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ON LOOP SPACES WITHOUT p TORSION II

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Let (X, μ) be a 1-connected H-space such that $H^*(\Omega X; Q_p)$ is torsion free. We study the torsion in $H^*(X; Q_p)$ as well as its algebra structure. In particular we characterize lack of torsion in $H^*(\Omega X; Q_p)$ in terms of the module of indecomposables $Q(H^*(X; Q_p))$. We also study the Steenrod module structure of $Q(H^*(X; Z_p))$.

1. Introduction. In this paper we will study 1-connected H-spaces (X, μ) which have the homotopy type of a CW complex of finite type. Let p be a prime and Q_p be the integers localized at the prime p. Let ΩX be the loop space of X. We will assume that $H^*(\Omega X; Q_p)$ is torsion free and study the consequences for $H^*(X; Q_p)$. Our main results generalize those established in [5] where we worked with the stronger hypothesis that X has the homotopy type of a finite CW complex. We will assume familiarity with [5].

Let T be the torsion subgroup of $H^*(X; Q_p)$. It is an ideal of $H^*(X; Q_p)$. Let $F = H^*(X; Q_p)/T$. We first prove

THEOREM 1.1. *Let (X, μ) be a 1-connecsed H-space such that $H^*(\Omega X; Q_p)$ is torsion free. Then $H^*(X; Q_p)$ has no higher p torsion and F is a free commutative algebra.*

The arguments used in establishing 1.1 enable us to characterize lack of p torsion in X in terms of the cohomology of X. Let Z_p be the integers reduced mod p. Let $\rho: Q_p \to Z_p$ be the reduction mod p map. We will also use ρ to denote the induced cohomology map $\rho: H^*(X; Q_p) \to H^*(X; Z_p)$. The action of the Steenrod algebra $A^*(p)$ on $H^*(X; Z_p)$ induces a Steenrod module structure on $Q(H^*(X; Z_p))$. Let $K = \sum_{m \geq 1} \beta_p \mathscr{P}^m Q(H^{2m+1}(X; Z_p))$. Note that for $p = 2$, K is trivial. Let $\bar{Q} = Q(H^{\text{even}}(X; Z_p))/K$. The map ρ induces a map $\alpha: Q(H^{\text{even}}(X; Q_p)) \to \bar{Q}$. The quotient map $H^*(X; Q_p) \to F$ induces a map $\beta: Q(H^*(X; Q_p)) \to Q(F)$.

THEOREM 1.2. *Let (X, μ) be a 1-connected H-space. Then $H^*(QX; Q_p)$ is torsion free if, and only if, $Q(H^*(X; Q_p))$ contains a torsion free submodule M such that:*

(a) *α induces an isomorphism $M \otimes Z_p \cong \bar{Q}$*

(b) *β induces an isomorphism $M \cong Q^{\text{even}}(F)$.*

As in [5] one of the principal tools used to prove results such as 1.1 and 1.2 is the Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. In particular we will show

THEOREM 1.3. *Let (X, μ) be a 1-connected H-space such that $H^*(\Omega X; Q_p)$ is torsion free. Then, in the Eilenberg-Moore spectral sequence $\{E_r\}$ converging to $H^*(\Omega X; Z_p)$ we have $E_p = E_\infty$.*

If we assume that (X, μ) is a finite H-space, or indeed, just that the rational cohomology $H^*(X; Q)$ is an exterior algebra, then we can recover the main results of [5] from Theorems 1.1, 1.2, and 1.3. In particular, we obtain from 1.2 the fact that $H^*(\Omega X; Q_p)$ is torsion free if, and only if, $Q(H^{\text{even}}(X; Z_p)) = K$. On the other hand, if $H^*(X; Q)$ is a polynomial algebra then 1.2 implies that $H^*(\Omega X; Q_p)$ is torsion free if, and only if, $H^*(X; Q_p)$ is a torsion free polynomial algebra. This paper arose from an effort to combine these two results. For, in general, $H^*(X; Q)$ is a tensor product of an exterior algebra and a polynomial algebra.

We also deduce one further result about finite H-spaces.

THEOREM 1.4. *Let (X, μ) be a 1-connected H-space such that $H^*(\Omega X; Q_p)$ is torsion free and $H^*(X; Z_p)$ is finitely generated as an algebra. Then $H^*(X; Z_p)$ is a finite dimensional vector space if, and only if, X is 2-connected.*

If ΩX has no p torsion then K is the kernal of the loop map (see 1.3 and 3.3) and hence is a sub-Steenrod module of $Q(H^*(X; Z_p))$. We will study this Steenrod module structure for p odd. Our interest in K lies in the fact that T can be determined from K via a Bockstein spectral sequence argument (see [2]). Hence our next theorem can be viewed as structure theorems for T as well. Given an integer n with p-adic expansion $n = \Sigma n_s p^s$ we say n is *binary* (with respect to p) if $n_s = 0$ or 1 for each s. Given a binary integer n define $q(n)$ and $r(n)$ as follows. Let N be the minimal integer s such that $n_s = 0$ in the p-adic expansion $\Sigma n_s p^s$ of n. Then let

$$q(n) = \begin{cases} 0 & \text{if } N = 0 \\ \sum_{i=0}^{N-1} p^i & \text{if } N > 0 \end{cases}$$

$$r(n) = \frac{n - q(n)}{p}.$$

THEOREM 1.5. *Let p be odd. Let (X, μ) be a 1-connected H-space such that $H^*(\Omega X; Q_p)$ is torsion free. Then*

(a) $K^{2n} = 0$ *unless n is binary*

(b) *For n binary $K^{2n} = \mathscr{P}^{r(n)} K^{2q(n)+2r(n)}$. In particular $K^{2n} = 0$ unless $n \equiv 1 \bmod p$.*

Variations of 1.4 and 1.5 are also known to James Lin. In §2 we will discuss Hopf algebras. In §3 we will discuss loop maps and their relation to the Eilenberg-Moore spectral sequences. In §4 we will reduce the proof of Theorems 1.1 and 1.2 to that of 1.3. In §5 we will prove Theorem 1.3. In §6 we will prove Theorem 1.4. In §7 we will prove Theorem 1.5.

All spaces are assumed to have the homotopy type of CW complexes of finite type. All spaces will have basepoints and all maps will preserve basepoints.

