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WEAK* GENERATORS OF H^∞ AND l^1

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We prove that a weak* generator of H^∞ has distinct radial limits. As a corollary, we show that a weak* generator of l^1 must be univalent on the closed unit disc.

A. Introduction. For each bounded domain E in the plane, let $H^\infty(E)$ be the Banach algebra of functions that are bounded and analytic on E with norm $\|f\|_\infty = \sup|f(z)|\,(z \in E)$. We shall denote the unit disc $\{|z| < 1\}$ by U, and we shall write $H^\infty(U) = H^\infty$.

We identify the space l^1 of absolutely convergent sequences with the set

$$\{f(z) = \sum_0^\infty a_n z^n \mid \|f\|_1 = \sum_0^\infty |a_n| < \infty\}\,.$$

The space l^1 becomes a Banach algebra under the usual pointwise operations and the indicated norm.

Definition. An element f of a topological algebra $\mathscr{A}$ is said to *generate* $\mathscr{A}$ if the set

$$P(f) = \{p(f) \mid p \text{ is a polynomial}\}$$

is dense in $\mathscr{A}$.

In [6], D. Sarason proved that if f is a weak* generator of H^∞, then the radial limits of f are distinct almost everywhere. We use Sarason's characterization of the weak* generators of H^∞ [7] to prove that if f is a weak* generator of H^∞, then the radial limits of f are distinct everywhere. As a corollary, we will see that every weak* generator of l^1 is univalent on $\{|z| \leqq 1\}$. We conclude by exhibiting a univalent function in H^∞ with distinct radial limits which is not a weak* generator of H^∞.

B. Weak* topology. Let $\mathscr{B}$ be a Banach space with dual space $\mathscr{B}^*$. For each vector subspace $\mathscr{M}$ of $\mathscr{B}^*$, let $\mathscr{M}^1$ be the subspace consisting of each point of $\mathscr{B}^*$ that is a weak* limit of a sequence of points of $\mathscr{M}$. Inductively, define $\mathscr{M}^\sigma$ for each countable ordinal number σ by

$$\mathscr{M}^\sigma = [\cup \mathscr{M}^\xi]^1 \quad (\xi < \sigma)\,.$$

Banach proved that if $\mathscr{B}$ is separable, then there exists a smallest countable ordinal number σ_0 such that $\mathscr{M}^{\sigma_0}$ is the weak* closure of $\mathscr{M}$. The number σ_0 is called the *order* of $\mathscr{M}$ (see [1] p. 213).

Because each of l^1 and H^∞ is the dual of a separable Banach space, we can apply the construction above to the weak* topology on each of l^1 and H^∞. The following two propositions are easy to verify.

PROPOSITION 1. *A sequence* $\{f_n\}$ *in* l^1 *converges to* 0 *(weak*) if and only if there is a number* M *with* $\|f_n\|_1 \leqq M$ *for all* n *and* $\lim_{n\to\infty} f_n(z) = 0$ *for each* $z \in U$.

PROPOSITION 2. *A sequence* $\{f_n\}$ *in* H^∞ *converges to* 0 *(weak*) if and only if there is a number* M *with* $\|f_n\|_\infty \leqq M$ *for all* n *and* $\lim_{n\to\infty} f_n(z) = 0$ *for each* $z \in U$.

By observing that $\|f\|_\infty \leqq \|f\|_1$ for each f in l^1, we obtain the following corollary to Propositions 1 and 2.

COROLLARY 1. *If* $f_n \in l^1$ *for* $n = 1, 2, 3, \cdots$, *and the sequence* $\{f_n\}$ *converges to* 0 *in the weak* topology of* l^1, *then it also converges to* 0 *in the weak* topology of* H^∞.

If we use Corollay 1 repeatedly with the construction outlined at the beginning of this section, we can prove the following proposition.

PROPOSITION 3. *If a subspace* $\mathscr{M}$ *of* l^1 *is weak* dense in* l^1, *then* $\mathscr{M}$ *is weak* dense in* H^∞.

COROLLARY 2. *If* f *is a weak* generator of* l^1, *then* f *is a weak* generator of* H^∞.

C. Complex function theory. Most of the material in this section may be found in Sarason's article on weak* generators of H^∞ ([7]).

