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THE EXACT BERGMAN KERNEL AND THE KERNELS OF SZEGÖ TYPE

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A relation between the magnitudes of the exact Bergman kernel and a product of two kernels of Szegö type is given. The method is turned to the establishment of a positive definiteness of a period matrix of a product of two kernels of Szegö type. The positive definiteness leads to some completeness theorems of such products.

1. Introduction. Let G denote an n-ply connected regular region with boundary components $\{C_\nu\}_{\nu=1}^{n}$. Let $K^E(z, \bar{z}_1)$ and $\hat{K}(z, \bar{z}_1)$ denote the exact Bergman kernel and the Szegö kernel of G, respectively. Let $\{Z_\nu(z)dz\}_{\nu=1}^{n-1}$ denote a basis of analytic differentials which are real along ∂G. Then the following identity is known:

$$4\pi\hat{K}(z, \bar{z}_1)^2 = K^E(z, \bar{z}_1) + \sum_{\nu=1}^{n-1}\sum_{\mu=1}^{n-1} C_{\nu\mu}\overline{Z_\nu(z_1)}Z_\mu(z) ,$$

for some uniquely determined constants $C_{\nu\mu}$. D. A. Hejhal [3] established the positive definiteness of the matrix $\|C_{\nu\mu}\|$ by means of the representations of $C_{\nu\mu}$ in terms of the theta function. In this paper, we shall establish a similar result in a very general situation by means of the pure theory of kernel functions in a sense. Our result leads to a variety of the completeness theorems of the kernels of Szegö type which are established in the paper [6].

In § 2, we state notation and preliminary facts and in § 3, the main theorem is given. In § 4 and § 5, completeness of the products of two kernels of Szegö type is discussed. These two sections are considered as a continuation of the paper [6]. In the final § 6, we refer to the case of the Szegö kernel with characteristic of an arbitrary compact bordered Riemann surface which is established by J. D. Fay [1].

2. Notation and preliminary facts. Let S denote the interior of a compact bordered Riemann surface $\bar{S}$ with boundary contours $\{C_\nu\}_{\nu=2n+1}^{2n+m}$ and genus n. Let $\{C_\nu\}_{\nu=1}^{2n+m-1}$ denote a canonical homology basis. Let M denote the Hilbert space of analytic differentials $f(z)dz$ which are regular in S and have finite norms:

$$\left(\iint_S |f(z)|^2 dx\, dy\right)^{1/2}$$

$<\infty (z=x+yi)$. Let $K(z, \bar{z}_1)dz$ and $K^E(z, \bar{z}_1)dz$ denote the Bergman kernel of class M and the exact Bergman kernel of S, respectively. Let $L(z, z_1)dz$ and $L^E(z, z_1)dz$ denote the adjoint L-kernels of theirs, respectively. They are analytic on $\bar{S}$ except for z_1 where they have a double pole:

$$\left\{\frac{1}{\pi}\frac{1}{(z-z_1)^2} + \text{regular terms}\right\}dz .$$

Further they satisfy the following relations:

$$(2.1)\quad -\overline{K(z, \bar{z}_1)dz} = L(z, z_1)dz \text{ and } -\overline{K^E(z, \bar{z}_1)dz} = L^E(z_1, z)dz \text{ along } \partial S .$$

The K-kernels $K(z, \bar{z}_1)$ and $K^E(z, \bar{z}_1)$ are Hermitian and $L(z, z_1)$ is symmetric, but $L^E(z, z_1)$ is not symmetric, in general (cf. [7], pp. 126-137).

Let $W(z, t)$ denote the meromorphic function which has the Green function $g(z, t)$ of S with pole at $t(\in S)$ as the real part of $W(z, t)$. The differential $\mathrm{id}W(z, t)$ is positive along ∂S and has $N = 2n + m - 1$ zeros $\{t_\nu\}$ in S. For simplicity, we assume that all the zeros t_ν are simple. In other cases, we can modify the following arguments slightly. Further we shall use the same notation for a point on $\bar{S}$ and a fixed local parameter around there. For an arbitrary integer q, let $H_2^q(S)$ denote the Hilbert space of analytic differentials $f(z)(dz)^q$ of order q on S with finite norms:

$$\left(\frac{1}{2\pi}\int_{\partial S}|f(z)(dz)^q|^2(\mathrm{id}W(z, t))^{1-2q}\right)^{1/2} < \infty ,$$

where $f(z)$ means the Fatou boundary value of f at $z \in \partial S$ in the obvious sense. Let $K_{q,t,\rho}(z, \bar{z}_1)(dz)^q$ denote the kernel function of the class $H_2^q(S)$ which is characterized by the following reproducing property:

$$f(z_1) = \frac{1}{2\pi}\int_{\partial S} f(z)(dz)^q\overline{K_{q,t,\rho}(z, \bar{z}_1)(dz)^q}\rho(z)(\mathrm{id}W(z, t))^{1-2q}$$
$$\text{for all } f(z)(dz)^q \in H_2^q(S) .$$