In closing it should be added that, between the writing of [5] and the writing of this paper, much has been learned about torsion in the loop space of finite H-spaces. In particular (see [8]) it is known that the loop space of a 1-connected finite H-space has no odd torsion.

I would also like to thank the referee for his comments. In particular they resulted in a rewriting of §4.

2. **Hopf algebras.** In this section we will discuss Hopf algebras over G where $G = Z_p$, Q or Q_p. A general reference for Hopf algebras is [10]. All modules will be graded, connected, and of finite type. Given a module M we define its dual M^* by the rule $(M^*)^m = \mathrm{Hom}(M^m; G)$. For $G = Z_p$, if M is a Steenrod module then M^* inherits a Steenrod module structure as well.

When A is a Hopf algebra we use $Q(A)$ and $P(A)$ to indicate indecomposables and primitives respectively. For $G = Z_p$ or Q, $P(A)$ and $Q(A^*)$ are dual modules. However, for $G = Q_p$, this is not necessarily so, even when A is torsion free, since $Q(A)$ may not be torsion free.

For Hopf algebras over Z_p and, in particular, for the concept of a Borel decomposition we refer to §1 of [5].

For Hopf algebras over Q we only remark that if A is commutative and associative then A is isomorphic, as an algebra, to a tensor produce $\bigotimes_{\iota \in I} A_\iota$ where each A_i is either an exterior algebra or a polynomial algebra generated by a single element a_i. The tensor product is called a Hopf decomposition of A and the elements $\{a_i\}$ are called generators of the decomposition.

For the rest of this section we will consider Hopf algebras over Q_p. Given such a Hopf algebra we can tensor with Z_p or Q and produce Hopf algebras over these fields. We can use these derived Hopf algebras to study the original Hopf algebra over Q_p. It is trivial that:

LEMMA 2.1. *If A is a Hopf algebra over Q_p then*

$$Q(A) \otimes Z_p \cong Q(A \otimes Z_p) \quad \text{and} \quad Q(A) \otimes Q \cong (A \otimes Q)\ .$$

LEMMA 2.2. *Let A be a Hopf algebra over Q_p which is commutative, associative, and torsion free. Then $Q(A^m)$ is torsion free unless $m \equiv 0$ mod $2p$.*

Proof. By 2.1 it suffices to show that the rank of $Q(A^m \otimes Z_p)$ as a Z_p module equals the rank of $Q(A^m \otimes Q)$ as a Q module if $m \not\equiv 0 \bmod 2p$. A subset B of $A \otimes Z_p$ (or of $A \otimes Q$) is a simple system of generators if the set $\{b_1^{r_1} b_1^{r_2} \cdots b_n^{r_n} \mid b_i \in B, 0 \leqq r_i \leqq 1$ if $|b_i|$ odd, $0 \leqq r_i \leqq p-1$ if $|b_i|$ even$\}$ is a Z_p basis of $A \otimes Z_p$ (or a Q basis of $A \otimes Q$). Pick a Borel decomposition of $A \otimes Z_p$ with generators $\{a_i\}_{i \in I}$ and a Hopf decomposition of $A \otimes Q$ with generators $\{b_j\}_{j \in J}$. The graded sets $S_1 = \{a_i^{p^s} \neq 0 \mid i \in I, s \geqq 0\}$ and $S_2 = \{b_j^{p^s} \neq 0 \mid j \in J, s \geqq 0\}$ are simple systems of generators for $A \otimes Z_p$ and $A \otimes Q$ respectively. But A is torsion free. Hence S_1 and S_2 must be isomorphic as graded sets. And, if $m \not\equiv 0 \bmod 2p$, then the elements of S_1 and S_2 of dimension m represents a basis for $Q(A^m \otimes Z_p)$ and $Q(A^m \otimes Q)$ respectively.

REMARK. Lemma 2.2 is still valid if we replace the assertion that A is a Hopf algebra by the hypotheses that $A \otimes Z_p$ and $A \otimes Q$ are Hopf algebras. This version will be used in the proof of Lemma 4.4.

The next result is a corollary of 2.2.

LEMMA 2.3. *Let A be a Hopf algebra over Q_p which is bicommutative, biassociative and torsion free. Then $P(A^m)$ and $Q(A^{*m})$ are dual Q_p modules unless $m \equiv 0 \bmod 2p$.*

It is straightforward that:

LEMMA 2.4. *Let A be a Hopf algebra over Q_p which is torsion free. Then $P(A) \otimes Q = P(A \otimes Q)$.*

It then follows that:

LEMMA 2.5. *Let A be a Hopf algebra over Q_p which is bicommutative, biassociative, and torsion free. Then the natural map $\gamma: P(A^m) \rightarrow Q(A^m)$ is an isomorphism unless $m \equiv 0 \bmod 2p$.*

Proof. Suppose $m \not\equiv 0 \bmod 2p$. By 2.2 both $P(A^m)$ and $Q(A^m)$

are torsion free. Hence, it suffices to show that $\gamma \otimes Q$ is an isomorphism and $\gamma \otimes Z_p$ is a monomorphism. The former follows from 2.1, 2.4, and 4.18 of [10]. The latter follows from 2.1, the fact that $P(A) \otimes Z_p \subset P(A \otimes Z_p)$ and 4.21 of [10].

LEMMA 2.6. *Let A be a Hopf algebra over Q_p which is bicommutative, biassociative, and torsion free. Then*

$$P(A^m) \otimes Z_p \cong P(A^m \otimes Z_p)$$

unless $m \equiv 0 \bmod 2p$.

Proof. By the proof of 2.5 we have a commutative diagram

$$\begin{array}{ccc} & P(A^m \otimes Z_p) & \\ \nearrow & & \searrow \\ P(A^m) \otimes Z_p & \longrightarrow & Q(A^m) \otimes Z_p \end{array}$$

where the bottom map is an isomorphism and the other maps are monomorphisms.

