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RATIONAL APPROXIMATION AND THE GROWTH OF ANALYTIC CAPACITY

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Let X be a compact set in the complex plane $\mathbf{C}$. Denote by $R(X)$ the closure in the supremum norm of the rational functions with poles off X and by $A(X)$ the set of continuous functions, which are analytic on the interior of X. The analytic capacity of a set S is denoted by $\gamma(S)$. For the definition of γ see below. Let $B_z(\delta) = \{\zeta \in \mathbf{C}; |z-\zeta| < \delta\}$ and let ∂X denote the boundary of X. Vitushkin has proved that $R(X) = A(X)$ if

$$\varliminf_{\delta \to 0} \frac{\gamma(B_z(\delta)\setminus X)}{\delta} > 0 \text{ for all } z \in \partial X.$$

Let ψ be a function from $\mathbf{R}^+$ to $\mathbf{R}^+$, where $\mathbf{R}^+ = \{x \in \mathbf{R}; x \geqq 0\}$. We now ask the following questions. If $\lim_{\delta\to 0} \psi(\delta) = 0$, is it possible to find a compact set X such that $R(X) \neq A(X)$ and such that $\gamma(B_z(\delta)\setminus X) \geqq \delta\psi(\delta)$ for all $z \in \partial X$ and for all δ, $0 < \delta < \delta_z$? If the answer is yes, can the answer still be yes, if $\lim_{\delta\to 0}\psi(\delta) = 0$ is replaced by $\varlimsup_{\delta\to 0}\psi(\delta) > 0$? The answers of these questions can be found in Theorem 1 and Theorem 2.

DEFINITION. Let K be a compact subset of $\mathbf{C}$. Then $\gamma(K) = \sup |f'(\infty)|$, where the supremum is taken over all functions f such that f is analytic on the unbounded component of $\mathbf{C}\setminus K$, $|f(z)| \leqq 1$ for all $z \in \mathbf{C}$ and $f(\infty) = 0$. Let S be an arbitrary subset of $\mathbf{C}$. Then $\gamma(S) = \sup \gamma(K)$, where the supremum is taken over all compact subsets of S.

For further information about this capacity see for instance **[2]**, **[3]**, **[4]** and **[5]**.

THEOREM 1. *Let $\delta_n \searrow 0$ when $n \to \infty$. Suppose that*

$$\varliminf_{n\to\infty} \frac{\gamma(B_z(\delta_n)\setminus X)}{\delta_n} > 0 \textit{ for all } z \in \partial X.$$

Then $R(X) = A(X)$.

THEOREM 2. *Let ψ be a function from $\mathbf{R}^+$ to $\mathbf{R}^+$. Suppose that $\lim_{\delta\to 0}\psi(\delta) = 0$. Then there exists a compact set X such that*

(a) $R(X) \neq A(X)$

and

(b) $\gamma(B_z(\delta)\backslash X) \geqq \psi(\delta)\delta$ *for all* $z \in \partial X$ *and for all* δ, $0 < \delta < \delta_z$.

REMARK. Theorem 1 gives the following. Let ψ be a function from $\mathbf{R}^+$ to $\mathbf{R}^+$. Suppose that $\overline{\lim}_{\delta\to 0}\psi(\delta) > 0$ and suppose that $\gamma(B_z(\delta)\backslash X) \geqq \psi(\delta)\delta$ for all $z \in \partial X$ and for all δ, $0 < \delta < \delta_z$. Then $R(X) = A(X)$.

2. The proofs. Theorem 1 can be proved in the same way as the theorem of Vitushkin mentioned in the introduction. See **[4]**, Ch. 2, §4. We omit the proof.

In **[1]** A. M. Davie constructed a compact set X such that every point of ∂X is a peak point for $R(X)$, but $R(X) \neq A(X)$. Our proof of Theorem 2 is a refinement of Davie's construction. We start by formulating two lemmas. The first lemma is well-known (see for instance **[2]**, p. 199). The second lemma is due to Carleson. For a proof see **[1]**.

LEMMA 1. *Let L be a compact set on a line. Then*

$$\gamma(L) \geqq \tfrac{1}{4}\{\textit{the length of } L\}.$$

LEMMA 2. *Let E be a a perfect subset of the real line and I the closed interval* $[0,1]$. *Then we can find a continuous function on* $\mathbf{C}$, *analytic outside* $I \times E$, *such that* $f(\infty) = 0$, $f'(\infty) = \frac{1}{4}$ *and* $|f(z)| \leqq 1$ *for all* $z \in \mathbf{C}$.

If $x \in \mathbf{R}$, let $[x]$ denote the greatest integer less than or equal to x.

Proof of Theorem 2. We may assume that $\psi(\delta)$ is a strictly increasing function. Put $a_n = 16\psi(2^{-n+1})$, $n = 1, 2, 3, \cdots$. Then $a_n \searrow 0$ when $n \to \infty$.