Here ρ is a positive continuous function on ∂S. Let $L_{q,t,\rho}(z, z_1)(dz)^{1-q}$ denote the adjoint L-kernel of $K_{q,t,\rho}(z, \bar{z}_1)(dz)^q$. Then $L_{q,t,\rho}(z, z_1)(dz)^{1-q}$ is a meromorphic differential on S of order $1-q$ with one simple pole at z_1 with residue 1 (in the obvious sense) and satisfies the relation

$$(2.2)\quad \overline{K_{q,t,\rho}(z, \bar{z}_1)(dz)^q}\rho(z)(\mathrm{id}W(z, t))^{1-2q} = \frac{1}{i}L_{q,t,\rho}(z, z_1)(dz)^{1-q} \text{ along } \partial S .$$

We note that $K_{q,t,\rho}(z, \bar{z}_1)$ and $L_{q,t,\rho}(z, z_1)$ are continuous on ∂S.

From (2.2), we have

$$\overline{K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz} = -L_{q,t,\rho}(z, z_1)L_{1-q,t,\rho^{-1}}(z, z_1)dz \text{ along } \partial S . \tag{2.3}$$

Let $\{Z_\nu(z)dz\}_{\nu=1}^N$ denote a basis of analytic differentials on $\bar{S}$ which are real along ∂S such that $Z_\nu(z) = \int_{C_\nu} L(\zeta, z)d\zeta$. Then from (2.1) and (2.3), we obtain the identities

$$K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1) = \pi K^E(z, \bar{z}_1) + \sum_{\nu=1}^N \sum_{\mu=1}^N C_{\nu\mu}\overline{Z_\nu(z_1)}Z_\mu(z) \tag{2.4}$$

and

$$L_{q,t,\rho}(z, z_1)L_{1-q,t,\rho^{-1}}(z, z_1) = \pi L^E(z_1, z) - \sum_{\nu=1}^N \sum_{\mu=1}^N \overline{G_{\nu\mu}}Z_\nu(z_1)Z_\mu(z) , \tag{2.5}$$

for some uniquely determined constants $\{C_{\nu\mu}\}$. Our first objective is to show the positive definiteness of the matrix $||C_{\nu\mu}||$. By setting $\int_{C_\mu} Z_\nu(z)dz = P_{\nu\mu}$, from (2.4), we obtain

$$\sum_\nu \sum_\mu C_{\nu\mu}\overline{P_{\nu\alpha}}P_{\mu\beta} = \int_{C_\alpha}\left(\int_{C_\beta} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\right)\overline{dz_1} \quad \text{for } \alpha,\ \beta = 1, 2, 3, \cdots, N . \tag{2.6}$$

Since the matrix $||P_{\nu\mu}||$ is nonsingular (cf. [7], pp. 93-97 and pp. 109-110), we shall show the positive definiteness of (2.6).

Here we note that especially $K_{0,t,1}(z, \bar{z}_1)$ and $K_{1,t,1}(z, \bar{z}_1)dz$ are the Rudin kernels (cf. [5]). If S is a bounded regular region on the plane, then we can identify functions and differentials on S. Hence we can write the reproducing property of $K_{q,t,\rho}(z, \bar{z}_1)(dz)^q$ as follows:

$$f(z_1) = \frac{1}{2\pi}\int_{\partial S} f(z)\overline{K_{q,t,\rho}(z, \bar{z}_1)}\rho(z)\left(\frac{\partial g(z, t)}{\partial \nu}\right)^{1-2q} ds_z \quad \text{for all } f \in H_2^0(S) , \tag{2.7}$$

where $\partial/\partial\nu$ denotes the inner normal derivative with respect to S. Therefore we can regard $K_{q,t,\rho}(z, \bar{z}_1)$ as the Szegö kernel with weight $\rho(z)(\partial g(z, t)/\partial\nu)^{1-2q}$ for the (Hardy) space $H_2^0(S)$. By this interpretation of (2.7), we can consider the kernel $K_{q,t,\rho}(z, \bar{z}_1)$ for an arbitrary real value of q and $K_{1/2,t,1}(z, \bar{z}_1)/2\pi$ is the classical Szegö kernel. For a real value of q, we can take a more general interpretation for (2.7) and we shall refer to this in § 6, again.

