PACIFIC JOURNAL OF MATHEMATICS
Vol. 70, No. 2, 1977

ON SOME NEW GENERALIZATIONS OF SHANNON'S INEQUALITY

PAL FISCHER

Let $A_n = \{P \in R^n\colon P = (p_1, p_2, \cdots, p_n)$, where $\sum_{i=1}^n p_i = 1$ and $p_i > 0$ for $i = 1, 2, \cdots, n\}$ and let $B_n = \{P \in A_n\colon p_1 \geqq p_2 \geqq \cdots \geqq p_n\}$. We show that the inequality

$$(1)\qquad \sum_{i=1}^n p_i f(p_i) \geqq \sum_{i=1}^n p_i f(q_i)$$

for all $P, Q \in B_n$ and some integer $n \geqq 3$, implies that $f(p) = c \log p + d$, where c is an arbitrary nonnegative number and d is an arbitrary real number. We show, furthermore, that if we restrict the domain of the inequality (1) to those $P, Q \in B_n$ for which $P > Q$ (Hardy-Littlewood-Pólya order), then any function that is convex and increasing satisfies (1).

1. Let $P, Q \in A_n$. Then the inequality

$$(2)\qquad \sum_{i=1}^n p_i \log p_i \geqq \sum_{i=1}^n p_i \log q_i$$

holds with equality iff $P = Q$ **[9]**. The inequality (2) has numerous applications in information theory **[1]**. Conversely, it was proved in **[3]**, that the so-called Shannon-inequality

$$(3)\qquad \sum_{i=1}^n p_i f(p_i) \geqq \sum_{i=1}^n p_i f(q_i)$$

for all $P, Q \in A_n$ and some integer $n \geqq 3$, implies that

$$f(p) = c \log p + d \quad \text{for} \quad p \in (0, 1)$$

where c is some nonnegative number and d is some real number.

The inequality (3) has other interpretations, too. Let us mention the following. Let $E_1, E_2, \cdots, E_n$ be a mutually exclusive and complete system of the events of an experiment with the probability distribution $(p_1, p_2, \cdots, p_n)$ with positive probabilities. Let $q_1, q_2, \cdots, q_n$ be the estimates of these probabilities ($q_i > 0$ for $i = 1, \cdots, n$). If the ith event occurs then the experiment results in the payment of $f(q_i)$. The pay-off function f must be chosen in such a way that the expected pay-off is maximized if the estimates coincide with the a priori distribution.

In some cases it is natural to modify this prescribed model. If the forecaster knows that $p_1 \geqq p_2 \geqq \cdots \geqq p_n$, then evidently the estimation will be made so that $q_1 \geqq q_2 \geqq \cdots \geqq q_n$. That way we restrict the domain of the inequality (1). A further restriction for the domain can be made by allowing P, Q pairs such that $P, Q \in B_n$ and $P > Q$, i.e.

$$\sum_{i=1}^{k} p_i \geqq \sum_{i=1}^{k} q_i \qquad (k = 1, 2, \cdots, n), \tag{4}$$

in addition to the conditions that define B_n.

The subject of this paper is to investigate the inequality (1) under these two types of restrictions. Similar types of inequalities are the topics of some recent papers, **[4]**, **[5]**, **[6]**, **[7]**.

2. In this section we consider the inequality (1) for all $P, Q \in B_n$ in the case when $n = 2$. In that case (1) reduces to

$$pf(p) + (1-p)f(I-p) \geqq pf(q) + (1-p)f(1-q) \tag{5}$$

for all $1 > p \geqq \frac{1}{2}$, $1 > q \geqq \frac{1}{2}$, $f\colon (0,1) \rightarrow R$. It is easy to see that

$$f(1-q) \geqq f(1-p) \text{ implies that } f(p) \geqq f(q). \tag{6}$$

By changing the roles of p and q in (5) and adding the thus obtained inequality to (5), we obtain

$$(p-q)[f(p) - f(q)] \geqq (q-p)[f(1-q) - f(1-p)]. \tag{7}$$

Assume that $1 > p > q \geqq \frac{1}{2}$ and $f(p) < f(q)$, then by (6) we get a contradiction. Therefore f is increasing on $[\frac{1}{2}, 1)$, and by using (6) again we see that f is increasing on $(0,1)$. Since in the previous argument 1 can be replaced by any positive number, and 0 can be replaced by any positive number $b < a/2$, we have shown the first part of the following theorem.