Let G be a bounded domain, and let G_∞ be the unbounded component of the complement of the closure of G.

DEFINITION. The *Caratheodory hull* of G is the complement of the closure of G_∞; we shall denote it by G^*:

$$G^* = C\backslash(G_\infty)^- .$$

Analytically,

$$G^* = \operatorname{Int}\{z \mid |p(z)| \leqq \sup_{w \in G} |p(w)| \text{ for all polynomials } p\} .$$

The components of G^* are simply connected. We let G^1 denote the component of G^* that contains G. The notation G^1 is suggestive of the fact that a function f in H^∞ is a sequential weak* generator of H^∞ (that is, $P(f)^1 = H^\infty$) if and only if $G = G^1$, where $G = f(U)$ (see Theorem 2 below).

DEFINITION. Let E be a simply connected domain containing G. The *relative hull of G in E*, or the *E-hull of G*, is the interior of the set

$$\{z \in E \mid |f(z)| \leqq \sup_{w \in G} |f(w)| \text{ for all } f \in H^\infty(E)\} .$$

DEFINITION. For each countable ordinal number σ, define a simply connected domain G^σ as follows:

(a) if σ has an immediate predecessor $\sigma - 1$, then G^σ is the component of the $G^{\sigma-1}$-hull of G that contains G;

(b) if σ has no immediate predecessor, then G^σ is the component of the interior of $\cap\, G^\xi(\xi < \sigma)$ that contains G.

We shall need the following theorems.

THEOREM 1 (Sarason [6]). *If f is a weak* generator of H^∞, it is univalent on U, and its radial limits $\lim_{r\to 1} f(re^{i\theta})$ are distinct almost everywhere.*

THEOREM 2 (Sarason [7]). *If $f \in H^\infty$ is univalent on U, with $G = f(U)$, then f is a weak* generator of H^∞ of order σ if and only if $G^\sigma = G$ and $G^\xi \neq G$ for $\xi < \sigma$.*

THEOREM 3 (Phragmen-Lindelof). *Suppose Ω is a Jordan domain and $h \in H^\infty(\Omega)$. Suppose further that h is continuous on $\partial\Omega\backslash\{p\}$, where $p \in \partial\Omega$, and that $|h(w)| \leqq m$ for each $w \in \partial\Omega\backslash\{p\}$. Then $|h(w)| \leqq m$ for all w in Ω.*

THEOREM 4 (Lindelof). *Let Ω be a domain whose boundary $\partial\Omega$ is a Jordan curve Γ, and let p be a point on Γ. Suppose that $F \in H^\infty(\Omega)$, that F is continuous at all points of Γ except possibly at p, and that $F(w)$ approaches limits L_1 and L_2 as w approaches the point p along Γ from two sides. Then $L_1 = L_2$, and F is continuous at p.*

D. Main result.

THEOREM 5. *Let f be a weak* generator of H^∞, and suppose $\lim_{r\to 1} f(re^{i\alpha}) = \lim_{r\to 1} f(re^{i\beta})$. Then $e^{i\alpha} = e^{i\beta}$.*

Proof. Let $G = f(U)$, let σ_0 be the order of f as a weak* generator of H^∞, and suppose that

$$\lim_{r\to 1} f(re^{i\alpha}) = \lim_{r\to 1} f(re^{i\beta}) = \wp$$

but $e^{i\alpha} \neq e^{i\beta}$.

Let $\Gamma_\alpha = \{f(re^{i\alpha}) \mid 0 \leqq r \leqq 1\}$ and $\Gamma_\beta = \{f(re^{i\beta}) \mid 0 \leqq r \leqq 1\}$. Because the function f is univalent on U (Theorem 1), the sets Γ_α and Γ_β are Jordan arcs in G^- with only the points $f(0)$ and $\wp$ in common. Thus, the set $\Gamma = \Gamma_\alpha \cup \Gamma_\beta$ is a closed Jordan cnrve; and $\Gamma\backslash\{\wp\} \subseteqq G$. Let Ω be the bounded component of the complement of Γ. Our goal is to show that $\Omega \subseteqq G$.

(a) $\Omega \subseteqq G^1$.

