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MAXIMAL SUBMONOIDS OF THE TRANSLATIONAL HULL

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Maximal submonoids of a semigroup have recently attracted attention in semigroup literature. This is particularly true for the semigroup $\mathscr{B}(X)$ of binary relations on a set. The interesting results of Zareckiĭ in this direction point to the fact that some of these statements pertain to the more general situation of the translational hull of a Rees matrix semigroup. More generally, we consider here maximal submonoids of the translational hull of a regular semigroup.

The first, and the main, theorem in this paper says that if ω is an idempotent bitranslation of a regular semigroup S, then $\omega\Omega(S)\omega \cong \Omega(\omega S\omega)$; here $\omega\Omega(S)\omega$ is a maximal submonoid of $\Omega(S)$. The second theorem pertains to subdirect irreducibility of certain subsemigroups of the translational hull of a Rees matrix semigroup. Finally, the third theorem concerns regular semigroups in which every maximal submonoid is a retract. These results have a number of consequences. The paper ends with several examples of concrete semigroups to which some of the preceding results are applied.

We start with a list of needed definitions and simple results. Let S be a semigroup. A function λ (resp. ρ), written on the left (resp. right) is a *left* (resp. *right*) *translation* of S if $\lambda(xy) = (\lambda x)y$ (resp. $(xy)\rho = x(y\rho)$) for all $x, y \in S$. The set $\Lambda(S)$ (resp. $P(S)$) of all left (resp. right) translations of S under composition $(\lambda\lambda)'x = \lambda(\lambda' z)$ (resp. $x(\rho\rho') = (x\rho)\rho'$) is a semigroup. The pair $(\lambda, \rho) \in \Lambda(S) \times P(S)$ is a *bitranslation* of S if $x(\lambda y) = (x\rho)y$ for all $x, y \in S$; the subsemigroup of $\Lambda(S) \times P(S)$ consisting of all bitranslations is the *translational hull* $\Omega(S)$ of S. Its elements will be usually written as $\omega = (\lambda, \rho)$, where ω is considered as a bioperator on S. For any $s \in S$, the function λ_s (resp. ρ_s) defined by $\lambda_s = sx$ (resp. $x\rho_s = xs$) for all $x \in S$, is the *inner left* (resp. *right*) *translation* and $\pi_s = (\lambda_s, \rho_s)$ is the *inner bitranslation* of S induced by s. The set $\Pi(S) = \{\pi_s \mid s \in S\}$ is an ideal of $\Omega(S)$ called its *inner part*. The mapping $\pi: s \rightarrow \pi_s$ is the *canonical homomorphism* of S into $\Omega(S)$. It is one-to-one if and only if S is *weakly reductive*. In such a case for any $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$, $s \in S$, we have $(\lambda s)\rho = \lambda(s\rho)$, and thus all parentheses may be omitted.

An element $s \in S$ is *regular* if $s = sts$ for some $t \in S$; if also $t = tst$, then t is an *inverse* of s. A semigroup in which every element is regular is a *regular semigroup*. Note that every regular element has an inverse, and that a regular semigroup is weakly reductive, and hence the canonical homomorphism above is one-to-one. A semigroup S is *completely regular* if every element of S has an inverse with which it commutes (equivalently, S is a union of groups).

An element e of S is idempotent if $e^2 = e$; the set of all *idempotents* of S will be denoted by E_S. If $e \in E_S$, then the set $eSe = \{ese \mid s \in S\}$ is the set of all elements of S having e as a (two-sided) identity, and is thus called a *maximal submonoid* of S (since a semigroup with an identity element is called a *monoid*). It is easy to see that every maximal submonoid of a regular semigroup is again a regular semigroup. If $\omega = (\lambda, \rho) \in E_{\Omega(S)}$, the above definitions and conventions yield

$$(1) \qquad \omega S \omega = \{\lambda s \rho \mid s \in S\} = \{s \in S \mid s = \lambda s = s\rho\} .$$

If I is an ideal of S, then S is an (ideal) *extension* of I; S is a *dense extension* of I if the equality relation on S is the only congruence on S whose restriction to I is the equality relation; if S is a maximal dense extension of I, then I is a *densely embedded ideal* of S. For a weakly reductive semigroup S, $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$.

The proofs of the above statements as well as the concepts used in the paper but not defined can be found in the book [5]. This reference as well as the survey article [2] contain a comprehensive collection of results concerning the translational hull.

2. The main theorem.

This result gives a suitable isomorphic copy of maximal submonoids of the translational hull of a regular semigroup.

THEOREM 1. *Let S be a regular semigroup. If $\omega \in E_{\Omega(S)}$, then the function χ defined by*

$$\chi: (\varphi, \psi) \longrightarrow (\varphi|_{\omega S \omega}, \psi|_{\omega S \omega}) \quad ((\varphi, \psi) \in \omega\Omega(S)\omega) ,$$

is an isomorphism of $\omega\Omega(S)\omega$ onto $\Omega(\omega S \omega)$.

