subset m_\phi \subset n_\phi \subset H_\phi$ into $L^2(\mu_l)$ which intertwines the two-sided representation $L \times R$ of $G \times G$ on H_ϕ with the two-sided regular representation U of $G \times G$ on $L^2(\mu_l)$ given by $(U(t_1, t_2)f)(s) = f(t_1^{-1}, st_2)\Delta(t_2)^{1/2}$. Now by Theorem 3 we know that π is quasi-contained in the regular representation and hence that the von Neumann algebra $R(\pi)$ is isomorphic to a direct summand of the von Neumann algebra generated by the left regular representation. Therefore, by the unicity of semi-invariant weights and the argument of the first part of the proof of Theorem 3, we know that if L_f is left convolution by f, a continuous function on G with compact support, and if e is the unit in $R(\pi)$ viewed as a central projection in the von Neumann algebra of the left regular representation, that $eL_f = L_f e = eL_f e$ is in n_ϕ and that $\phi((eL_f)^*eL_f) \leqq f^*f(1)$ where 1 is the identity element in G. But under the quasi-equivalence of π with a summand of $L_2(\mu_l)$, the operator eL_f is nothing other than $T(f) = \int f(s)\pi(s)d\mu_l(s)$, and so $T(f) \in n_\phi$. Also note that $T(f)^* \in n_\phi$. Moreover, for any $x \in m_\phi$,

$$\phi(T(f)^*x) = \phi\left(\left(\int \overline{f(s)}\pi(s)^{-1}dg\right)x\right) = \int m_x(s)\bar{f}(s)ds.$$

If we read this in terms of inner products this says that $f \to T(f)$ is the adjoint of the operator $x \to m_x$. Thus m defined on s_ϕ has a densely defined adjoint and so is closeable. But now m intertwines $L \times R$ on H_ϕ with U on $L^2(\mu_l)$. Now $L \times R$ is irreducible on H_ϕ, and it follows

since m is closeable that m is a multiple of an isometry and hence after proper normalization of ϕ extends to a unitary intertwining operator of H_ϕ into $L^2(\mu_l)$. Evidently its range is contained in $H(\pi)$ and since $H(\pi)$ is irreducible under U, its range is all of $H(\pi)$. This completes the proof except for the statement that s_ϕ is all of m_ϕ.

To see this, let $x \in m_\phi$ and choose a sequence $x(n) \in s_\phi$ with $x(n) \to x$ in H_ϕ. Then $m_{x(n)}(\cdot) = \phi(\pi(\cdot)^{-1}x(n))$ converges to a limit q in $L^2(\mu_l)$. But now as above, if f is continuous and compactly supported on G, $T(f) \in n_\phi$ and therefore $\phi(T(f)^*x(n))$ converges to $\phi(T(f)^*x)$. As above, this means that

$$\int \bar{f}(s)m_{x(n)}(s)d\mu_l \to \int \bar{f}(s)\phi(\pi(s)^{-1}x)d\mu_l$$

and it follows that $q(s) = \phi(\pi(s)^{-1}x)$ a.e. so that $\phi(\pi(\cdot)^{-1}x)$ is square integrable and hence $x \in s_\phi$.

REMARK. As one can see from the above arguments in Theorems 3 and 4, the objects m_x ($x \in m_\phi$) that we have called matrix coefficients are really matrix coefficients of the right representation R which is the contragradient of L or, equivalently, π. The matrix coefficients of L or π are simply the complex conjugates of those of R, specifically $(L(g)u, v) = \overline{(R(g)Ju, Jv)}$ for $u, v \in H_\phi$. As an alternative to the m_x's one could consider instead functions of the form $\phi(a^*\pi(g)b)$, $a, b \in n_\phi$ which by a simple calculation becomes $\Delta(g)^{-1/2}\phi((Ja)^*(Jb)\pi(g)^{-1}) = \Delta(g)^{1/2}\bar{m}_c(g)$ where $c = (Ja)^*Jb$ provided, of course, that Ja and Jb are in n_ϕ. We have chosen the m_x as our basic objects since formulas come out more elegantly this way.