3. Positive definiteness of $K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)$.

THEOREM 3.1. *The matrix*

$$\left\| \int_{C_\nu}\left(\int_{C_\mu} K_{q,t,\rho}(z, \bar{z}_1) K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\right)\overline{dz_1} \right\|^{N\times N}$$

is positive definite.

Proof. Let $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ denote some complete orthonormal systems such that $K_{q,t,\rho}(z, \bar{z}_1) = \sum_j \overline{\varphi_j(z_1)}\varphi_j(z)$ and $K_{1-q,t,\rho^{-1}}(z, \bar{z}_1) = \sum_k \overline{\psi_k(z_1)}\psi_k(z)$. Then we have

$$\int_{C_\nu}\left(\int_{C_\mu} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\right)\overline{dz_1}$$
$$= \sum_j \sum_k \int_{C_\mu}\varphi_j(z)\psi_k(z)dz\overline{\int_{C_\nu}\varphi_j(z_1)\psi_k(z_1)dz_1}\ .$$

Here we see easily that the double sequence converges absolutely. Let m denote the double index (jk) and we set

$$A = \begin{vmatrix} A_1^{(1)} & A_2^{(1)} & A_3^{(1)} & \cdots & A_m^{(1)} & \cdots \\ A_1^{(2)} & A_2^{(2)} & A_3^{(2)} & \cdots & A_m^{(2)} & \cdots \\ A_1^{(3)} & A_2^{(3)} & A_3^{(3)} & \cdots & A_m^{(3)} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ A_1^{(N)} & A_2^{(N)} & A_3^{(N)} & \cdots & A_m^{(N)} & \cdots \end{vmatrix}$$

where

$$A_m^{(\nu)} = \int_{C_\nu}\varphi_j(z)\psi_k(z)dz(\nu = 1, 2, 3, \cdots, N)\ .$$

Further we set $X = (X_1, X_2, X_3, \cdots, X_N) \in \boldsymbol{C}^N$. Then we obtain

$$XA\bar{A}^t\bar{X}^t = \sum_m |X_1A_m^{(1)} + X_2A_m^{(2)} + X_3A_m^{(3)} + \cdots + X_NA_m^{(N)}|^2 \geqq 0\ .$$

Here equality holds if and only if

$$X_1A_m^{(1)} + X_2A_m^{(2)} + \cdots + X_NA_m^{(N)} = 0 \quad \text{for all } m\ . \tag{3.1}$$

Hence we obtain, for any z_1 and $z_2 \in S$,

$$\sum_{\nu=1}^{N} X_\nu\int_{C_\nu} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_2)dz = 0\ .$$

From the identity

$$K_{q,t,\rho}(z, \bar{z}_1) = \frac{1}{2\pi}\int_{\partial S} K_{q,t,\rho}(\zeta, \bar{z}_1)(d\zeta)^q\overline{K_{q,t,\rho}(\zeta, \bar{z})(d\zeta)^q}\rho(\zeta)(\mathrm{id}\,W(\zeta, t))^{1-2q}\ ,$$

we have, by exchanging the variables,

$$\int_{\partial S}\Big[\Big(\sum_{2n+1}^{N}\omega_\nu(z)X_\nu\Big)K_{1-q,t,\rho^{-1}}(z,\bar z_2)(dz)^{1-q} + \Big(\frac{1}{2\pi}\sum_{\nu=1}^{2n}X_\nu\int_{C_\nu}\overline{K_{q,t,\rho}(z,\bar\zeta)(dz)^q}K_{1-q,t,\rho^{-1}}(\zeta,\bar z_2)d\zeta\Big) \times \rho(z)(\mathrm{id}\,W(z,t))^{1-2q}\Big]K_{q,t,\rho}(z,\bar z_1)(dz)^q = 0 \text{ for all } z_1 \text{ and } z_2\in S\,.$$