Proof. Let $\omega = (\lambda, \rho)$ and note that

$$(2) \qquad \omega\Omega(S)\omega = \{(\varphi, \psi) \in \Omega(S) \mid \varphi = \lambda\varphi = \varphi\lambda,\ \lambda = \rho\psi = \psi\rho\} .$$

Next let $(\varphi, \psi) \in \omega\Omega(S)\omega$. For any $x \in \omega S \omega$, using (1) and (2) we have

$$\varphi x = (\lambda\varphi)(x\rho) = \lambda(\varphi x)\rho \in \omega S \omega$$

so that $\varphi|_{\omega S\omega}$ maps $\omega S\omega$ into itself. Similarly $\psi|_{\omega S\omega}$ has the same property. It then follows without difficulty that χ is a homomorphism of $\omega\Omega(S)\omega$ into $\Omega(\omega S\omega)$.

Next let $(\varphi, \psi), (\varphi', \psi') \in \omega\Omega(S)\omega$ and assume that $(\varphi, \psi)\chi = (\varphi', \psi')\chi$. Let $x \in S$; there exists $u \in S$ such that $\lambda x = (\lambda x)u(\lambda x)$. Then $\lambda(xu)\rho \in \omega S\omega$ and

$$\begin{aligned}\varphi x = (\varphi\lambda)x = \varphi(\lambda x) = \varphi[(\lambda x)u(\lambda x)] &= [\varphi(\lambda(xu)\rho)]x \\ = [\varphi'(\lambda(xu)\rho)]x = \varphi'[(\lambda x)u(\lambda x)] &= \varphi'(\lambda x) = (\varphi'\lambda)x = \varphi' x\end{aligned}$$

so that $\varphi = \varphi'$; analogously $\psi = \psi'$. Consequently χ is one-to-one.

Next let $(\varphi, \psi) \in \Omega(\omega S\omega)$. Define φ' and ψ' on S by

$$\begin{aligned}\varphi' x &= [\varphi(\lambda(xu)\rho)]x \quad \text{if} \quad \lambda x = (\lambda x)u(\lambda x)\ , \\ x\psi' &= x[(\lambda(vx)\rho)\psi] \quad \text{if} \quad x\rho = (x\rho)v(x\rho)\ .\end{aligned}$$

We will show first that the definition of φ' is independent of the choice of the element u. Hence assume that

$$\lambda x = (\lambda x)u(\lambda x) = (\lambda x)t(\lambda x)\ .$$

Then

$$\lambda(xu)\rho = (\lambda x)(u\rho) = (\lambda x)t(\lambda x)(u\rho) = [\lambda(xt)\rho][\lambda(xu)\rho]$$

so that

$$\begin{aligned}[\varphi(\lambda(xu)\rho)]x &= \{\varphi[((\lambda(xt)\rho)(\lambda(xu)\rho)]\}x = [\varphi(\lambda(xt)\rho)][\lambda(xu)\rho]x \\ &= [\varphi(\lambda(xt)\rho)](\lambda x)u(\lambda x) = [\varphi(\lambda(xt)\rho)](\lambda x)\end{aligned} \tag{3}$$

which evidently implies independence of φ' on the choice of u. Similarly the definition of ψ' is independent of the choice of v.

Now let $x, y \in S$, $\lambda x = (\lambda x)u(\lambda x)$, $\lambda(xy) = \lambda(xy)w\lambda(xy)$. Using (3), we obtain

$$\begin{aligned}(\varphi' x)y &= [\varphi(\lambda(xu)\rho)]xy = [\varphi(\lambda(xu)\rho)](\lambda x)y \\ &= [\varphi(\lambda(xu)\rho)]\lambda(xy)w\lambda(xy) = [\varphi(\lambda(xu)\rho)][\lambda(xyw)\rho]xy \\ &= \{\varphi[(\lambda(xu)\rho)(\lambda(xyw)\rho)]\}xy \\ &= \{\varphi[(\lambda(xu)\rho)(\lambda x)(yw)\rho)]\}xy \\ &= \{\varphi[(\lambda x)u(\lambda x)(yw)\rho]\}xy \\ &= \{\varphi[(\lambda x)(yw)\rho]\}xy = [\varphi(\lambda(xyw)\rho]xy = \varphi'(xy)\ .\end{aligned}$$

Hence φ' is a left translation of S, a symmetric proof shows that ψ' is a right translation of S.