THEOREM 1. *The general solution of the inequality*

$$p_1 f(p_1) + p_2 f(p_2) \geqq p_1 f(q_1) + p_2 f(q_2) \tag{8}$$

for all $p_1 \geqq p_2 \geqq b$, $q_1 \geqq q_2 \geqq b$, $p_1 + p_2 = a$, $q_1 + q_2 = a$, $f: (b, a-b) \rightarrow R$, *is* *increasing in the interval* $(b, a-b)$, *where* a *and* b *are fixed nonnegative numbers,* $b < a/2$. *Furthermore, if* f *is differentiable at a point* $p \in (b, a-b)$, *then it is differentiable at* $a-p$, *too, and*

$$pf'(p) = (a-p)f'(a-p). \tag{9}$$

Proof. According to our previous remark, we have to show only the second part of this theorem.

Without loss of generality we can assume that $p > a/2$. Setting first $p_1 = p+h$, $q_1 = p$ and setting secondly $p_1 = p$ and $q_1 = p+h$ in the inequality (8), where $|h|$ is sufficiently small, we obtain

$$\begin{aligned} \frac{p+h}{a-p-h}[f(p+h)-f(p)] &\geqq f(a-p)-f(a-p-h) \\ &\geqq \frac{p}{a-p}[f(p+h)-f(p)]. \end{aligned} \tag{10}$$

Dividing (10) by $h \neq 0$, and tending with h to 0, we obtain the proof of this theorem.

3. In this section we consider the inequality (1) for all $P, Q \in B_n$ in the case where $n \geqq 3$. We prove the following theorem.

THEOREM 2. *Let* $f\colon (0,1) \to R$. *Then* f *satisfies the inequality*

$$\sum_{i=1}^{n} p_i f(p_i) \geqq \sum_{i=1}^{n} p_i f(q_i) \tag{11}$$

for all $P, Q \in B_n$, *where* n *is a fixed positive integer,* $n \geqq 3$ *only if* f *has the form* $f(p) = c\log p + d$, *where* c *is an arbitrary nonnegative number and* d *is an arbitrary real number.*

Proof. First we show that f is increasing in $(0,1)$. Let $p_3 = \cdots = p_n = q_3 = \cdots = q_n < 1/n$. Then (11) reduces to

$$p_1 f(p_1) + p_2 f(p_2) \geqq p_1 f(q_1) + p_2 f(q_2) \tag{12}$$

for all $p_1 \geqq p_2 \geqq p_3$, $q_1 \geqq q_2 \geqq p_3$ such that $p_1 + p_2 = q_1 + q_2 = 1-(n-2)p_3$. By Theorem 1 we can conclude that f is increasing in the interval $(p_3, 1-(n-1)p_3)$. Since we can choose p_3 to be arbitrarily small positive number we see that f is increasing in the interval $(0,1)$.

Secondly we show that f is differentiable in $(0,1)$. Assume that there exists a point p_0 where f is not differentiable, then by choosing $a > p_0$, $p_3 = q_3 = \cdots = p_n = q_n < \min(p_0, a - p_0)$, we see by Theorem 1 that f is not differentiable at $a - p_0$. By changing a and $p_3 = \cdots = q_n$ adequately the set of points $a - p_0$ forms a set of positive measure, but this is impossible because, according to a theorem of Lebesgue, an increasing function is differentiable almost everywhere.

Finally, by Theorem 1 we can conclude, that $pf'(p) = \frac{1}{2}f'(\frac{1}{2})$ for all $p \in (0, 1)$, that is

$$pf'(p) \equiv c \text{ for all } p \in (0,1), \tag{13}$$

where c is nonnegative, since f is increasing. From (13) this theorem follows immediately.

4. In this section we make further restrictions to the domain of the inequality (1). We shall need the following lemma.

LEMMA 1. *Let $a_1, a_2, \cdots, a_n$ be a sequence of reals such that*

$$s_k = \sum_{i=1}^{k} a_i \geqq 0 \quad \text{for} \quad 1 \leqq k \leqq n. \tag{14}$$

Let $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$ be a sequence of reals. Then $\sum_{i=1}^{n} \lambda_i a_i \geqq 0$.

Proof. This lemma is implicitly contained in Lemmas 3 and 5 in **[7]**.

Now we shall prove a theorem, which is analogous to a result of L. Fuchs **[8]**.