Here $\omega_\nu(z)$ denotes the harmonic measure of C_ν on S. Let $\Omega(z)dz$ be a nonvanishing analytic differential on $\bar S$. Then since the set of kernels $\{K_{q,t,\rho}(z,\bar z_1)(dz)^q\,|\,z_1\in S\}$ is complete in $H_2^q(S)$, we have

$$\int_{\partial S}f(z)dz\Big[\Big(\sum_{2n+1}^{N}\omega_\nu(z)X_\nu\Big)K_{1-q,t,\rho^{-1}}(z,\bar z_2)(dz)^{1-q} + \Big(\frac{1}{2\pi}\sum_{\nu=1}^{2n}X_\nu\int_{C_\nu}\overline{K_{q,t,\rho}(z,\bar\zeta)(dz)^q}K_{1-q,t,\rho^{-1}}(\zeta,\bar z_2)d\zeta\Big) \times \rho(z)(\mathrm{id}\,W(z,t))^{1-2q}\Big](\Omega(z)dz)^{q-1} = 0 \quad \text{for all } f(z)dz\in H_2^1(S)\,.$$

Hence from the theorem of Cauchy-Read [4], we obtain, by a function $F_{z_2}\in H_2^0(S)$,

$$\Big[\Big(\sum_{2n+1}^{N}\omega_\nu(z)X_\nu\Big)K_{1-q,t,\rho^{-1}}(z,\bar z_2)(dz)^{1-q} + \Big(\frac{1}{2\pi}\sum_{\nu=1}^{2n}X_\nu\int_{C_\nu}\overline{K_{q,t,\rho}(z,\bar\zeta)(dz)^q} \times K_{1-q,t,\rho^{-1}}(\zeta,\bar z_2)d\zeta\Big)\rho(z)(\mathrm{id}\,W(z,t))^{1-2q}\Big](\Omega(z)dz)^{q-1} = F_{z_2}(z) \text{ a.e. on } \partial S\,.$$

From the relation (2.2),

$$\Big[\Big(\sum_{2n+1}^{N}\omega_\nu(z)X_\nu\Big)K_{1-q,t,\rho^{-1}}(z,\bar z_2)(dz)^{1-q} + \frac{1}{2\pi i}\sum_{\nu=1}^{2n}X_\nu\int_{C_\nu}L_{q,t,\rho}(z,\zeta)(dz)^{1-q} \times K_{1-q,t,\rho^{-1}}(\zeta,\bar z_2)d\zeta\Big](\Omega(z)dz)^{q-1} = F_{z_2}(z) \quad \text{a.e. on } \partial S\,.$$

At first, from the Lusin-Riesz-Privalow theorem, we see that all the $\{X_\nu\}_{\nu=2n+1}^{N}$ are zero. Next since $L_{q,t,\rho}(z,\zeta)$ has a simple pole at $z=\zeta$, from the property of Cauchy integral, we see that the constants $\{X_\nu\}_{\nu=1}^{2n}$ are also zero. Thus we have completed the proof of the theorem.

From the theorem, we obtain

COROLLARY 3.1. *The matrix* $\|C_{\nu\mu}\|$ *is positive definite. Especially we have the inequality*

$$K_{q,t,\rho}(z,\bar z)K_{1-q,t,\rho^{-1}}(z,\bar z) > \pi K^E(z,\bar z) \quad \textit{for all } z\in S\,.$$

COROLLARY 3.2. *The periods as anti-analytic differentials in* $z_1 \in S$,

$$\left\{\int_{C_\nu} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\right\}_{\nu=1}^{N}$$

are linearly independent.

Here we note that Theorem 3.1 and its corollaries are valid for an arbitrary real value of q, when S is a bounded regular region in the plane and we regard $K_{q,t,\rho}(z, \bar{z}_1)$ as the weighted Szegö kernel.

4. Completeness of $\{K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz \,|\, z_1 \in S\}$ **in** M. Let $\{Z_j\}_{j=1}^{\infty}$ ($Z_j \neq Z_{j'}$, for $j \neq j'$) denote any point set of S such that $\lim_{j\to\infty} Z_j = Z_0$ for some $Z_0 \in S$. Then, as we see from the reproducing property, the set of kernels $\{K_{q,t,\rho}(z, \bar{Z}_j)(dz)^q\}_{j=1}^{\infty}$ is complete in $H_2^q(S)$. In the following two sections we shall show that the set $\{K_{q,t,\rho}(z, \bar{Z}_j)K_{1-q,t,\rho^{-1}}(z, \bar{Z}_j)dz\}_{j=1}^{\infty}$ is complete in both M and $H_2^1(S)$. These theorems are a variety of the completeness theorems which are given in the paper [6].