DEFINITION. If π is a square integrable representation we shall call the semi-invariant weight ϕ normalized as in Theorem 4 the *formal degree* of π. Thus we see that the formal degree which classically was a positive number becomes first a semi-invariant positive operator as in [2] and now even more generally becomes a semi-invariant weight on $R(\pi)$.

(4) We now turn to the case when $R(\pi)$ is semi-finite so that the semi-invariant weight ϕ is of the form $\phi(x) = \mathrm{Tr}(D^{1/2}xD^{1/2})$ where Tr is a trace on $R(\pi)$ and $D^{1/2}$ a uniquely determined semi-invariant positive operator affiliated to $R(\pi)$. We let $L^1(\mathrm{Tr})$ and $L^2(\mathrm{Tr})$ be the spaces of integrable and square integrable operators relative to the trace [7]. The natural action of $R(\pi)$ on $L^2(\mathrm{Tr})$ by left multiplication is of course the *GNS* construction applied to the weight Tr, and we so realize $R(\pi)$. Then D and all its powers can be realized as self-adjoint

operators on $L^2(\mathrm{Tr})$. Of especial importance to us will be the (dense) domain of definition of the operator $D^{-1/2}$, $\mathrm{dom}(D^{-1/2})$. Now let $p = \{x, x \in R(\pi) \cap L^2(\mathrm{Tr}), \text{range}(x^*) \subset \mathrm{dom}(D^{-1/2})\}$, where range (x^*) is the range of x^* as an operator on $L^2(\mathrm{Tr})$, and let q be the span of elements of the form y^*x, $x, y \in p$. Note that p and q are linear spaces invariant under right and left multiplication by $\pi(g)$, $g \in G$. Now if $x \in p$, the closed operator $D^{-1/2}x^*$ is everywhere defined, hence bounded and so $xD^{-1/2}$ is bounded on its domain and hence has a continuous extension to a bounded operator $f(x)$ which is necessarily in $R(\pi)$. Evidently $\phi(f(x)^*f(x)) = \mathrm{Tr}(x^*x)$ so $f(x) \in n_\phi$ and f is an isometry. In the same way, the map l defined on q by $l(y^*x) = (D^{-1/2}y^*xD^{-1/2})^-$ is well defined and sends q into m_ϕ. Finally, note that $q = q^*$ and that if $u \in q$, then $D^{-1/2}u$ is well defined and is bounded and is square integrable relative to the trace, for if $u = y^*x$, $x, y \in p$, then $D^{-1/2}u = (D^{-1/2}y^*)(x)$. Since $D^{-1/2}y^*$ is bounded and x is square integrable and bounded, the product is bounded and square integrable as asserted.

In the context of a semi-finite primary representation, we are also interested in functions $n_u(g) = \mathrm{Tr}(\pi(g)^{-1}u)$ which also have a claim to be called matrix coefficients of π. It is evident that n_u is well defined and is a continuous function on G for any $u \in L^1(\mathrm{Tr})$. We begin the study of such matrix coefficients by considering the case $u \in q \subset L^1(\mathrm{Tr})$. We have the following result.

THEOREM 5. *If $u \in q = q^*$, n_u is square integrable with respect to left Haar measure and if $u, v \in q$,*

$$\int n_u(g)\overline{n_v(g)}d\mu_l = \mathrm{Tr}((D^{-1/2}u^*)^*D^{-1/2}v^*)$$

$$= (D^{-1/2}v^*, D^{-1/2}u^*)_{L^2(\mathrm{Tr})}.$$

Proof. This is simply a matter of rephrasing Theorem 4 and we omit the details.

The above result leaves unanswered the question of providing necessary and sufficient conditions for n_u to be square integrable and we turn to this question now. Let

$$S = \{u: u \in L^1(\mathrm{Tr}) \cap L^2(\mathrm{Tr}), n_u \in L^2(\mu_l)\}.$$

Then we have the following result.