Let $x, y \in S$, $x\rho = (x\rho)s(x\rho)$, $\lambda y = (\lambda y)z(\lambda y)$. Then

$$\begin{aligned} x(\varphi' y) &= x[\varphi(\lambda(yz))\rho)]y = x[(\lambda\varphi)(\lambda(yz)\rho)]y \\ &= x\{\lambda[(\varphi(yz)\rho)]\}y = (x\rho)[\varphi(\lambda(yz)\rho)]y \\ &= (x\rho)s(x\rho)[\varphi(\lambda(yz)\rho)]y = x[\lambda(sx)\rho][\varphi(\lambda(yz)\rho)]y \\ &= x[(\lambda(sx)\rho)\psi][\lambda(yz)\rho]y = x[(\lambda(sx)\rho)\psi](\lambda y)z(\lambda y) \\ &= x[(\lambda(sx)\rho)\psi](\lambda y) = x\{[(\lambda(sx)\rho)\psi]\rho\}y \\ &= x[(\lambda(sx)\rho)(\psi\rho)]y = x[(\lambda(sx)\rho)\psi]y = (x\psi')y \end{aligned}$$

which implies that $(\varphi', \psi') \in \Omega(S)$.

Further, for $x \in S$ and $\lambda x = (\lambda x)u(\lambda x)$, we have

$$\begin{aligned} (\lambda\varphi')x &= \lambda(\varphi' x) = \lambda\{[\varphi(\lambda(xu)\rho)]x\} = [(\lambda\varphi)(\lambda(xu)\rho)]x \\ &= [\varphi(\lambda(xu)\rho)]x = \varphi' x , \\ (\varphi'\lambda)x &= \varphi'(\lambda x) = [\varphi(\lambda(xu)\rho))]x = \varphi' x \end{aligned}$$

which proves that $\varphi' = \lambda\varphi' = \varphi'\lambda$; analogously $\psi' = \rho\psi' = \psi'\rho$. Consequently $(\varphi', \psi') \in \omega\Omega(S)\omega$.

Finally let $x \in \omega S\omega$, $x = xux$. Recall formula (3); then

$$\begin{aligned} \varphi' x &= [\varphi(\lambda(xu)\rho)]x = \{\varphi[(\lambda x)(u\rho)]\}x = \{\varphi[(x\rho)(u\rho)]\}x \\ &= \{\varphi[x(\lambda u\rho)]\}x = \varphi[x(\lambda u\rho)x] = \varphi[(x\rho)(u\rho)x] \\ &= \varphi[xu(\lambda x)] = \varphi(xux) = \varphi x \end{aligned}$$

so that $\varphi'|_{\omega S\omega} = \varphi$, analogously $\psi'|_{\omega S\omega} = \psi$. Therefore $(\varphi', \psi')\chi = (\varphi, \psi)$ and χ maps $\omega\Omega(S)\omega$ onto $\Omega(\omega S\omega)$.

COROLLARY 1. *Let S be a regular semigroup. If $\omega \in E_{\Omega(S)}$, then $\omega\Omega(S)\omega \cap \Pi(S)$ is a densely embedded ideal of $\omega\Omega(S)\omega$.*

Proof. Let $\pi: S \to \Omega(S)$ be the canonical homomorphism. It is easy to verify that

$$\Pi(\omega S\omega) \cong \omega S\omega \cong \pi(\omega S\omega) = \omega\Omega(S)\omega \cap \Pi(S) .$$

On the other hand, $\Pi(\omega S\omega)$ is a densely embedded ideal of $\Omega(\omega S\omega)$, which is in turn isomorphic to $\omega\Omega(S)\omega$ by the theorem.

COROLLARY 2. *If $\Omega(S)$ is a regular semigroup, and $\omega \in E_{\Omega(S)}$, then $\Omega(\omega S\omega)$ is a regular semigroup.*

Proof. This follows from the theorem since $\Omega(\omega S\omega) \cong \omega\Omega(S)\omega$ and any maximal submonoid of a regular semigroup is regular.

LEMMA 1. *If S is a regular semigroup and $\omega \in E_{\Omega(S)}$, then $\omega S\omega$ is a regular semigroup.*

Proof. Let $x \in \omega S \omega$ and x' be an inverse of x. Then

$$x = xx'x = (x\rho)x'(\lambda x) = x(\lambda x')\rho)x$$

which shows that $\omega S \omega$ is regular.

COROLLARY. *If S is an inverse semigroup* (resp. a semilattice of groups) *and $\omega \in E_{\Omega(S)}$, then both $\omega S \omega$ and $\Omega(\omega S \omega)$ are inverse semigroups* (resp. semilattices of groups).

Proof. In view of the lemma, the assertion follows easily from ([5], V.4.6) (resp. V.6.6).

3. **Rees matrix semigroups.** The theorem of this section relates subdirect irreducibility of a maximal subgroup of a Rees matrix semigroup S with that of a number of subsemigroups of $\Omega(S)$. We start with a general discussion and a string of lemmas.