As in the representations (2.4) and (2.5), we obtain

$$K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1) = \pi K(z, \bar{z}_1) + \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} C_{\nu\mu}^{*}\overline{Z_\nu(z_1)}Z_\mu(z) \tag{4.1}$$

and

$$L_{q,t,\rho}(z, z_1)L_{1-q,t,\rho^{-1}}(z, z_1) = \pi L(z, z_1) - \sum_{\nu=1}^{N}\sum_{\mu=1}^{N} C_{\nu\mu}^{*}Z_\nu(z_1)Z_\mu(z) \,. \tag{4.2}$$

Here we note that the constants $C_{\nu\mu}^{*}$ are real symmetric by virtue of Hermitian of the K-kernels and symmetry of $L_{q,t,\rho}(z, z_1)L_{1-q,t,\rho^{-1}}(z, z_1)$ and $L(z, z_1)$. Note that $L_{q,t,\rho}(z, z_1) = -L_{1-q,t,\rho^{-1}}(z_1, z)$([5]). As to the constants $C_{\nu\mu}^{*}$, we obtain

LEMMA 4.1. *The matrix*

$$\left\|\sum_{\alpha} C_{\nu\alpha}^{*}P_{\alpha\mu} - \pi\delta_{\nu\mu}\right\|^{N\times N}$$

is nonsingular.

Proof. Suppose that

$$\begin{aligned}\sum_{\mu} Y_\mu\int_{C_\mu} &K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\\ &= \sum_{\mu} Y_\mu\int_{C_\mu}[\pi K(z, \bar{z}_1) + \sum_{\alpha}\sum_{\beta} C_{\alpha\beta}^{*}\overline{Z_\alpha(z_1)}Z_\beta(z)]dz \equiv 0 \,.\end{aligned}$$

Then from the identity

$$\int_{C_\mu} K(z, \bar{z}_1)dz = -\iint_S K(z, \bar{z}_1)\overline{Z_\mu(z)}dx\,dy$$
$$= -\overline{Z_\mu(z_1)} \text{ (cf. [7], pp. 102-105)},$$

we have

$$-\sum_\mu \pi Y_\mu \overline{Z_\mu(z_1)} + \sum_\mu \sum_\alpha \sum_\beta Y_\mu C^*_{\alpha\beta} \overline{Z_\alpha(z_1)} P_{\beta\mu} \equiv 0 .$$

Hence we obtain

$$-\pi Y_\alpha + \sum_\beta \sum_\mu C^*_{\alpha\beta} P_{\beta\mu} Y_\mu = 0 \qquad \text{for} \quad \alpha = 1, 2, 3, \cdots, N .$$

Thus Corollary 3.2 is equivalent to the lemma.

Now we obtain

THEOREM 4.1. *The set* $\{K_{q,t,\rho}(z, \bar{Z}_j)K_{1-q,t,\rho^{-1}}(z, \bar{Z}_j)dz\}_j$ *is complete in* M.

Proof. Suppose that for any $f(z)dz \in M$,

$$\iint_S f(z)\overline{K_{q,t,\rho}(z, \bar{Z}_j)K_{1-q,t,\rho^{-1}}(z, \bar{Z}_j)}dx\,dy = 0 \qquad \text{for all } j .$$

Hence from (4.1) and the identity

$$\iint_S f(z)\overline{Z_\mu(z)}dx\,dy = -\int_{C_\mu} f(z)dz \text{ (cf. [7], p. 102)}, \tag{4.3}$$

we have

$$\pi f(z_1) \equiv \sum_\alpha \sum_\beta C^*_{\alpha\beta} Z_\alpha(z_1) \int_{C_\beta} f(z)dz .$$

Thus from the identities

$$\pi \int_{C_\mu} f(z_1)dz_1 = \sum_\alpha \sum_\beta C^*_{\alpha\beta} P_{\alpha\mu} \int_{C_\beta} f(z)dz \quad \mu = 1, 2, 3, \cdots, N ,$$

and Lemma 4.1, we obtain the desired result.

5. Completeness of $\{K_{q,t,\rho}(z, \bar{Z}_j)K_{1-q,t,\rho^{-1}}(z, \bar{Z}_j)dz\}_j$ in $H^1_2(S)$.

The following lemma is essential for our purpose:

LEMMA 5.1. *For the critical points* $\{t_\nu\}_{\nu=1}^N$ *of the Green function* $g(z, t)$, *the matrix*

$$\left\| \int_{C_\mu} L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu)dz \right\|^{N\times N}$$

is nonsingular.

If not all t_ν is simple, we modify the above matrix slightly, as usual.