THEOREM 3. *Let $x_1 \geqq \cdots \geqq x_n$, $y_1 \geqq \cdots \geqq y_n$ be arbitrary real numbers, and let $p_1, \cdots, p_n$ be arbitrary nonnegative numbers. Then the inequality*

$$\sum_{i=1}^{n} p_i f(x_i) \geqq \sum_{i=1}^{n} p_i f(y_i) \tag{15}$$

holds for every continuous convex and increasing function f: $[(\min(x_n, y_n), \max(x_1, y_1)] \to R$ if

$$\sum_{i=1}^{k} p_i x_i \geqq \sum_{i=1}^{k} p_i y_i \quad \text{for} \quad k = 1, \cdots, n. \tag{16}$$

Proof. Since f is a continuous convex and increasing function on the interval $[(\min(x_n, y_n), \max(x_1, y_1)]$, there is on the interval

$[\min(x_n, y_n), \max(x_1, y_1))$ an increasing and nonnegative function h such that

$$f(p) = c + \int_{\alpha}^{p} h(q)\,dq \tag{17}$$

where $\alpha \in (\min(x_n, y_n), \max(x_1, y_1))$. It may be necessary to interpret (17) as an improper integral when p is an endpoint.

In order to show the inequality (15), we have to establish that

$$\sum_{i=1}^{n} p_i(f(x_i) - f(y_i)) = \sum_{i=1}^{n} p_i \int_{y_i}^{x_i} h(q)\,dq \geqq 0, \tag{18}$$

or equivalently, we have to show that

$$\sum_{x_i \geqq y_i} p_i \int_{y_i}^{x_i} h(q)\,dq \geqq \sum_{x_i < y_i} p_i \int_{x_i}^{y_i} h(q)\,dq. \tag{19}$$

Using the fact that h is increasing, we see that

$$\sum_{x_i < y_i} p_i \int_{x_i}^{y_i} h(q)\,dq \leqq \sum_{x_i < y_i} p_i(y_i - x_i)h(y_i) \tag{20}$$

and

$$\sum_{x_i \geqq y_i} p_i \int_{y_i}^{x_i} h(q)\,dq \geqq \sum_{x_i \geqq y_i} p_i(x_i - y_i)h(y_i). \tag{21}$$

In order to prove (18) it is sufficient to show that

$$\sum_{x_i \geqq y_i} p_i(x_i - y_i)h(y_i) \geqq \sum_{x_i \geqq y_i} p_i(y_i - x_i)h(y_i), \tag{22}$$

that is to show that

$$\sum_{i=1}^{n} (x_i - y_i)p_i h(y_i) \geqq 0. \tag{23}$$

Lemma 1 yields (23) by letting $p_i(x_i - y_i) = a_i$ and $h(y_i) = \lambda_i$ for $1 \leqq i \leqq n$.

The following lemma can be found implicitly in Mitrinović [**10**, pp. 337–338].

LEMMA 2. *Let $b_1, b_2, \cdots, b_n$ be real numbers, let $a_1 \geqq \cdots \geqq a_n \geqq 0$, and let*

$$\sum_{i=1}^{k} a_i \leqq \sum_{i=1}^{k} b_i \quad \text{for} \quad k = 1, \cdots, n. \tag{24}$$

Then $\sum_{i=1}^{n} a_i^2 \leqq \sum_{i=1}^{n} a_i b_i \leqq \sum_{i=1}^{n} b_i^2$.

Now, we can prove the main result of this section.

THEOREM 4. *Any increasing and convex function $f\colon (0,1) \to R$ satisfies the inequality*

$$\sum_{i=1}^{n} p_i f(p_i) \geqq \sum_{i=1}^{n} p_i f(q_i) \tag{25}$$

for all $P, Q \in B_n$, $P > Q$.

Proof. Let $P, Q \in B_n$, $P > Q$, then by Lemma 2

$$\sum_{i=1}^{n} p_i q_i \leqq \sum_{i=1}^{n} p_i^2.$$

Therefore, according to Theorem 3, by setting $x_i = p_i$, $y_i = q_i$ into (16) $(i = 1, \cdots, n)$ we see that any increasing and convex function f satisfies (25), since f is continuous on $[\min(p_n, q_n), p_1]$.

5. In the previous section we have shown that any convex and increasing function satisfies (1) for all $P, Q \in B_n$ such that $P > Q$. In this section we establish the same result by an alternative proof without the use of any additional lemmas.