Throughout this section we fix a (regular) Rees matrix semigroup $S = \mathscr{M}^0(I, G, M; P)$. We outline briefly a construction of $\Omega(S)$, see ([5], V.3). For a partial transformation α on I, whose domain is denoted by $d\alpha$, and a function φ mapping $d\alpha$ into G, the mapping λ defined by

$$\lambda(i, g, \mu) = (\alpha i, (\varphi i)g, \mu) \quad \text{if} \quad i \in d\alpha$$

and $\lambda(i, g, \mu) = 0$ otherwise, is a left translation of S; analogously

$$(i, g, \mu)\rho = (i, g(\mu\psi), \mu\beta) \quad \text{if} \quad \mu \in d\beta$$

and $(i, g, \mu)\rho = 0$ otherwise, is a right translation of S; they are linked if and only if

$$(4) \qquad \begin{cases} i \in d\alpha,\ p_{\mu(\alpha i)} \neq 0 \Longleftrightarrow \mu \in d\beta,\ p_{(\mu\beta)i} \neq 0 \\ \qquad\qquad \Longrightarrow p_{\mu(\alpha i)}(\varphi i) = (\mu\psi)p_{(\mu\beta)i}\,. \end{cases}$$

In such a case, we write $\omega = (\lambda, \rho) \sim (\alpha, \varphi; \beta, \psi)$. Conversely, every bitranslation of S is of this form for unique parameters $\alpha, \varphi, \beta, \psi$. It is easy to verify that $\omega^2 = \omega$ if and only if

$$\alpha|_{r\alpha} = \iota_{r\alpha},\ \varphi|_{r\alpha}\colon r\alpha \longrightarrow 1\,, \quad \beta|_{r\beta} = \iota_{r\beta},\ \psi|_{r\beta}\colon r\beta \longrightarrow 1$$

where $r\alpha$ is the range of α, $\iota_{r\alpha}$ is the identity mapping on $r\alpha$, 1 is the identity of G, etc. With this notation, we have

LEMMA 2. *If $\omega \in E_{\Omega(S)}$, then $\omega S \omega = \mathscr{M}^0(r\alpha, G, r\beta; P^\omega)$ where P^ω is the restriction of P to $r\beta \times r\alpha$.*

Proof. Indeed, for $0 \neq (i, g, \mu) \in S$, we have

$$\begin{aligned}(i, g, \mu) \in \omega S\omega &\Longleftrightarrow (i, g, \mu) = \lambda(i, g, \mu) = (i, g, \mu)\rho \\ &\Longleftrightarrow (i, g, \mu) = (\alpha i, (\varphi i)g, \mu) = (i, g(\mu\psi), \mu\beta) \\ &\Longleftrightarrow i = \alpha i, \varphi i = 1, \mu\psi = \mu, \mu\beta = \mu \\ &\Longleftrightarrow i \in \boldsymbol{r}\alpha, \mu \in \boldsymbol{r}\beta .\end{aligned}$$

By Lemma 1, $\omega S\omega$ is regular, hence the sandwich matrix P^{ω} has a nonzero element in each row and each column.

If the sandwich matrix P has no two distinct rows (or columns) which have the corresponding entries simultaneously nonzero, then P (and also S) is said to *have no contractions*, see ([3], § 6). The importance of this notion stems from the fact that these are precisely completely 0-simple semigroups all of whose proper congruences are contained in $\mathscr{H}$.

LEMMA 3. *Let the notation be as in Lemma 2. If P has no contractions, then neither does P^{ω}.*

Proof. Let $i, j \in \boldsymbol{r}\alpha$ and assume that

$$p_{\mu i} \neq 0 \Longleftrightarrow p_{\mu j} \neq 0 \quad (\mu \in \boldsymbol{d}\beta) . \tag{5}$$

Let $\mu \in M$ be such that $p_{\mu i} \neq 0$. Now $i \in \boldsymbol{r}\alpha$ implies that $i \in \boldsymbol{d}\alpha$ and $\alpha i = i$ since $\alpha^2 = \alpha$. Hence $i \in \boldsymbol{d}\alpha$ and $p_{\mu(\alpha i)} \neq 0$ which by (4) implies that $\mu \in \boldsymbol{d}\beta$ and $p_{(\mu\beta)i} \neq 0$. Here $\mu\beta \in \boldsymbol{r}\beta$ and $p_{(\mu\beta)i} \neq 0$ so that by (5), we have $p_{(\mu\beta)j} \neq 0$. But then $\mu \in \boldsymbol{d}\beta$ and $p_{(\mu\beta)j} \neq 0$ and hence $j \in \boldsymbol{d}\alpha$ and $p_{\mu(\alpha j)} \neq 0$ by (4). Since $\alpha j = j$, it follows that $p_{\mu j} \neq 0$. By symmetry, we conclude that

$$p_{\mu i} \neq 0 \Longleftrightarrow p_{\mu j} \neq 0 \quad (\mu \in M) ,$$

which by hypothesis that P has no contractions implies that $i = j$. One proves symmetrically that for $\mu, \nu \in \boldsymbol{r}\beta$,

$$p_{\mu i} \neq 0 \Longleftrightarrow p_{\nu i} \neq 0 \quad (i \in \boldsymbol{r}\alpha)$$

implies $\mu = \nu$. Therefore P^{ω} has no contractions.