Proof. We proceed as in Theorem 2.1 in [6], but in this case the proof is more delicate. From the reproducing property of $K_{1,t,1}(z, \bar{z}_1)dz$, we have

$$K_{q,t,\rho}(z_1, \bar{z})K_{1-q,t,\rho^{-1}}(z_1, \bar{z}) = \frac{1}{2\pi}\int_{\partial S} \frac{K_{q,t,\rho}(\zeta, \bar{z})K_{1-q,t,\rho^{-1}}(\zeta, \bar{z})d\zeta\overline{K_{1,t,1}(\zeta, \bar{z}_1)d\zeta}}{\mathrm{id}W(\zeta, t)} .$$

Hence from (2.3) and the residue theorem, we obtain

$$(5.1) \quad \left(\frac{K_{1,t,1}(z, \bar{z}_1)}{W'(z, t)}\right)' = -K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1) - \sum_\nu \left(\frac{K_{1,t,1}(t_\nu, \bar{z}_1)}{W''(t_\nu, t)}\right)L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu) ,$$

and

$$(5.2) \quad K_{1,t,1}(z, \bar{z}_1) = -\left\{\int_t^z K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_1)d\zeta + \sum_\nu \frac{K_{1,t,1}(t_\nu, \bar{z}_1)}{W''(t_\nu, t)}\int_t^z L_{q,t,\rho}(\zeta, t_\nu)L_{1-q,t,\rho^{-1}}(\zeta, t_\nu)d\zeta\right\}W'(z, t) .$$

From (5.1), we obtain

$$(5.3) \quad \sum_\nu \frac{K_{1,t,1}(t_\nu, \bar{z}_1)}{W''(t_\nu, t)}\int_{C_\mu} L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu)dz = -\int_{C_\mu} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz \quad \mu = 1, 2, 3, \cdots, N .$$

Let $\{X_\nu\}$ be *any* solution of system (5.3); i.e.,

$$\sum_\nu X_\nu\int_{C_\mu} L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu)dz = -\int_{C_\mu} K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz \quad \mu = 1, 2, 3, \cdots, N ,$$

and we shall define

$$\tilde{K}_{1,t,1}(z, \bar{z}_1) = -\left\{\int_t^z K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_1)d\zeta + \sum_\nu X_\nu \int_t^z L_{q,t,\rho}(\zeta, t_\nu)L_{1-q,t,\rho^{-1}}(\zeta, t_\nu)d\zeta\right\}W'(z, t) .$$

Then $\tilde{K}_{1,t,1}(z, \bar{z}_1)dz \in H_2^1(S)$ and from the definition of X_ν and the identity (2.5), we see that

$$(K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_1) + \sum_\nu X_\nu L_{q,t,\rho}(\zeta, t_\nu)L_{1-q,t,\rho^{-1}}(\zeta, t_\nu))d\zeta$$

is exact. (We assume that $t_\nu \notin C_\mu$ for all ν and μ.) For any analytic differential $f(z)dz$ on $\bar{S}$ (in fact in S such that $f(z)dz \in H_2^1(S)$), we set

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{\partial S} \frac{f(z)dz\overline{\tilde{K}_{1,t,1}(z, \bar{z}_1)dz}}{\mathrm{id}W(z, t)} \\ &= \frac{1}{2\pi i}\int_{\partial S} f(z)\left[\overline{\int_t^z (K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_1)} \right. \\ &\qquad \left. \overline{+ \sum_\nu X_\nu L_{q,t,\rho}(\zeta, t_\nu)L_{1-q,t,\rho^{-1}}(\zeta, t_\nu))d\zeta}\right] dz . \end{aligned}$$

Let D_ν be a tiny disc of radius r_ν in the plane of a local parameter at t_ν, with center t_ν and let D denote the union of the D_ν, $\nu = 1, 2, 3, \cdots, N$. Then from the Green's formula, we obtain

$$\begin{aligned} I - \sum_\nu \frac{1}{2\pi i}\int_{\partial D_\nu} & f(z)\left[\overline{\int_t^z (K_{q,t,\rho}(\zeta, \bar{z}_1)K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_1)}\right. \\ & \left. \overline{+ \sum_\nu X_\nu L_{q,t,\rho}(\zeta, t_\nu)L_{1-q,t,\rho^{-1}}(\zeta, t_\nu))d\zeta}\right] dz \\ = \frac{1}{\pi}\iint_{S-D} & f(z)\overline{(K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)} \\ & \overline{+ \sum_\nu X_\nu L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu))}dx\, dy . \end{aligned}$$

By letting $\sum_\nu |r_\nu|$ tend to zero, we obtain

$$\begin{aligned} I = & \frac{1}{\pi}\iint_S f(z)\overline{K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)}dx\, dy \\ & + \frac{1}{\pi}\sum_\nu \overline{X_\nu}\, \underset{t_\nu}{p.v.}\iint_S f(z)\overline{L_{q,t,\rho}(z, t_\nu)L_{1-q,t,\rho^{-1}}(z, t_\nu)}dx\, dy , \end{aligned}$$

(cf. (2.5) and [7], pp. 118-120). Hence *if* we can show that $I = f(z_1)$, we obtain the lemma as in Theorem 2.1 in [6].