THEOREM 6. *Let $u \in L^2(\mathrm{Tr})$; then $u \in S \Leftrightarrow u$ (equivalently*

$u*) \in L^1(\mathrm{Tr})$, and $u* \in \mathrm{dom}(D^{-1/2})$. If $u, v \in S$, then $(n_u, n_v) = (D^{-1/2}v*, D^{-1/2}u*)$.

Proof. We define the operator N on S into $L_2(G)$ by $N(u) = n_u$. The set S is dense in $L^2(\mathrm{Tr})$ as q is and is two-sided invariant under $\pi(g)$. Moreover, if $\rho(t_1, t_2)u = \pi(t_1)u\pi(t_2)^{-1}$ for $u \in L^2(\mathrm{Tr})$ and $(u(t_1, t_2)f)(g) = f(t_1^{-1}gt_2)\Delta(t_2)^{1/2}$ is as before, it is clear that

$$N\rho(t_1, t_2)u = \Delta(t_2)^{-1/2}U(t_1, t_2)N(u) \quad \text{for} \quad u \in S.$$

Now $N(u)$ is immediately seen to be a matrix coefficient of the representation σ of G on $L_2(\mathrm{Tr})$ given by left multiplication by π. But σ is quasi-equivalent to π, and it follows by Rosenberg's reasoning [6] that $N(u) \in H(\pi) \subset L^2(\mu_l)$.

Now $R(\pi)$ can be realized as a von Neumann algebra operating on $H(\pi)$ so that $\pi(g) \in R(\pi)$ is left translation by g^{-1}. Let $b \to \lambda(b)$ be this realization, $b \in R(\pi)$. Then the formulas in the previous paragraph say that $N(bu) = \lambda(b)N(u)$ for $u \in S$, and b a finite linear combination of the operators $\pi(g)$, $g \in G$. But now if b_n is a sequence of such linear combinations converging to a limit $b \in R(\pi)$, σ strongly, and with $|b_n| \leq k < \infty$, then $N(b_n u)(g) = \mathrm{Tr}(\pi(g)^{-1}b_n u)$ converges to $\mathrm{Tr}((g)^{-1}bu)$ for each $g \in G$. But $\lambda(b_n)N(u)$ converges to $\lambda(b)N(u)$ as λ is normal, and therefore $\mathrm{Tr}(\pi(g)^{-1}bu) = (\lambda(b)u)(g)$ is in $L^2(\mu_l)$ as a function of g. But $bu \in L^1(\mathrm{Tr})$ and so $bu \in S$ and $N(bu) = \lambda(b)N(u)$.

Now let $u_n \in S$ and $u_n \to u$ in $L^2(\mathrm{Tr})$, and suppose $N(u_n)$ has a limit v in $H(\pi)$. Then for any $b \in L^2(\mathrm{Tr}) \cap R(\pi)$, $bu_n \in L^1(\mathrm{Tr})$ and $bu_n \to bu$ in $L^1(\mathrm{Tr})$ and so $N(bu_n)(g) \to \mathrm{Tr}(\pi(g)^{-1}bu)$ for each $g \in G$. But $N(bu_n) = \lambda(b)N(u_n)$ converges to $\lambda(b)v$ in $H(\pi)$. But this says that $\lambda(b)v$ is determined by u alone for each $b \in L^2(\mathrm{Tr}) \cap R(\pi)$, and that u determines v uniquely. This says of course that N is closeable as a densely defined operator $S \to H(\pi)$; and if $\bar{N}$ is its closure, we have by continuity $\bar{N}\rho(t_1, t_2) = \Delta(t_2)^{-1/2}U(t_1, t_2)\bar{N}$.