Alternative proof. Since f is increasing and convex, f has an integral representation of the form

$$f(p) = \alpha_0 + \int_{\alpha}^{p} h(t)dt, \qquad \alpha \in (0,1) \tag{26}$$

where h is nonnegative and increasing function on $(0,1)$. In order to prove (25), we have to show that

$$\Sigma p_i [f(p_i) - f(q_i)] = \Sigma p_i \int_{q_i}^{p_i} h(t)dt \geqq 0;$$

equivalently, we have to show that

$$\sum_{p_i \geqq q_i} p_i \int_{q_i}^{p_i} h(t)dt \geqq \sum_{p_i < q_i} p_i \int_{p_i}^{q_i} h(t)dt.$$

We see that

$$\sum_{p_i \geqq q_i} (p_i - q_i) = \sum_{q_i > p_i} (q_i - p_i)$$

and

$$\sum_{\substack{p_i \geqq q_i \\ 1 \leqq i \leqq k}} (p_i - q_i) \geqq \sum_{\substack{q_i > p_i \\ 1 \leqq i \leqq k}} (q_i - p_i)$$

for $1 \leqq k \leqq n$. Let i_1 be the smallest index i for which $q_i > p_i$, let i_2 be the smallest index i such that $i_2 > i_1$ and $p_i > q_i$, let i_3 be the smallest index i such that $i_3 > i_2$ and $q_i > p_i$, and so on. We shall show that

$$\sum_{i=1}^{i_1-1} p_i \int_{q_i}^{p_i} h(t)dt \geqq \sum_{i=i_1}^{i_2-1} p_i \int_{p_i}^{q_i} h(t)dt. \tag{27}$$

Furthermore if $\sum_{i=1}^{i_1-1}(p_i - q_i) > \sum_{i=i_1}^{i_2-1}(q_i - p_i)$, we shall show that

$$\sum_{i=1}^{i_1^*-1} p_i \int_{q_i^*}^{p_i} h(t)dt \geqq \sum_{i=i_1}^{i_2-1} p_i \int_{p_i}^{q_i} h(t)dt, \tag{28}$$

where $q_i^* = q_i$ for all $i \leqq i_1^* - 1$, except possibly for $i = i_1^* - 1$, and $i_1^* - 1$ and $q_{i_1-1}^*$ are determined by the relation

$$\sum_{i=1}^{i_1^*-1} (p_i - q_i^*) = \sum_{i=i_1}^{i_2-1} (q_i - p_i).$$

(This last relation determines i_1^* uniquely, unless $p_i = q_i$ for some i, but in this latter case we can choose any of these indices.)

To prove (28), we remark that any interval of the type (q_i^*, p_i) is to the right of the interval (p_i, q_i) for $i_1 \leqq i \leqq i_2 - 1$ and that $\{p_i\}$ is a nonnegative and decreasing sequence. These things, together with the fact that h is a nonnegative and increasing function, prove (28). The next step of the proof consists in showing that

$$\begin{aligned} p_{i_1^*-1} \int_{q_{i_1-1}}^{q_{i_1-1}^*} h(t)dt + \sum_{i=i_1^*}^{i_1-1} p_i \int_{q_i}^{p_i} h(t)dt + \sum_{i=i_2}^{i_3-1} p_i \int_{q_i}^{p_i} h(t)dt \\ \geqq \sum_{i=i_3}^{i_4-1} p_i \int_{p_i}^{q_i} h(t)dt \end{aligned} \tag{29}$$

and in proving the analogue of inequality (28) if

$$\sum_{i=1}^{i_1-1}(p_i-q_i)+\sum_{i=i_2}^{i_3-1}(p_i-q_i)>\sum_{i=i_1}^{i_2-1}(q_i-p_i)+\sum_{i=i_3}^{i_4-1}(q_i-p_i).$$

By repeating the argument we obtain the proof of this theorem.

6. With the aid of simple examples one can see that (1) may fail for some $P, Q \in B_n$ with $P > Q$ if we assume that f is merely increasing, or that f is merely convex, or merely concave.

We shall next present, some related inequalities which may be of independent interest.

We need the following lemma.

LEMMA 3. *Let $u_1, u_2, \cdots, u_n$ be a sequence of real numbers such that*

$$S_k=\sum_{i=1}^{k} u_i \geqq 0 \quad \textit{for} \quad 1 \leqq k \leqq n-1 \tag{30}$$

and

$$S_n=\sum_{i=1}^{n} u_i = 0. \tag{31}$$

Let $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, then

$$\sum_{i=1}^{n} \lambda_i u_i \leqq 0. \tag{32}$$

Proof. The relation (31) implies that $\lambda_1 u_1 + \lambda_1 \sum_{i=2}^{n} u_i = 0$. Clearly $\sum_{i=k}^{n} u_i \leqq 0$ for $1 \leqq k \leqq n$. Therefore

$$\lambda_1 \sum_{i=2}^{n} u_i \geqq \lambda_2 \sum_{i=2}^{n} u_i, \quad \text{i.e.}$$

$$0 \geqq \lambda_1 u_1 + \lambda_2 u_2 + \lambda_2 \sum_{i=3}^{n} u_i.$$

By repeating the argument we obtain the proof of this lemma.