The next result is of general interest for extensions of regular semigroups.

LEMMA 4. *Let V be an extension of a regular semigroup S. Then every congruence on S contained in $\mathscr{H}$ can be extended to a congruence on V.*

Proof. Let σ be a congruence on S contained in $\mathscr{H}$ and τ be the equivalence relation on V whose classes are the σ-classes and singletons $\{v\}$ with $v \in V \backslash S$. Then τ is a congruence if and only if for any $v \in V$, $a, b \in S$, $a\sigma b$ implies $va\sigma vb$ and $av\sigma bv$. Let $a, b \in S$ be such that $a\sigma b$. The hypothesis implies that $a\mathscr{H}b$, and thus $a = bx$ for some $x \in S$. Let b' be an inverse of b. Then

$$a = bx = (bb'b)x = bb'(bx) = bb'a ,$$

and thus for any $v \in V$, we have

$$va = v(bb'a) = (vbb')a\sigma(vbb')b = vb$$

since $vbb' \in S$. A symmetric argument can be used to show that $av\sigma bv$. Consequently τ is a congruence and is obviously an extension of σ.

LEMMA 5. *Let V be a dense extension of a semigroup S. If S is subdirectly irreducible, then so is V. The converse holds if every congruence on S can be extended to a congruence on V.*

Proof. This is a part of ([5], III.5.19 Exerc. 5).

We can now prove the desired result.

THEOREM 2. *Let $S = \mathscr{M}^0(I, G, M; P)$ and assume that P has no contractions. Let $\omega \in E_{\Omega(S)}$ and V be a subsemigroup of $\Omega(S)$ such that*

$$\omega\Omega(S)\omega \cap \Pi(S) \subseteqq V \subseteqq \omega\Omega(S)\omega .$$

Then G and V are simultaneously subdirectly reducible or irreducible.

Proof. We have mentioned above that the hypothesis that P has no contractions is equivalent to S having all proper congruences contained in $\mathscr{H}$ ([3], Proposition 6.2). Any one of the numerous descriptions of congruences on a Rees matrix semigroup can be used to easily show that the lattice of all congruences on S contained in $\mathscr{H}$ is isomorphic to the lattice of all congruences (and thus normal subgroups) on G. Under our hypothesis this means that G is subdirectly irreducible if and only if S is.

By Lemma 3, the matrix P^ω has no contractions. The above argument for S is now valid for $\omega S\omega$ in view of Lemma 2. Hence G and $\omega S\omega$ are simultaneously subdirectly irreducible or not. By Lemma 1, $\omega S\omega$ is regular. It follows that

$$\omega S\omega \cong \omega\Omega(S)\omega \cap \Pi(S) \tag{6}$$

as in the proof of Corollary 1 to Theorem 1. According to the last reference, we also have that $\omega\Omega(S)\omega \cap \Pi(S)$ is a densely embedded ideal of $\omega\Omega(S)\omega$. Hence by ([5], III.5.6), V given in the statement of the theorem is a dense extension of $\omega\Omega(S)\omega \cap \Pi(S)$. Since the last semigroup has no contractions, its proper congruences are contained in $\mathscr{H}$, so by Lemma 4, are extendible to V. But then Lemma 5 asserts that $\omega\Omega(S)\omega \cap \Pi(S)$ is subdirectly irreducible if and only if V is.

Now a combination of the statements concerning G and $\omega S\omega$, (6), and $\omega\Omega(S)\omega \cap \Pi(S)$ and V, establishes the theorem.

Note that for $\omega = (\iota_S, \iota_S)$, the identity bitranslation, we may take $V = \Pi(S)$ (and $\Pi(S) \cong S$), or $V = \Omega(S)$. Also for any nonzero idempotent e of S, the bitranslation $\omega = (\lambda_e, \rho_e)$ gives for $\omega S\omega$ the maximal subgroup G_e of S with identity e (and $G_e \cong G$). Also observe that we have used Theorem 1 via its Corollary 1.

4. Retracts. A subsemigroup T of a semigroup S is a *retract* (of S) if there exists a homomorphism φ of S onto T which leaves all elements of T fixed; φ is then a *retraction*. We discuss here regular semigroups in which all its maximal submonoids are retracts. A related condition will be expressed by means of bitranslations; for this reason we introduce

DEFINITION. Let S be a semigroup and $(\lambda, \rho) \in E_{\Omega(S)}$ such that $(\lambda x)\rho = \lambda(x\rho)$ for all $x \in S$ (so we can write $\lambda x\rho$ without ambiguity). The mapping

$$[\lambda, \rho]: x \longrightarrow \lambda x\rho \quad (x \in S)$$

is said to be *induced* by (λ, ρ).