We then write the polar decomposition $\bar{N} = WK$ where the domain of K is the domain of $\bar{N}$, and where W is a unitary equivalence between the (irreducible) representations ρ and U, and where $K\rho(t_1, t_2) = \Delta(t_2)^{-1/2}\rho(t_1, t_2)K$. Then K as an operator on $L^2(\mathrm{Tr})$ is affiliated to $R(\pi)'$ since it commutes with $\pi(g)$ operating as left multiplications, and is semi-invariant under right multiplications. It follows that JKJ is an operator affiliated to $R(\pi)$ and is semi-invariant of degree $\Delta^{1/2}$. Therefore it is precisely $D^{-1/2}$ up to a multiple as this is the only semi-invariant operator of this degree (Theorems 1 and 2). Thus up to scalars $K(u) = (D^{-1/2}u*)*$ and the domain of K, which is the domain of $\bar{N}$ as already noted, is precisely the set of u such that $u*$ is in the domain of

$D^{-1/2}$. We know now that $S^* \subset L^1(\mathrm{Tr}) \cap \mathrm{dom}(D^{-1/2})$ and that the orthogonality relations hold up to a scalar as W is unitary; that the normalization is correct so that $K = D^{-1/2}$ follows from Theorem 5.

It remains to show that $S^* = L^1(\mathrm{Tr}) \cap \mathrm{dom}(D^{-1/2})$. If u is in the right hand side, $\bar{N}(u)$ is well defined and it follows as above that $\lambda(b)\bar{N}(u)(g) = \mathrm{Tr}(\pi(g)^{-1}bu)$ for $b \in L^2(\mathrm{Tr}) \cap R(\pi)$. Now let b_n be a bounded sequence with $b_n \to 1$ σ-strong with $b_n \in L^2(\mathrm{Tr})$. Then $\lambda(b_n)\bar{N}(u) \to \bar{N}(u)$ and $\mathrm{Tr}(\pi(g)^{-1}b_n u) \to \mathrm{Tr}(\pi(g)^{-1}u)$ for each g as $u \in L^1(\mathrm{Tr})$. Thus $N(u)(g) = \mathrm{Tr}(\pi(g)^{-1}u)$ a.e., and this says that $u \in S$ as desired. This completes the proof.

We can reformulate Theorem 6 in terms of right Haar measure as follows (proof omitted).

THEOREM 7. *Let* $u \in L^2(\mathrm{Tr}) \cap L^1(\mathrm{Tr})$. *Then* $n_u \in L^2(\mu_r)$ *iff* $u \in \mathrm{dom}(D^{-1/2})$ *and then for such* u *and* v

$$\int n_u(g)\overline{n_v(g)}d\mu_r(g) = (D^{-1/2}u, D^{-1/2}v).$$

In conclusion it is instructive to see that we recapture the necessary and sufficient condition in [2] and [4] for square integrability of the ordinary matrix coefficients of an irreducible square integrable representation. Let π be such and let x, y be vectors in the Hilbert space H of π, and let $c_{x,y} = (x, \pi(\cdot)y)$. Then $L^2(\mathrm{Tr})$ is of course the Hilbert–Schmidt operators and we let $T(x, y)(w) = (w, y)x$ be the rank one operator associated to x and y. A simple calculation shows that $c_{x,y} = n_{T(x,y)}$ in the notation of Theorems 5 and 6. Now $T(x, y)^* = T(y, x)$ and $T(y, x) \in \mathrm{dom}(D^{-1/2})$ (viewing $D^{-1/2}$ as an operator on $L^2(\mathrm{Tr})$) iff $y \in \mathrm{dom}(D^{-1/2})$ with $D^{-1/2}$ viewed here as an operator on H; if this is the case, $D^{-1/2}T(y, x) = T(D^{-1/2}y, x)$. Thus Theorem 6 implies that $c_{x,y}$ is square integrable with respect to left Haar measure iff $y \in \mathrm{dom}(D^{-1/2}) \subset H$ and if $y', y \in \mathrm{dom}(D^{-1/2})$, then

$$\begin{aligned}
\langle c_{x,y}, c_{x',y'} \rangle &= \langle T(D^{-1/2}y', x'), T(D^{-1/2}y, x) \rangle \\
&= \mathrm{Tr}(T(x, D^{-1/2}y)T(D^{-1/2}y', x')) \\
&= (D^{-1/2}y', D^{-1/2}y)\mathrm{Tr}(T(x, x')) \\
&= (x, x')(D^{-1/2}y', D^{-1/2}y)
\end{aligned}$$

which is exactly as given in [2] and [4].

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