3. Loop maps. In this section we will discuss loopmaps for a 1-connected H-space (X, μ). Let $\delta\colon \Sigma\Omega X \to X$ be the adjoint map to the identity map $1\colon \Omega X \to \Omega X$. For each of $G = Z_p$, Q, or Q_p, δ gives rise to the loop maps

$$\Omega^*\colon Q(H^m(X; G)) \longrightarrow P(H^{m-1}(X; G))$$
$$\Omega^*\colon Q(H_{m-1}(\Omega X; G) \longrightarrow P(H_m(X; G))$$

We will use the same symbols Ω^* and Ω_* for each case of G. It will be clear from the context what coefficients are involved. For each case of G, Ω^* and Ω_* are adjoint in the sense that $\langle \Omega_*(a), b\rangle = \langle a, \Omega^*(b)\rangle$ for any $a \in Q(H_*(\Omega X; G))$ and $b \in Q(H^*(X; G))$.

Let ρ be as in §1. Let ι denote the inclusion map $\iota\colon Q_p \to Q$ as well as the map induced in homology and cohomology. We have identities of the form $\rho\Omega^* = \Omega^*\rho$, $\Omega_*\rho = \rho\Omega_*$, $\iota\Omega^* = \Omega^*\iota$, and $\iota\Omega^* = \Omega_*\iota$. These identities enable us to study the case $G = Q_p$ via the cases $G = Z_p$ or Q.

We can obtain strong restrictions when $G = Z_p$ on Q by using the machinery of spectral sequences. In each of these cases there exists an Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; G)$ and another converging to $H_*(X; G)$. The above loop maps can then be redefined in terms of the appropriate Eilenberg-Moore spectral sequence. For Ω^* and the spectral sequence converging to $H^*(\Omega X; G)$ see [3], §2 of [5] and [12]. For Ω_* and the spectral sequence converging to $H_*(X; G)$, at least for the case $G = Z_p$, see [3]. Using this machinery we can deduce

LEMMA 3.1. *For* $G = Q$, Ω^* *and* Ω_* *are isomorphisms.*

LEMMA 3.2. (i) $\Omega^*: Q(H^{\mathrm{odd}}(X; Z_p)) \to P(H^{\mathrm{even}}(\Omega X; Z_p))$ *is injective*

$\Omega_*: Q(H_{\mathrm{odd}}(\Omega X; Z_p)) \to P(H_{\mathrm{even}}(X; Z_p))$ *is injective.*

(ii) $\Omega^*: Q(H^{\mathrm{even}}(X; Z_p)) \to P(H^{\mathrm{odd}}(\Omega X; Z_p))$ *is surjective*

$\Omega_*: Q(H_{\mathrm{even}}(\Omega X; Z_p)) \to P(H_{\mathrm{odd}}(X; Z_p))$ *is surjective.*

In both 3.1 and 3.2 the results for Ω^* and for Ω_* are equivalent. For 3.1 in the case Ω^* see [12]. For 3.2 in the case Ω_* see [3].

In the case $G = Z_p$, the kernel of Ω^* is related to the Z_p module K defined in §1. Let $\{E_r\}$ be the Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. Then

LEMMA 3.3. $K \subset \ker \Omega^*$ *with equality if, and only if,* $E_p = E_\infty$.

Although it is not explicitly stated as a lemma in [5], 3.3 is used in the proof of 1.1 of [5]. See that proof for an implicit proof of 3.3.

4. **Proof of Theorems 1.1 and 1.2.** In this section we will show that Theorems 1.1 and 1.2 hold if Theorem 1.3 holds. Then, in the next section, we will prove Theorem 1.3. Our reason for proving the theorems in this order is that, in the proof of 1.3 for the case $p = 2$, we want to assume that 1.1 holds up to a certain dimension in $H^*(X; Z_2)$. Since it will be trivial that 1.3 holds in the desired range of dimensions the proofs in this section show that it is indeed valid to suppose that 1.1 holds as well. For, although, for convenience, we only prove the absolute case, it will be apparent that the proofs of this section can be modified to show that 1.1 and 1.2 hold up to certain dimension in $H^*(X; Z_p)$ if 1.3 holds up to the same dimension.

Assume for the rest of this section that (X, μ) is a 1-connected H-space such that $E_p = E_\infty$ whenever $H^*(\Omega X; Q_p)$ is torsion free.

Before proving 1.1 and 1.2 we make a few remarks about the cohomology Bockstein spectral sequence. As shown in [1] the cohomology spectral sequence $\{B_r\}$ is a spectral sequence of Hopf algebras with $B_1 = H^*(X; Z_p)$ and $B_\infty = H(X; Q_p)/T \otimes Z_p$. The map ρ induces a map $\rho_r: H^*(X; Q_p) \to B_r$ of algebras for all r. The differential d_r acts trivially on all elements in Image ρ_r and Image $d_r = \rho_r(T_r)$ where T_r is the torsion subgroup of $H^*(X; Q_p)$ consisting of elements of order p^r. It is this spectral sequence plus the refinements of it as developed in [2] which will be the major tool used in proving 1.1 and 1.2.

(A) *Proof of Theorem* 1.1. We assume that $H^*(\Omega X; Q_p)$ is torsion free. Also, until further notice we assume that p is odd. Since $E_p = E_\infty$ it follows from 3.2(ii) and 3.3 that the loop map induces an isomorphism

$$(4.1) \qquad \Omega^*: \bar{Q} \cong P(H^{\text{odd}}(\Omega X; Z_p)) .$$

It is this consequence of 1.3 which we will be using in the proof of 1.1.

We being our proof of 1.1 by studying a spectral sequence $\{E_r\}$ which passes from B_1 to B_2. It is the spectral sequence induced from the augmentation filtration on B_1. As shown in [2] this is a spectral sequence of commutative, associative, primitively generated Hopf algebras. The augmentation filtration on B_1 induces a filtration on $B_2 = H(B_1)$ (not necessarily the augmentation filtration). With respect to these filtrations we have $E_1 = E^0(B_1)$ and $E_\infty = E^0(B_2)$. Since the filtration on B_1 is the augmentation filtration it follows that B_1 and $E^0(B_1)$ are isomorphic as algebras. In addition $Q(B_1)$ is a differential submodule of $E^0(B_1)$, it generates $E(B_1)$ as an algebra, and all of its elements are primitive. (See in particular 1 and 2 of [2] for further details on these facts.) Considering the action of d_1 on $Q(B_1) \subset E_1$ we have

LEMMA 4.2. $K = \text{Image}\, d_1$.