From (4.1), (4.2), (4.3), and the fact

$$\underset{\xi}{p.v.}\iint_S f(z)\overline{L(z, \xi)}dx\, dy = 0$$
$$\text{for all} \quad f(z)dz \in M \text{ (cf. [7], pp. 121-126)},$$

we have

$$I = f(z_1) + \frac{1}{\pi}\Big[\sum_\alpha \sum_\beta C^*_{\alpha\beta} Z_\alpha(z_1)\Big(-\int_{C_\beta} f(z)dz\Big)$$
$$- \sum_\nu \sum_\alpha \sum_\beta \overline{X_\nu} C^*_{\alpha\beta} \overline{Z_\alpha(t_\nu)}\Big(-\int_{C_\beta} f(z)dz\Big)\Big].$$

Hence we must show that

$$(5.4) \qquad \sum_\alpha C^*_{\alpha\beta} Z_\alpha(z_1) - \sum_\alpha \sum_\nu \overline{X_\nu} C^*_{\beta\alpha} \overline{Z_\alpha(t_\nu)} = 0 \quad \beta = 1, 2, 3, \cdots, N.$$

From the definition of X_ν, by making use of (4.1) and (4.2), we obtain

$$\pi \sum_\nu X_\nu Z_\mu(t_\nu) - \sum_\nu \sum_\alpha \sum_\beta X_\nu C^*_{\alpha\beta} Z_\alpha(t_\nu) P_{\beta\mu}$$
$$= \pi \overline{Z_\mu(z_1)} - \sum_\alpha \sum_\beta C^*_{\alpha\beta} \overline{Z_\alpha(z_1)} P_{\beta\mu}$$

and hence

$$(5.5) \qquad \pi\Big(\sum_\nu X_\nu Z_\mu(t_\nu) - \overline{Z_\mu(z_1)}\Big)$$
$$- \sum_\beta \sum_\alpha \Big(\sum_\nu X_\nu Z_\alpha(t_\nu) - \overline{Z_\alpha(z_1)}\Big) C^*_{\alpha\beta} P_{\beta\mu} = 0 \quad \mu = 1, 2, 3, \cdots, N.$$

Hence from Lemma 4.1, we obtain

$$\sum_\nu X_\nu Z_\mu(t_\nu) - \overline{Z_\mu(z_1)} = 0 \quad \mu = 1, 2, 3, \cdots, N.$$

Thus from the regularity of the matrix $\| P_{\beta\mu} \|$ and (5.5), we obtain the desired result (5.4).

Now as in the proof of Theorem 2.2 in [6], we obtain

THEOREM 5.1. *The set* $\{K_{q,t,\rho}(z, \bar{Z}_j) K_{1-q,t,\rho^{-1}}(z, \bar{Z}_j) dz\}_j$ *is complete in* $H^1_2(S)$.

Here we note that for the set

$$\left\{\frac{\partial^n K_{q,t,\rho}(z, \bar{z}_1) K_{1-q,t,\rho^{-1}}(z, \bar{z}_1) dz}{\partial \bar{z}_1^n}\right\}_{n=0}^\infty$$

for any fixed $z_1 \in S$, Theorems 4.1 and 5.1 are valid, as we see easily from those theorems. The circumstances are similar for Theorems 2.2 and 3.3 in [6].

6. Szegö kernels with characteristic. Let $U = \{U_\alpha\}$ be a covering of S such that to each U_α there is associated a unique local uniformizing parameter z_α: $U_\alpha \to C$ and z_α and z_β are analytically related on $U_\alpha \cap U_\beta$. Then, for a real value of q, a differential $f(z)(dz)^q$ of order q is defined as a collection of variables which satisfy the transformation laws

$$\psi_\alpha = \psi_\beta\left(\frac{dz_\beta}{dz_\alpha}\right)^q \quad \text{in} \quad U_\alpha \cap U_\beta \,.$$

Of course, this function $(dz_\beta/dz_\alpha)^q$ is not unique, in general and we must show that we can choose them consistently; i.e., such that

$$\left(\frac{dz_\beta}{dz_\alpha}\right)^q\left(\frac{dz_\gamma}{dz_\beta}\right)^q\left(\frac{dz_\alpha}{dz_\gamma}\right)^q \equiv 1 \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma \,.$$

This is a crucial point, in the treatment of differentials $f(z)(dz)^q$ for a noninteger value of q (cf. [8], pp. 249-251 and [2], pp. 215-218). In the case of $q = 1/2$, which is considered as an especially important case, N. S. Hawley and M. Schiffer [2] for the first time had investigated "the Szegö kernel" of half-order differentials and then D. A. Hejhal [3] in the case of planar regions and J. D. Fay [1] in the case of arbitrary compact bordered Riemann surfaces had investigated.