Our next result will show, among other things, that the function $f(p)=1/p^{\alpha}$ $(\alpha \geqq 1)$ satisfies the inequality (1) for all $P, Q \in B_n$, $P > Q$.

THEOREM 5. *Let $P, Q \in B_n$, $P > Q$. Then*

$$(33) \qquad \sum_{i=1}^{n} \frac{p_i}{q_i^{\alpha}} \leqq \sum_{i=1}^{n} \frac{1}{q_i^{\alpha-1}} \leqq \sum_{i=1}^{n} \frac{1}{p_i^{\alpha-1}} \quad \text{where} \quad \alpha \geqq 1.$$

Proof. The right-hand side of (33) is, of course, well-known; we mention it only because it shows that $1/p^{\alpha}$ for $\alpha \geqq 1$ satisfies (1) for all $P, Q \in B_n$, $P > Q$. It is a special case of a theorem of Hardy–Littlewood–Pólya, which says that if $P > Q$, $p_1 \geqq p_2 \geqq \cdots \geqq p_n$, $q_1 \geqq \cdots \geqq q_n$, and $f: [p_n, p_1] \to R$ is a continuous and convex function, then

$$(34) \qquad \sum_{i=1}^{n} f(p_i) \geqq \sum_{i=1}^{n} f(q_i).$$

Let $u_i = p_i - q_i$ for $i = 1, 2, \cdots, n$. It is easy to see that $\sum_{i=1}^{k} u_i \geqq 0$ for $1 \leqq k \leqq n-1$ and $\sum_{i=1}^{n} u_i = 0$. Thus

$$\sum_{i=1}^{n} \frac{p_i}{q_i^{\alpha}} = \sum_{i=1}^{n} \frac{u_i}{q_i^{\alpha}} + \sum_{i=1}^{n} \frac{1}{q_i^{\alpha-1}} \leqq \sum_{i=1}^{n} \frac{1}{q_i^{\alpha-1}}$$

since by Lemma 3 $\sum_{i=1}^{n} u_i/q_i^{\alpha} \leqq 0$.

Finally we note two special cases of the previous theorem. If $P, Q \in B_n$ and $P > Q$, then

$$(35) \qquad \sum_{i=1}^{n} \frac{p_i}{q_i} \leqq n,$$

and

$$(36) \qquad \sum_{i=1}^{n} \frac{p_i}{q_i^2} \leqq \sum_{i=1}^{n} \frac{1}{q_i}.$$

REFERENCES

1. R. Ash, *Information Theory*, Interscience Publishers, 1965.
2. P. Fischer, *On the inequality* $\Sigma p_i f(p_i) \geqq \Sigma p_i f(q_i)$, Metrika, **18** (1972), 199–208.
3. ———, *On the inequality* $\Sigma g(p_i) f(p_i) \geqq \Sigma g(p_i) f(q_i)$, Aequationes Math., **10** (1974), 23–33.
4. ———, *On the inequality* $\Sigma p_i f(p_i)/f(q_i) \leqq 1$, Canad. Math. Bull. **17** (1974), 193–199.
5. ———, *Sur l'inégalité* $\Sigma[p_i f(p_i) + q_i f(q_i)] \geqq [p_i f(q_i) + q_i f(p_i)]$, Period. Hungar., **5** (1974), 87–92.
6. ———, *On the inequality* $\Sigma p_i f(p_i)/f(q_i) \geqq 1$, Pacific J. Math., **60** (1975), 65–74.
7. P. Fisher and J. A. R. Holbrook, *Matrices doubly stochastic by blocks*, Canad. J. Math., **29** No. 3 (1977), 559–571.
8. L. Fuchs, *A new proof of an inequality of Hardy–Littlewood–Pólya*, Mat. Tidsskr. B (1947), 53–54.

9. P. G. Kirmser, *Problem E* 1274, Amer. Math. Monthly, **64** (1957).
10. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, 1970.

Received April 20, 1977.

UNIVERSITY OF GUELPH
GUELPH, ONTARIO, CANADA