Proof. We can identify the action of d_1 on $Q(B_1)$ with the action of the Bockstein β_p on $Q(H^*(X; Z_p))$. By 3.2(i) and the fact that $H^*(\Omega X; Q_p)$ is torsion free it follows that $\beta_p: Q(H^{\text{even}}(X; Z_p)) \to Q(H^{\text{odd}}(X; Z_p))$ is trivial. Similary, this time using 4.1, $\beta_p: Q(H^{\text{odd}}(X; Z_p)) \to Q(H^{\text{even}}(X; Z_p)) \to \bar{Q}$ is trivial. Now, by definition, $K \subset \text{Image}\, \beta_p$. Then, by the above, $K = \text{Image}\, \beta_p$.

It follows from 4.2 that any element of $\bar{Q}$ defines a nonzero element in E_2 and thus in $Q(E_2)$ as well. Considering $\bar{Q}$ as laying in $Q(E_2)$ we wish to show

LEMMA 4.3. $\bar{Q} \cong Q^{\text{even}}(E_2)$.

Proof. Let $A \subset E_1$ be the primitively generated Hopf algebra generated by $Q^{\text{odd}}(B_1)$ and $d_1 Q^{\text{odd}}(B_1)$. Hence A is a differential sub

Hopf algebra of E_1. Because p is odd A is isomorphic as a differential Hopf algebra to a tensor product $\otimes A_i$ of differential Hopf algebras where $A_i = E(e_i)$ or $A_i = E(e_i) \otimes P(de_i)/\langle (de_i)^{p^s} \rangle (1 \leqq s \leqq \infty)$. Thus

(i) $H(A)$ is an exterior algebra on odd dimensional generators. Further, by 4.2, d_1 acts trivially on $Q^{\text{even}}(B_1) \subset E_1$. Since $Q(B_1)$ generates E_1 as an algebra it follows that the induced map $H(A) \to H(E_1)$ is injective. In other words

(ii) $H(A)$ is a sub Hopf algebra of $H(E_1)$.

We now study the quotient Hopf algebras $E_1//A$ and $H(E_1)//H(A)$. By 4.2 the elements of $\bar{Q}$ represent elements in $E_1//A$ and thus in $Q(E_1//A)$. Further, by the definition of A plus 4.2,

(iii) $\bar{Q} \cong Q(E_1//A)$.

The induced differential acts trivially on $E_1//A$. Thus $H(E_1//A) = E_1//A$. By (iii) the induced map $H(E_1) \to H(E_1//A) = E_1//A$ is surjective. This map factors through $H(E_1)//H(A)$ to give a surjective map $f: H(E_1)//H(A) \to E_1//A$. We now use f to show

(iv) $H(E_1)//H(A)$ and $E_1//A$ are isomorphic as Hopf algebras.

For f induces a surjective map $1 \otimes f: H(A) \otimes H(E_1)//H(A) \to H(A) \otimes E_1//A$. But $H(A) \otimes H(E_1)//H(A) \cong H(E_1)$ as a Z_p module by 4.4 of [10]. Also the Serre spectral sequence of the extension $Z_p \to A \to E_1 \to E_1//A \to Z_p$ converges from $H(A) \otimes E_1//A$ to $H(E_1)$. Thus it follows by a counting argument that $1 \otimes f$ must be an isomorphism. Thus f is an isomorphism.

Finally, by 3.11 of [10] plus (iv) we have an exact sequence of Z_p modules

(v) $Q(H(A)) \to Q(H(E_1)) \to Q(E_1//A) \to 0$.

The lemma now follows from (i), (ii), (iii), (iv) and (v).

We will prove Theorem 1.1 by showing by induction on dimension that

(a) $E_2 = E_\infty$

(b) $B_2 = B_\infty$

(c) B_∞ is a free algebra.

Since F is torsion free and $F \otimes Q = H^*(X; Q)$ is a free commutative algebra, condition (*c*) on $B_\infty = F \otimes Z_p$ is equivalent to asserting that

(c′) F is a free algebra.

Thus 1.1 follows from (a), (b), (c).

Consider the following condition:

(d) $\bar{Q}$ has a set of representatives in B_1 which survive to B_∞ and represent a basis of $Q^{\text{even}}(B_r)$ for $r \geqq 2$.

We will prove (a), (b), (c) by using the following

LEMMA 4.4. *If condition* (d) *holds up to dimension* $2n$ *then conditions* (a), (b), *and* (c) *hold up to dimension* $2n$.

Proof. We will only prove the absolute case. It will be obvious that the dimension restrictions can be inserted into the proof. So assume that (d) holds in all dimensions.

Condition (c). Since F is torsion free and $F \otimes Q$ is a free algebra condition (c′) follows from showing that $Q(F)$ is torsion free. For $Q^{\text{odd}}(F)$ this follows from 2.2 and the remark which follows it. For $Q^{\text{even}}(F)$ this follows from (d). For we have a commutative diagram

$$(*)\qquad \begin{array}{ccc} Q(H^{\text{even}}(X; Q_p)) & \xrightarrow{\Omega^*} & P(H^{\text{odd}}(\Omega X; Q_p)) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\rho} \\ \bar{Q} & \xrightarrow{\cong} & P(H^{\text{odd}}(\Omega X; Z_p)) \end{array}$$

where α and ρ are as in §1. By (d) α is surjective. By (*), 2.6, and the fact that $P(H(\Omega X; Q_p))$ is torsion free it follows that $Q(H^{\text{even}}(X; Q_p))$ contains a torsion free submodule M such that α induces an isomorphism $M \otimes Z_p \cong \bar{Q}$ and Ω^* induces an isomorphism $M \cong P(H^{\text{odd}}(\Omega X; Q_p))$. But the loop map factors through $Q(F)$

$$(**)\qquad \begin{array}{ccc} Q(H^{\text{even}}(X; Q_p)) & \xrightarrow{\Omega^*} & P(H^{\text{odd}}(\Omega X; Q_p)) \\ {\scriptstyle\beta}\searrow & & \nearrow \\ & Q^{\text{even}}(F) & \end{array}$$

Thus β restricted to M is injective. But by (d) B restricted to M is also surjective. Thus $Q^{\text{even}}(F) \cong M$ is torsion free.