LEMMA 6. *Consider the following conditions on a semigroup S.*

(a) *For any $a, b \in S$, $e \in E_S$, $eabe = eaebe$.*

(b) *Every maximal submonoid of S is a retract.*

(c) *Every idempotent inner bitranslation on S induces a retraction.*

Then (a) *and* (b) *are equivalent*; (c) *implies* (a); *and* (a) *implies* (c) *if S is weakly reductive.*

Proof. Straightforward.

Recall that an idempotent semigroup satisfying the condition

(a) in Lemma 6 is called a *regular band.* We are now ready for the theorem of this section.

THEOREM 3. *Let S be a regular semigroup. If S satisfies condition* (a) *in Lemma* 6, *then it also satisfies the following conditions.*

(d) *S is completely regular.*

(e) *Every idempotent bitranslation induces a retraction.*

(f) *Idempotents of S form a regular band.*

Proof. (d). Let a' be an inverse of an element a of S. Then

$$a = (aa')aa'(aa')a = (aa')a(aa')a'(aa')a \in a^2Sa$$

which by ([5], IV.1.6) implies that S is completely regular.

(e) Let $(\lambda, \rho) \in E_{\Omega(S)}$, $x, y \in S$. Using part (d), for any element $z \in S$, we let z' be the inverse of z in the maximal subgroup of S containing z. We compute

$$\begin{aligned} \lambda(xy)\rho &= [\lambda(xy)\rho][\lambda(xy)\rho]'[\lambda(xy)\rho] \\ &= \{[(\lambda x)(\lambda x)'](\lambda x)(y\rho)[\lambda(xy)\rho]'[(\lambda x)(\lambda x)']\}(\lambda x)(y\rho) \\ &= \{[(\lambda x)(\lambda x)'](\lambda x)[(\lambda x)(\lambda x)'](y\rho)[\lambda(xy)\rho]'[(\lambda x)(\lambda x)']\}(\lambda x)(y\rho) \\ &= \{[(\lambda x)(\lambda x)'](\lambda x\rho)[(\lambda x)(\lambda x)'](y\rho)[\lambda(xy)\rho]'[(\lambda x)(\lambda x)']\}(\lambda x)(y\rho) \qquad (7) \\ &= \{[(\lambda x)(\lambda x)'](\lambda x\rho)(y\rho)[\lambda(xy)\rho]'[(\lambda x)(\lambda x)']\}(\lambda x)(y\rho) \\ &= (\lambda x)(\lambda y\rho)[\lambda(xy)\rho]'[\lambda(xy)\rho] \\ &= (\lambda x\rho)(\lambda y\rho)[\lambda(xy)\rho]'[\lambda(xy)\rho] ; \end{aligned}$$

analogously

$$\lambda(xy)\rho = [\lambda(xy)\rho][\lambda(xy)\rho]'(\lambda x\rho)(\lambda y\rho) . \qquad (8)$$

On the other hand,

$$\begin{aligned} (\lambda x\rho)(\lambda y\rho) &= (\lambda x\rho)(\lambda y\rho)[(\lambda x\rho)(\lambda y\rho)]'(\lambda x\rho)(\lambda y\rho) \\ &= [(\lambda x\rho)(\lambda x\rho)'](\lambda x\rho)(y\rho)[(\lambda x\rho)(\lambda y\rho)]'[(\lambda x\rho)(\lambda x\rho)'](\lambda x\rho)(\lambda y\rho) \\ &= [(\lambda x\rho)(\lambda x\rho)'](\lambda x\rho)[(\lambda x\rho)(\lambda x\rho)'](y\rho) \\ &\quad \times [(\lambda x\rho)(\lambda y\rho)]'[(\lambda x\rho)(\lambda x\rho)'](\lambda x\rho)(\lambda y\rho) \qquad (9) \\ &= [(\lambda x\rho)(\lambda x\rho)'](\lambda x)[(\lambda x\rho)(\lambda x\rho)'](y\rho) \\ &\quad \times [(\lambda x\rho)(\lambda y\rho)]'[(\lambda x\rho)(\lambda x\rho)'](\lambda x\rho)(\lambda x\rho) \\ &= (\lambda x\rho)(\lambda x\rho)'[\lambda(xy)\rho][(\lambda x\rho)(\lambda y\rho)]'(\lambda x\rho)(\lambda y\rho) . \end{aligned}$$

The conjunction of (7) and (9) shows that $\lambda(xy)\rho$ and $(\lambda x\rho)(\lambda y\rho)$ are $\mathscr{J}$-related. Since S is completely regular, they are contained in a completely simple subsemigroup of S. Hence (7) and (8) imply that they are also contained in the same maximal subgroup G of S. But then $[\lambda(xy)\rho]'[\lambda(xy)\rho]$ must the identity of G, which together with (7) shows that $\lambda(xy)\rho = (\lambda x\rho)(\lambda y\rho)$. This is evidently equivalent to the statement that the bitranslation (λ, ρ) induces a retraction.