Let f be an increasing function such that $f(-2 - \log a_n) = n$. Put

$$b_0 = 1$$

and

$$b_n = \min(e^{-f(n)}, \tfrac{1}{4}b_{n-1}) \quad \text{for} \quad n \geqq 1.$$

Let E be the usual Cantor set on the real axis such that the set E_n obtained in nth step consists of 2^n intervals of length b_n. Let $I = [0, 1]$.

Let n be fixed for a moment. There exists an integer k_n such that

$$b_n \geqq 2^{-k_n}. \tag{1}$$

Denote the intervals in E_n by $I_{n,i}$, $i = 1, 2, \cdots, 2^n$. In every $I \times I_{n,i}$ choose open disjoint discs with radius $2^{-k_n-3}e^{-n-1}$ in the following way. Every disc must not intersect $I \times E_{n+1}$ but every disc must touch $I \times E_{n+1}$. Moreover, the discs are arranged such that the centres of the discs lie on two horizontal lines in every $I_{n,i}$. There are 2^{k_n+3} centres on each line and the distance between two successive centres is 2^{-k_n-3}. Call the chosen discs $U_{n,j}$.

Repeat the construction for all n, $n = 1, 2, 3, \cdots$. Put

$$X = \overline{B_0(2)} \setminus \left(\bigcup_{n,j} U_{n,j} \right),$$

where $\overline{B_0(2)}$ denotes the closure of $B_0(2)$. X is a compact set and

$$\partial X = \partial B_0(2) \cup \left(\bigcup_{n,j} \partial U_{n,j} \right) \cup (I \times E).$$

It is easy to see that $\Sigma_{n,j} \operatorname{diam} U_{n,j} < \infty$. Lemma 2 and a standard argument give

$$R(X) \neq A(X).$$

See [2], p. 220.

(i) Let

$$z \in \partial B_0(2) \cup \left(\bigcup_{n,j} \partial U_{n,j} \right).$$

Lemma 1 gives for all $m \geqq m_z$

$$\gamma(B_z(2^{-m}) \setminus X) \geqq \tfrac{1}{4} 2^{-m} \geqq \tfrac{1}{4} a_m 2^{-m}.$$

(ii) Let $z \in I \times E$. Let m be a positive integer such that $a_m < e^{-2}$. The definition of f gives $f(-2 - \log a_m) = m$. Fix n such that $n = [-\log a_m] - 1$. If we use that f is an increasing function and the definition of b_n, we obtain

$$2^{-m} = e^{-f(-2-\log a_m)} \geqq e^{-f(-1+[-\log a_m])} = e^{-f(n)} \geqq b_n.$$

Thus

$$(2) \qquad 2^{-m} \geqq b_n.$$

One now easily shows that $B_z(2^{-m})$ contains disjoint discs U_{n,j_i}, $i = 1, 2, \cdots, 2^{k_n+2}2^{-m} - 2$, such that their centres are on one straight line. Lemma 1, (1) and (2) give

$$\gamma(B_z(2^{-m})\setminus X) \geqq \gamma\left(\bigcup_i U_{n,j_i}\right) \geqq \tfrac{1}{4}\{2^{k_n+2}2^{-m} - 2\}2^{-k_n-2}e^{-n-1}$$
$$= \tfrac{1}{4}e^{-n-1}\{2^{-m} - 2^{-k_n-1}\} \geqq \tfrac{1}{4}e^{-n-1}\{2^{-m} - \tfrac{1}{2}b_n\}$$
$$\geqq \tfrac{1}{4}e^{-n-1}\{2^{-m} - \tfrac{1}{2}2^{-m}\} = \tfrac{1}{8}2^{-m}e^{-n-1}.$$

Thus

$$\gamma(B_z(2^{-m})\setminus X) \geqq \tfrac{1}{8}2^{-m}e^{-n-1}.$$

If we use that $n = [-\log a_m] - 1$, we obtain

$$e^{-n-1} = e^{-[-\log a_m]} \geqq e^{\log a_m} = a_m.$$

Thus

$$\gamma(B_z(2^{-m})\setminus X) \geqq \tfrac{1}{8}a_m 2^{-m}.$$

Now (i) and (ii) give that for all $z \in \partial X$ there is a constant m_z such that

$$\gamma(B_z(2^{-m})\setminus X) \geqq \tfrac{1}{8}a_m 2^{-m} \text{ for all } m \geqq m_z.$$

The definition of a_m gives for all $z \in \partial X$ and for all $m \geqq m_z$

$$\gamma(B_z(2^{-m})\setminus X) \geqq 2\psi(2^{-m+1})2^{-m}.$$

If we use that ψ is increasing, we get

$$\gamma(B_z(\delta)\setminus X) \geqq \psi(\delta)\delta$$

for all $z \in \partial X$ and for all δ, $0 < \delta < \delta_z$.

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