Condition (b). We proceed by induction. It follows from (c) and (d) that B_r is a free algebra for $r \geqq 2$. Now suppose $B_r = B_2$ for $r \geqq 2$. If $B_r \neq B_{r+1}$ we can pick the minimal dimension for which there exists $x \in B_r$ such that $d_r(x) = y \neq 0$. Then by 4.21 of [10] y is either nondecomposable or a pth power. But y nondecomposable is not possible since (d) implies that d_r acts trivially on $Q(B_r)$. And $y = z^p$ is not possible since B_{r+1} is a free algebra and z would not generate a polynomial subalgebra of B_{r+1}. We conclude that $B_r = B_{r+1}$.

Condition (a). The argument from (b) will suffice provided we can show

(i) $\bar{Q}$ survives to E_∞ and $\bar{Q} \cong Q^{\text{even}}(E_r)$ for $r \geqq 2$

(ii) E_∞ is a free algebra.

Regarding (i) $\bar{Q}$ survives to E_∞ by condition (d). We show that

$\bar{Q} = Q^{\text{even}}(E_r)$ by induction on r. For $r = 2$ see 4.3. For general r we duplicate the arguments given in the proof of 4.3. Several minor modifications are necessary. We let A be the differential Hopf algebra generated by $P^{\text{odd}}(E_{r-1}) \subset E_r$. (Recall that, by 4.23 of [10], $P^{\text{odd}}(E_{r-1}) \cong Q^{\text{odd}}(E_{r-1})$.) Also, instead of 4.2, we use the fact that $Q^{\text{even}}(E_{r-1}) \cong \bar{Q}$ has representative which survive to E_r.

Regarding (ii) it suffices to show that $Q(B_2)$ and $Q(E_\infty)$ are isomorphic as graded Z_p modules; for by (b) and (c), $B_2 = B_\infty$ is a free algebra, while B_2 and $E_\infty = E^0(B_2)$ are isomorphic as Z_p modules. In even dimensions this follows from (i) and (d). In odd dimensions this can be seen by taking a simple system of generators S_1 and S_2 for B_1 and B_2 respectively. (See the proof of 2.2 for the meaning of simple system.) Since B_2 and E_∞ are isomorphic as Z_p modules it follows that S_1 and S_2 are isomorphic as graded sets. But the elements of S_1 and S_2 of odd dimension represent a basis of $Q^{\text{odd}}(B_2)$ and $Q^{\text{odd}}(E_\infty)$ respectively. (Again see 2.2).

We now prove that (d) is true in all even dimensions. We use induction. Suppose (d) is true in dimension $\leqq 2n - 2$. We will use the term "rank" in the sense of the dimension of a Z_p vector space. We have the following sequence of inequalities

$$\text{rank } Q^{2n}(E_2) \geqq \text{rank } Q^{2n}(E_\infty) \geqq \text{rank } Q^{2n}(B_2) \geqq \text{rank } Q^{2n}(B_\infty) \; . \tag{4.5}$$

The first inequality relation follows from the fact that X is 1-connected and that (a) holds in dimension $\leqq 2n - 2$. The second from the fact that $E_\infty = E^0(B_2)$. The third from the fact that (b) holds in dimension $\leqq 2n - 2$ and that X is 1-connected. Furthermore, we have strict equalities throughout (4.5) only if condition (d) holds in dimension $\leqq 2n$. For the equalities

$$\text{rank } Q^{2n}(E_2) = \text{rank } Q^{2n}(E_\infty) = \text{rank } Q^{2n}(B_2)$$

plus 4.3 ensures that $\bar{Q}^{2n}$ has a set of representatives in B_1 which survive to B_2 and represent a basis of $Q^{2n}(B_2)$. The equality

$$\text{rank } Q^{2n}(B_2) = \text{rank } Q^{2n}(B_\infty)$$

then implies the rest of (d).

Thus to establish condition (d) in dimension $2n$ it suffices to prove

LEMMA 4.6. rank $Q^{2n}(B_\infty) \geqq$ rank $Q^{2n}(E_2)$.

Proof. We have the following sequence of equalities or inequalities

$$\begin{aligned}\operatorname{rank} Q^{2n}(E_2) &= \operatorname{rank} \bar{Q}^{2n} \qquad \text{(by 4.3)}\\ &= \operatorname{rank} P(H^{2n-1}(\Omega X; Z_p)) \qquad \text{(by 4.1)}\\ &= \operatorname{rank} P(H^{2n-1}(\Omega X; Q)) \qquad \text{(by 2.4 and 2.6)}\\ &= \operatorname{rank} Q(H^{2n}(X; Q)) \qquad \text{(by 3.1)}\\ &= \operatorname{rank} Q^{2n}(F \otimes Q)\\ &\leqq \operatorname{rank} Q^{2n}(F \otimes Z_p) \qquad \text{(by 2.1)}\\ &= \operatorname{rank} Q^{2n}(B_\infty)\ .\end{aligned}$$

This completes the proof of 1.1 for p odd. For $p = 2$ we use the same basic argument. However it must be modified when dealing with B_1 and the spectral sequence $\{E_r\}$ converging from B_1 to B_2. The difficulty arises from the fact that $B_1 = H^*(X; Z_2)$ may contain odd dimensional elements x such that $x^2 \neq 0$. Thus the decomposition $\otimes A_i$ of the Hopf algebra A which appears in the proof of 4.3 may be more complicated than for p odd. On the other hand this type of difficulty disappears in B_r for $r > 1$. For, if $x \in H^{2m+1}(X; Z_2)$ then $x^2 = Sq^{2m+1} = Sq^1 Sq^{2m}(x)$. Since Sq^1 is the first Bockstein differential it follows that the square of any odd dimensional element in $B_2 = H(B_1)$ is trivial.

In dealing with B_1 we proceed in the same basic way as we did for p odd. We first use the argment employed in 4.3 to show

LEMMA 4.7. $\bar{Q} \cong Q^{\text{even}}(E_3)$.

By which we mean that $\bar{Q} \subset E_1$ survives to E_3 and projects isomorphically onto $Q^{\text{even}}(E_3)$. The argument given will also show that

LEMMA 4.8. $x^2 = 0$ *if* $x \in E_3$ *is odd dimensional.*

Thus the difficulty with odd dimensional squares disappears at the E_3 level. Then, using 4.7 in place of 4.3, we can proceed, as we did for p odd, to establish (a), (b), (c) and (d), only we replace condition (a) by the condition

(a′) $$E_3 = E_\infty\ .$$

Hence once we have proved 4.7 and 4.8 we will be done.