In this final section, we shall see that our results are valid even if in the case of the Szegö kernels with characteristic on an arbitrary compact bordered Riemann surface.

In order to save space, we shall use the same notations and results freely in Chapter VI in [1].

For any even half period $e \in T_0$, the Szegö kernel $\sigma_e(\bar{x}, y)$ of a compact bordered Riemann surface $\bar{R}$ with characteristic $[e]$ is defined as follows:

$$\sigma_e(\bar{x}, y) = \frac{1}{2\pi i}\,\frac{\theta[e](y-\bar{x})}{\theta[e](0)E(y, \bar{x})} \,. \tag{6.1}$$

Here $\theta[e](z)$ is the first order theta function with characteristic $[e]$, $E(y, \bar{x})$ is the prime form and $\bar{x}$ is the symmetric point of x on the double C of R ([1], p. 124, Proposition 6.14 and cf. Chapters I and II). Then $\sigma_e(\bar{x}, y)$ is holomorphic in $\bar{x}$ and y except for a pole along $y = x$ and satisfies

$$\sigma_e(\bar{x}, y) = -\sigma_e(y, \bar{x}) = -\overline{\sigma_e(x, \bar{y})} \qquad \text{for all } x, y \in C \,. \tag{6.2}$$

For any section Φ of L_e holomorphic on $\bar{R}$, $\sigma_e(\bar{x}, y)$ has the reproducing property

$$\Phi(x) = \int_{\partial R} \Phi(y)\overline{\sigma_e(\bar{x}, y)} \qquad \text{for all } x \in R \,.$$

Again $\sigma_e(\bar{x}, y)$ is represented by $\sum_j \overline{\Phi_j(x)}\Phi_j(y)$ for a complete orthonormal system $\{\Phi_j\}$ of holomorphic sections of L_e on $\bar{R}$. Further we see that $\sigma_e(\bar{x}, y)^2$ is a single-valued analytic differential on $C \times C$ except for a double pole along $y = x$ and satisfies

$$(6.3)\qquad \overline{\sigma_e(\bar{x}, y)^2} = -\overline{\left(\frac{\theta[e](y-\bar{x})^2}{4\pi^2\theta[e](0)^2E(y,\bar{x})^2}\right)} = -\frac{\theta[e](\bar{y}-x)^2}{4\pi^2\theta[e](0)^2E(\bar{y},x)^2} = \sigma_e(x,y)^2 \quad \text{along} \quad y \in \partial R\ .$$

Hence in our case, we obtain the following representations as in (4.1) and (4.2):

$$(6.4)\qquad 4\pi^2\sigma_e(\bar{z}_1, z)^2 = \pi K^E(z, \bar{z}_1) + \sum_{\nu=1}^{N}\sum_{\nu=1}^{N} \tilde{C}_{\nu\mu}\overline{Z_\nu(z_1)}Z_\mu(z)$$

and

$$(6.5)\qquad -4\pi^2\sigma_e(z_1, z)^2 = \pi L^E(z_1, z) - \sum_{\nu=1}^{N}\sum_{\nu=1}^{N} \overline{\tilde{C}_{\nu\mu}}Z_\nu(z_1)Z_\mu(z)\ .$$

Therefore for $4\pi^2\sigma_e(\bar{z}_1, z)^2$, we obtain the same results as

$$K_{q,t,\rho}(z, \bar{z}_1)K_{1-q,t,\rho^{-1}}(z, \bar{z}_1)dz\,\overline{dz_1}\ ,$$

by a slight modification. In the proof of the main Theorem 3.1, for example we need a nonvanishing section Φ_0 of L_e holomorphic on $\bar{S}$, but the existence of such a Φ_0 is clear. As to this fact, we recall that there exists a single-valued function which is analytic and nonvanishing on $\bar{S}$ except for poles and zeros of prescribed order at prescribed points (cf. [4], § 2.5).

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