(f) It suffices to show that idempotents of S form a subsemigroup. Using a Rees matrix representation of a completely simple semigroup T, it is an easy exercise to show that condition (a) in Lemma 6 implies that E_T is a subsemigroup of T. Since S is a semilattice of completely simple semigroups, ([5], IV.3.7) implies that E_S is a subsemigroup of S.

Comparing Lemma 6 with Theorem 3, we see that if in a regular semigroup every idempotent inner bitranslation induces a retraction, then so does every idempotent bitranslation. The semigroup S of all transformations on a set of two elements is regular and trivially satisfies condition (a); in this semigroup $\mathscr{H}$ is not a congruence. However, if S is a regular semigroup satisfying (a) in which $\mathscr{H}$ is a congruence, then it follows easily from ([4], Theorem 3.2) that S is a subdirect (even spined) product of a semilattice of groups and a regular band. Conversely, it is easy to see that a regular semigroup S which is a subdirect product of a semilattice of groups and a regular band must satisfy (a) and $\mathscr{H}$ is a congruence on S. It seems unlikely that conditions (d) and (f) in Theorem 3 imply condition (a).

One might conjecture that if a regular semigroup S satisfies condition (a) and $\Omega(S)$ is regular, then $\Omega(S)$ also satisfies (a). This, however, is far from being the case. If T is the semigroup of all transformations on a set with at least three elements, then the constants in T form an ideal S of T such that: (α) S is a left (if the transformations are written on the left) zero semigroup, thus regular and satisfying (a), (β) $\Omega(S) \cong T$ so that $\Omega(S)$ is a regular semigroup. If $\Omega(S)$ satisfied (a), then by Theorem 3, it would have to be completely regular. But T is not completely regular, so $\Omega(S)$ does not satisfy (a).

5. Examples. The following examples illustrate some of the applications of Theorems 1 and 2. The proofs of many assertions that follow are either omitted or can be found in [5].

(a) *The semigroup $\mathscr{T}(X)$ of transformations on a set X* (written on the left). For the constants $\mathscr{T}_0(X)$, we have

$$\mathscr{T}_0(X) \cong \mathscr{M}(X, 1, \{X\}; P)$$

with $P = (p_{Xa})$, $p_{Xa} = 1$ (right zero semigroup on X), 1 is a one element group,

$$\mathscr{T}(X) \cong \Omega(\mathscr{M}(X, 1, \{X\}, P)) .$$

For any $\alpha \in E_{\mathscr{T}(X)}$, we have

$$\alpha\mathscr{T}(X)\alpha \cong \Omega(\mathscr{M}(r\alpha, 1, \{r\alpha\}; P^{\alpha})) \cong \mathscr{T}(\alpha X) ,$$

where P^{α} is essentially the restriction of P.

(b) *The semigroup* $\mathscr{F}(X)$ *of partial transformations on a set* X (written on the left). For the (partial) constants $\mathscr{F}_0(X)$, we have

$$\mathscr{F}_0(X) \cong \mathscr{M}^0(X, 1, \mathfrak{P}(X); P_X)$$

where $\mathfrak{P}(X)$ is the set of all nonempty subsets of X, $P_X = (p_{Aa})$, $p_{Aa} = 1$ if $a \in A$, $p_{Aa} = 0$ if $a \notin A$;

$$\mathscr{F}(X) \cong \Omega(\mathscr{M}^0(X, 1, \mathfrak{P}(X); P_X)) . \tag{10}$$

For any $\alpha \in E_{\mathscr{F}(X)}$, we have

$$\alpha\mathscr{F}(X)\alpha \cong \Omega(\mathscr{M}^0(r\alpha, 1, r\beta; P^{\alpha})) \tag{11}$$

where β is a partial transformation on $\mathfrak{P}(X)$ with

$$\begin{aligned} d\beta &= \{B \subseteqq X \mid B \cap r\alpha \neq \varnothing\} , \\ B\beta &= \{x \in d\alpha \mid \alpha x \in B\} \quad \text{if} \quad B \in d\beta , \\ r\beta &= \{B \mid B \cap r\alpha \neq \varnothing\} , \end{aligned}$$

and P^{α} is essentially the restriction of P_X. It can be proved that

$$\mathscr{M}^0(r\alpha, 1, r\beta; P^{\alpha}) \cong \mathscr{M}^0(r\alpha, 1, \mathfrak{P}(r\alpha); P_{r\alpha}) \tag{12}$$

and thus (10)-(12) yield

$$\alpha\mathscr{F}(X)\alpha \cong \mathscr{F}(r\alpha) .$$

It can be shown that none of the Rees matrix semigroups here has contractions. Hence all these semigroups are subdirectly irreducible.