We first observe that $E_1 = E_2$. For it suffices to show that d_1 acts trivially on $Q(B_1) \subset E_1$. But this follows from 4.2 which is valid for $p = 2$ as well. Now, to prove 4.7 and 4.8, we proceed as we did for 4.3. We will prove properties (i), (ii), (iii), (iv) and (v) as established there, only for E_3 rather than for E_2. Two main differences arise. The first, as already mentioned, concerns the fact that the decomposition $\otimes A_i$ of A may be more complicated than for p odd. The second difference concerns $Q(B_1)$. Since $E_1 = E_2$ it follows that $Q(B_1)$ is a primitive submodule of E_2 which generates E_∞ as an algera. However $Q(B_1)$ is not necessarily invariant under the action of d_2 as it was for d_1. For the action of d_2 on $Q(B_1)$ reflects the action of Sq^1 on the indecomposables of B_1 modulo the triple decomposables. However, this fact, rather than complicating the situation further, will actually compensate for the difficulties with A.

We begin by proving an analogue of 4.2.

LEMMA 4.9. *$\bar{Q} = Q^{\text{even}}(B_1)$ surves to E_3.*

Proof. We know at least that if $x \in Q(B_1)$ then $d_2(x)$ is decomposable. Hence we need only show that d_2 acts trivially on $Q^{\text{even}}(B_1)$. But if $x \in Q^{\text{even}}(B_1)$ then $d_2(x)$ is an odd dimensional primitive and hence, by 4.21 of [10], is indecomposable if nonzero.

Of the properties (i), (ii), (iii), (iv) and (v) only (i) presents difficulties. Provided (i) is true, the other properties follow as they did in 4.3, except that the use of 4.2 is replaced by 4.9. So let A be the primitively generated differential Hopf algebra generated by $Q^{\text{odd}}(B_1) \subset E_2$. To show $H(A)$ is an exterior algebra let $\bigotimes_{i \in I} A_i$ be a Borel decomposition of A. Given $i \in I$, if $A_i = P(a_i)/\langle a_i^{2s+1} \rangle$ where $s > 0$, let $A_i' = E(a_i) \otimes P(b_i)/\langle b_i^{2s} \rangle$ where $|b_i| = |a_i^2|$. Otherwise let $A_i' = A_i$. Let $A' = \otimes A_i'$. There is the obvious Z_p module map γ: $A \to A'$. We can give A' a differential Hopf algebra stucture by requiring that γ is an isomorphism of differential coalgebras. Then to establish (i) it suffices to show that $H(A')$ is an exterior algebra. For the induced map $\gamma_*: H(A) \to H(A')$ is an isomorphism of coalgebras and thus $P(H(A)) \cong P(H(A'))$. But, by 4.23 of [10], the first even dimensional generator of $H(A)$ must be primitive. Hence $P^{\text{even}}(H(A)) = 0$ implies $Q^{\text{even}}(H(A)) = 0$. To show $H(A')$ is an exterior algebra it suffices to show A' is isomorphic, as a differential Hopf algebra, to a tensor product as in 4.3. This amount to showing that, for each factor A_i of A where $A_i = P(a_i)/\langle a_i^{2s+1} \rangle$ and $s > 0$, we have $a_i^2 = d_2(b)$ where $b \in A$. Now we can certainly find $b \in E_3$ such that $d_2(b) = a_i^2$. For, since the square of any odd dimensional element in $H^*(H; Z_2)$ lies in the image of Sq^1, a_i^2 cannot survive to E_∞. And, by our comments concerning the action of d_2 on E_2, a_i^2 cannot then survive to E_3. We now show that b can be chosen from A. Let E_2^{odd} and E_2^{even} be the primitively generated sub Hopf algebras generated by $Q^{\text{odd}}(B_1)$ and $Q^{\text{even}}(B_1)$ respectively. Then E_2 is isomorphic as a Hopf algebra to $E_2^{\text{odd}} \otimes E_2^{\text{even}}$. Hence in dimensions >0 E_2 is isomorphic as a Z_p module to $E_2^{\text{odd}} \oplus I$ where I is the ideal of E_2^{even} in E_2. But by 4.9 I is invariant under the action of d_2. On the other hand, $a_i^2 \in E_2^{\text{odd}}$. Thus given b such that $d_2(b) = a_i^2$, d_2 acts trivially on the component b' of b in I. Replacing b by $b - b'$ we have $b \in E_2^{\text{odd}} \subset A$.

(B) *Proof of Theorem* 1.2. Suppose ΩX has no p torsion. Then the module M constructed in the proof of 4.4 was shown to satisfy properties (a) and (b) of Theorem 1.2. The rest of this section will be devoted to proving the converse.

Suppose that there exists a torsion free module M satisfying (a) and (b) of 1.2. We will show that the Bockstein spectral $\{B_r\}$ for ΩX collapses. Our proof will be by induction on dimension. Suppose $B_1 = B_\infty$ in dimensions $<n$. Define k by the rule $n = 2k + 1$ if n is odd and $n = 2k$ if n is even.

LEMMA 4.10. rank $Q^{2k+1}(B_1) \geqq$ rank $Q^{2k+1}(B_\infty)$ *with equality only if* $B_1^n = B_\infty^n$.

Proof. It suffices to prove that, for $r \geqq 1$, rank $Q^{2k+1}(B_r) \geqq$ rank $Q^{2k+1}(B_{r+1})$ with equality only if $B_r^n = B_{r+1}^n$. We use the biprimitive spectral sequence $\{E_s\}$ defined in §4 of [2] where $E_1 =$ the biprimitive form of B_r and $E_\infty =$ the biprimitive form of B_{r+1}. By 2.7 of [2] $Q^{\text{odd}}(E_1) \cong Q^{\text{odd}}(B_r)$ and $Q^{\text{odd}}(E_\infty) \cong Q^{\text{odd}}(B_{r+1})$ as graded Z_p modules. The result then follows from 3.9 of [2] plus the fact that $E_1 = E_\infty$ in dimensions $<n$.

Theorem 1.2 will then follow if we can prove

LEMMA 4.11. rank $Q^{2k+1}(B_1) \leqq$ rank $Q^{2k+1}(B_\infty)$.

Proof. We have the following sequence of isomorphisms:

(i) $M \otimes Q \cong Q(H^{\text{even}}(X; A)) \cong P(H^{\text{odd}}(\Omega X; Q)) \cong Q(H^{\text{odd}}(\Omega X; Q))$.

The isomorphisms come from property (b) of 1.2, 3.1 and 4.18 of [10] respectively. We also have the following sequence of isomorphisms or surjective maps

(ii) $M \otimes Z_p \cong \bar{Q} \to P(H^{\text{odd}}(\Omega X; Z_p)) \cong Q(H^{\text{odd}}(\Omega X; Z_p))$.

That the maps are isomorphic or surjective follows from property (a) of 1.2, 3.2(ii), and 4.23 of [10] respectively. We then have the following sequence of inequalities

$$\begin{aligned} \text{rank } Q^{2k+1}(B_1) &= \text{rank } Q(H^{2k+1}(\Omega X; Z_p)) \\ &\leqq \text{rank } Q(H^{2k+1}(\Omega X; Q)) \qquad \text{(by (i) and (ii))} \\ &\leqq \text{rank } Q^{2k+1}(B_\infty) \qquad \text{(as in 4.6)}\,. \end{aligned}$$

5. Proof of Theorem 1.3. In this section we will prove Theorem 1.3. For p odd the result follows from Theorem 5.1 of [5]. Hence the rest of this section will be devoted to proving Theorem 1.3 for the case $p = 2$. We will work with Z_2 coefficients and (X, μ) will be a 1-connected H-space such that $H^*(\Omega X; Q_2)$ is torsion free. Our proof is a straightforward modification of that given in §5 of [5] to show that $E_2 = E_\infty$ when X is a finite H-space. We will assume familiarity with that proof.

We will also assume familiarity with the concept of transpotence elements in homological algebra. It will suffice for our purposes to define them as the nondecomposable elements in the -2 stem of $\mathrm{Tor}^{**}_{H^*(X;Z_2)}(Z_2; Z_2)$. $(=\mathrm{Tor}^{**})$. In order for Tor^{**} to have a transpotence element of bidegree $(-2, s)$ we must have $s = 2^{r+2}t$ for $r \geqq 0$, $t > 0$ and $H^*(X; Z_2)$ must possess a nondecomposable of dimension t and of height 2^{r+2} (see [12] for more details).

Our proof consists of three lemmas. The proof in [5] consists of the same three lemmas though they are not stated so explicitly.

LEMMA 5.1. *$E_2 = E_3$ and $E_4 = E_\infty$.*

LEMMA 5.2. *If $E_3 \neq E_4$ then there exists $0 \neq x \in P(H_{2n}(\Omega X; Z_2))$ where $x^2 = 0$, all elements in $A^*(2)$ of poitive degree act trivially on x, and $n = 2^{q+1}Q + 2^q - 2$ for q, $Q > 0$.*

LEMMA 5.3. *The properties possessed by x in 5.2 are incompatible.*

The first and third lemmas can be deduced as in [5]. Lemma 5.1 follows from 2.6 and 5.1 of [5] while the second half of §5 of [5] is devoted to proving precisely Lemma 5.3.

As for the proof of Lemma 5.2 it is based on [5] as well. We first observe that, by 5.2 of [5] we can find $0 \neq x \in P(H_{s-2}(\Omega X; Z_2))$ such that $x^2 = 0$ and $\theta(x) = 0$ for all $\theta \in A^*(p)$ of positive degree. We are left with showing that $n = s - 2$ is necessarily of the required form.

Now we obtain x by finding an element $y \in E_3^{**} = \mathrm{Tor}^{**}$ of the smallest possible total degree such that $d_3(y) \neq 0$. Then x is the dual of a transpotence element $z \in E_3^{-2,s} = \mathrm{Tor}^{-2,s}$ while x^2 is the dual of $y \in E_3^{-4,2s}$ (the differential acting nontrivially on y is what "kills" x^2). Suppose z is the transpotence of the nondecomposable $w \in H^t(X; Z_2)$ where w has height 2^{r+2}. Then $s = 2^{r+2}t$. We will be done if we can show

$$(5.4) \qquad t \text{ is odd}\,.$$

We know that $t \geqq 2$ since X is 1-connected. Hence $s \geqq 8$ and $s \leqq 2s - 4$. But since d_3 acts trivially on all elements in E_3^{**} of total degree less than $2s - 4$ it follows that $E_2 = E_\infty$ when the total degree is less than $2s - 4$. Therefore, Theorems 1.1 and 1.2 hold for $H^*(X; Z_2)$ in dimensions less than $2s - 4$. Thus, within this range of dimensions, any even dimension nondecomposable of $H^*(X; Z_2)$ generates a polynomial subalgebra of $H^*(X; Z_2)$. Hence t must be odd.

REMARK. The only point on which our proof of Lemma 5.2 differs from [5] is in the justification given for 5.4. In [5] we simply eliminated the possibility of even dimensional nondecomposables existing in dimension s or less.

6. Proof of Theorem 1.4. In this section we prove Theorem 1.4. Throughout this section we assume that (X, μ) is a 1-connected H-space such that $H^*(X\Omega, \mu)$ is torsion free and $H(X; Z_p)$ is finitely generated as an algebra.

By 1.1 any element in $H^*(X; Z_p)$ of dimension 2 generates a polynomial subalgebra of $H^*(X; Z_p)$. Hence, if $H^*(X; Z_p)$ is a finite algebra, then X must be 2-connected.

The rest of this section is devoted to proving the converse. So, assume X is 2-connected. Let N be the nilradical, that is, the elements of finite height in $H^*(X; Z_p)$. Then N is a Hopf ideal in $H^*(X; Z_p)$ (in particular, see formula (*) in 4.2 of [5]). Let $P = H^*(X; Z_p)/N$. It suffices to show P is trivial. We will do this by showing

LEMMA 6.1. *$Q(P^m) = 0$ unless $m \equiv 0 \bmod 2p$ and then, by a different argument, that*

LEMMA 6.2. *If $Q(P^m) \neq 0$ and $Q(P^i) = 0$ for $i < m$ then $m \equiv 2 \bmod 2p$.*

We will do the case $p = 2$ in detail. Now N is invariant under the action of the Steenrod algebra $A^*(2)$ (see 3.4 of [5]). Hence, the Steenrod module stucture of $H^*(X; Z_2)$ induces one on the polynomial Hopf algebra P.