(c) *The semigroup* $\mathscr{S}(V)$ *of linear transformations on a* (left) *vector space* V (written on the right). We will use the notation and results of ([6], I.2). The semigroup $\mathscr{S}_0(V)$ of linear transformations of rank $\leqq 1$ has the property

$$\mathscr{S}_0(V) \cong \mathscr{M}^0(I_{V^*}, \mathscr{M}\Delta^-, I_V; P)$$

and

$$\mathscr{S}(V) \cong \Omega(\mathscr{M}^0(I_{V^*}, \mathscr{M}\Delta^-, I_V; P)) .$$

For any $0 \neq \alpha \in E_{\mathscr{S}(V)}$, we have

$$\alpha\mathscr{S}(V)\alpha \cong \Omega(\mathscr{M}^0(I_{\alpha^*V^*}, \mathscr{M}\Delta^-, I_{V\alpha}; P^{\alpha})) \cong \mathscr{S}(V\alpha) .$$

It can be shown that the matrix P has no contractions. Consequently $\mathscr{M}\Delta^-$ (the multiplicative group of nonzero elements of the division ring Δ of the vector space V), $\mathscr{S}_0(V)$ and $\mathscr{S}(V)$ are simultaneously subdirectly reducible or irreducible.

(d) *Brandt semigroups* $S = \mathscr{M}^0(X, G, X; \Delta)$. For $0 \neq \omega \in E_{\Omega(S)}$, we have

$$\omega\Omega(S)\omega \cong \Omega(\mathscr{M}^0(r\alpha, G, r\alpha; \Delta)) .$$

Let $\mathscr{I}(X)$ be the semigroup of partial 1-1 transformations on X, and $\mathscr{I}_0(X)$ be the partial 1-1 constants on X. Then

$$\mathscr{I}_0(X) \cong \mathscr{M}^0(X, 1, X; \Delta)$$
$$\mathscr{I}(X) \cong \Omega(\mathscr{M}^0(X, 1, X; \Delta)) ,$$

and if $0 \neq \alpha \in E_{\mathscr{I}(X)}$, then

$$\alpha\mathscr{I}(X)\alpha \cong \Omega(\mathscr{M}^0(r\alpha, 1, r\alpha; \Delta)) \cong \mathscr{I}(r\alpha) .$$

None of these Rees matrix semigroups has contractions; hence G, $\mathscr{M}^0(X, G, X; \Delta)$, $\Omega(\mathscr{M}^0(X, G, X; \Delta))$ are simultaneously subdirectly reducible or irreducible. In particular both $\mathscr{I}_0(X)$ and $\mathscr{I}(X)$ are subdirectly irreducible.

(e) *The semigroup* $\mathscr{B}(X)$ *of binary relations on a set* X. For the semigroup $\mathscr{R}(X)$ of all rectangular binary relations on X, we have

$$\mathscr{R}(X) \cong \mathscr{M}^0(\mathfrak{P}(X), 1, \mathfrak{P}(X); P)$$

with $p_{AB} = 1$ if $A \cap B \neq \varnothing$ and $p_{AB} = 0$ otherwise. Further,

$$\mathscr{B}(X) \cong \Omega(\mathscr{M}^0(\mathfrak{P}(X), 1, \mathfrak{P}(X); P)) .$$

Let $0 \neq \sigma \in E_{\mathscr{B}(X)}$. Then

$$\sigma\mathscr{B}(X)\sigma \cong \Omega(\mathscr{M}^0(r\alpha, 1, r\beta; P^\sigma))$$

where α and β are partial transformations on X for which

$$d\alpha = \{A \subseteq X \mid (X \times A) \cap \sigma \neq \varnothing\} ,$$
$$\alpha A = \{x \in X \mid x\sigma y \text{ for some } y \in A\} \quad \text{if} \quad A \in d\alpha ,$$

and $d\beta$ and $B\beta$ are defined symmetrically, P^α is essentially the restriction of P; see [1]. We may let $Y = (r\beta \cup \{\varnothing\})\backslash(X\sigma)$ and

$$T = \mathscr{M}^0(Y, 1, r\beta; Q) \tag{13}$$

with $Q = (q_{AB})$, $q_{AB} = 1$ if $A \not\subseteq B$ and $q_{AB} = 0$ otherwise. Using some results of Zareckiĭ [7], one can show that

$$\mathscr{M}^0(r\alpha, 1, r\beta; P^\sigma) \cong \mathscr{M}^0(Y, 1, r\beta; Q)$$

so that

$$\sigma\mathscr{B}(X)\sigma \cong \Omega(\mathscr{M}^0(Y, 1, r\beta; Q)) .$$

None of these Rees matrix semigroups has contractions. Hence all these semigroups are subdirectly irreducible. In particular, this implies ([7], Proposition 4.4). Also Corollary 1 to Theorem 1 for $S = \mathscr{B}(X)$ implies ([7], Theorem 3.2). The semigroup T in (13) is particularly interesting since it can be constructed directly by means of a completely distributive lattice, which then yields an abstract characterization of maximal submonoids of $\mathscr{B}(X)$, see [7].

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