Proof of 6.1. Our proof is based on the idea of a contraction as defined by Thomas in §3 of [13]. First, we show that $P^{2s+1} = 0$ for all s. By 4.21 of [10] and the fact that P is finitely generated as an algebra it follows that P^{2s+1} contains nonzero primitive elements for at most a finite number of values for s. It follows that P has only trivial primitive elements of add dimension. For, let t be the maximum integer for which there exists $0 \neq a \in P^{2t+1}$ which is primitive. In particular $t > 0$ since X is 2-connected. But $a^2 = Sq^1 Sq^{2t}(a) \neq 0$. Therefore $b = Sq^{2t}(a)$ is nonzero and primitive which contradicts the maximality of t. Finally, it follows that P has only trivial elements in odd dimensions. For, if we pick the minimal s such that $P^{2s+1} \neq 0$ then every element in P^{2s+1} is primitive.

Secondly, we show that $P^{4s+2} = 0$ for all s. Let $\langle Sq^1 \rangle$ be the ideal in $A^*(2)$ generated by Sq^1. Then, by the first paragraph, all elements of $\langle Sq^1 \rangle$ act trivially on P and the action of $A^*(2)$ on P induces an action of $A^*(2)/\langle Sq^1 \rangle$ on $\bar{P}$. Define a polynomial Hopf algebra $\bar{P}$ by the rule $\bar{P}^m = P^{2m}$ for any n. Then, the action of $A^*(2)/\langle Sq^1 \rangle$ on P "induces" an action of $A^*(2)$ on $\bar{P}$ via the canonical isomorphism $A^*(2) \cong A^*(2)/\langle Sq^1 \rangle$. The argument in the first paragraph then shows that $\bar{P}^{2s+1} = 0$ for all s. Therefore $P^{4s+2} = 0$ for all s.

Proof of 6.2. By 1.1, $\beta_p Q(H^{\text{even}}(X; Z_p)) = 0$. We can then make a mod 2 version of the arguments in [14] to deduce Lemma 6.2. Such mod 2 arguments have been done in great detail by James Lin. (See [7].)

For p odd the proofs of 6.1 and 6.2 as given above go through with only minor modifications. In particular, N is only invariant under the action of the subalgebra $B \subset A^*(p)$ generated by the operations $\{\mathscr{P}^{p^s}\}_{s \geqq 0}$. Thus, we use B in place of $A^*(p)$.

7. Proof of Theorem 1.5. In this section we prove Theorem 1.5. The proofs are motivated by [6] and we will refer freely to that paper. For the rest of this section assume that (X, μ) is a 1-connected H-space such that $H^*(\Omega X; Q_p)$ is torsion free. Define $\gamma(s)$ for any integer $s \geqq 0$ by the rule $\gamma(0) = 0$ and $\gamma(s) = \sum_{i=1}^{s-1} p^i$ for $s > 0$. Let $\{Q_s\}_{s \geqq 0}$ be the Milnor elements in the Steenrod algebra $A^*(p)$. (See [9].) In particular $Q_0 = \beta_p$ and, for integers $m \geqq 0$, they satisfy the relation

$$Q_0 \mathscr{P}^m = \sum (-1)^s \mathscr{P}^{m-\gamma(s)} Q_s \tag{7.1}$$

(we use the convention that $\mathscr{P}^q = 0$ if $q < 0$).

We assume that p is odd. We will prove 1.5 by induction on dimension in K. Pick the minimal integer n such that $K^{2n} = Q_0 \mathscr{P}^m Q(H^{2m+1}(X; Z_p)) \neq 0$. Since $K^{2i} = 0$ if $i < n$ it follows from 7.1 that $m = \gamma(s)$ for some $s \geqq 1$. Thus $n = pm + 1 = \gamma(s+1)$ and both (a) and (b) of 1.5 are trivially satisfied.

Suppose 1.5 is satisfied for $i < n$ and $K^{2n} = Q_0 \mathscr{P}^m Q(H^{2m+1}(X; Z_p)) \neq 0$. By 7.1 we can find $k \geqq 0$ such that $\mathscr{P}^{m-\gamma(k)} Q_k Q(H^{2m+1}(X; Z_p)) \neq 0$.

Let $l = m + p^k$. Then $K^{2l} \neq 0$. We can assume $m \neq \gamma(k)$ since otherwise, as in the argument for the initial step, we are done. Thus $l < n$ and so, by the induction hypothesis, l is binary. To prove (a) it suffices to show

LEMMA 7.2. $m \equiv \gamma(k) \bmod p^{k+1}$.

For then l binary implies that m is binary and hence that $n = mp + 1$ is binary as well. To prove (b) it suffices to show

LEMMA 7.3. $K^{2n} = \mathscr{P}^{m-\gamma(k)} K^{2l}$.

For, by 7.2, $r(n) = m - \gamma(k)$ and $l = q(n) + r(n)$.

The proof of 7.2 and 7.3 depend on the following lemma (see §4 of [6]).

LEMMA 7.4. *For* $s \geqq 0$, *if* $n \equiv \gamma(s) \bmod p^s$ *and* $\mathscr{P}^q$ *acts nontrivially on* $(K^*)^{2n}$ *then* $q \equiv 0 \bmod p^s$. *Further, if* n *is binary then* q *is binary.*

Proof of 7.2. By induction. We will show for any $s \leqq k$ that $m \equiv \gamma(k) \bmod p^s$ implies $m \equiv \gamma(k) \bmod p^{s+1}$. Pick $s \leqq k$. Then $m \equiv \gamma(k) \bmod p^s$. Hence $n \equiv \gamma(k+1) \bmod p^{s+1}$. And, by 7.4, $\mathscr{P}^{m-\gamma(k)}$ acting nontrivially on $(K^*)^{2n}$ implies $m - \gamma(k) \equiv 0 \bmod p^{s+1}$.

Proof of 7.3. By 7.1 and the relation $K^{2n} = Q_0 \mathscr{P}^m K^{2m}$ it suffices to show $\mathscr{P}^{m-\gamma(s)}$ acts trivially on $(K^*)^{2n}$ unless $s = k$. This follows from 7.2. For, if $s < k$ then $m - \gamma(s) \not\equiv 0 \bmod p^k$. While if $s > k$ then $m - \gamma(s) \equiv (p-1)p^k \bmod p^{k+1}$ and hence $m - \gamma(k)$ is not binary.

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