Let G_∞ be the unbounded component of the complement of the closure of G. The curve Γ is contained in the set G^-; therefore $\Gamma \cap G_\infty = \varnothing$, and hence G_∞ is contained in the unbounded component of the complement of Γ. But then $\Omega \cap G_\infty = \varnothing$. Because the set Ω is open, $\Omega \cap (G_\infty)^- = \varnothing$; but then $\Omega \subseteqq C\backslash(G_\infty)^-$, which is the Caratheodory hull G^* of G. The set Ω is connected, $G \cap \Omega \neq \varnothing$, and $\Omega \subseteqq G^*$; therefore Ω is contained in the component of G^* that contains G; therefore $\Omega \subseteqq G^1$.

(b) $\Omega \subseteqq G^{\sigma-1}$ implies $\Omega \subseteqq G^\sigma$.

Suppose $h \in H^\infty(G^{\sigma-1})$; then $h \in H^\infty(\Omega)$ and h is continuous on $\partial\Omega\backslash\{\wp\}$. Let $m = \sup_{w\in G} |h(w)|$. Since $\partial\Omega\backslash\{\wp\} \subseteqq G$, we see that

$$|h(w)| \leqq m \qquad \text{for each } w \in \partial\Omega\backslash\{\wp\}\,.$$

The Phragmen-Lindelof Theorem, Theorem, 3, implies that

$$|h(w)| \leqq m \qquad \text{for each } w \in \Omega\,.$$

Thus

$$\Omega \subseteqq \{z \in G^{\sigma-1} \mid |h(z)| \leqq \sup_{w\in G} |h(w)| \quad \text{for all } h \in H^\infty(G^{\sigma-1})\}\,,$$

so that $\Omega \subseteqq G^{\sigma-1}$-hull of G. As before, the hypotheses that Ω is connected, $G \cap \Omega \neq \varnothing$, and $\Omega \subseteqq G^{\sigma-1}$-hull of G imply that Ω is contained in the component of the $G^{\sigma-1}$-hull of G which contains G, in other words they imply that $\Omega \subseteqq G^\sigma$.

(c) $\Omega \subseteqq G^\sigma$ if σ has no immediate predecessor.

Suppose σ has no immediate predecessor, and suppose that $\Omega \subseteqq G^\xi$ for all $\xi < \sigma$. Let $H = \cap G^\xi(\xi < \sigma)$. Then $\Omega \subseteqq H$, so that $\Omega \subseteqq \mathrm{Int}\,(H)$, since Ω is an open set. The set G^σ is the component of Int (H) that contains G. Finally, Ω is connected, $G \cap \Omega \neq \emptyset$, and $\Omega \subseteqq \operatorname{Int}(H)$, so that $\Omega \subseteqq G^\sigma$,

Consequently, $\Omega \subseteqq G^\sigma$ for each countable ordinal number σ. In particular, $\Omega \subseteqq G^{\sigma_0}$. By Theorem 2, $G^{\sigma_0} = G$, and therefore $\Omega \subseteqq G$.

To complete the proof, we consider the function $F = f^{-1}$ restricted to $G \cap \Omega^-$. The function F is bounded and analytic on Ω and continuous on $\partial\Omega = \Gamma$ except at the one point p. Also,

$$\lim_{\substack{w \to p \\ w \in \Gamma_\alpha}} F(w) = \lim_{\substack{w \to p \\ w \in \Gamma_\alpha}} f^{-1}(w) = e^{i\alpha}$$

and

$$\lim_{\substack{w \to p \\ w \in \Gamma_\beta}} F(w) = \lim_{\substack{w \to p \\ w \in \Gamma_\beta}} f^{-1}(w) = e^{i\beta} .$$

By the Lindelof theorem, Theorem 4, $e^{i\alpha} = e^{i\beta}$.

E. **Application to weak* generators of l^1.** By using the fact that evaluation at a point of $\{|z| \leqq 1\}$ is a bounded linear functional on l^1, one can easily verify that if a function f generates l^1, then f must be univalent on $\{|z| \leqq 1\}$. D. J. Newman and L. I. Hedberg have each established a sufficient condition for a function to generate l^1. Their results are as follows.

THEOREM (Newman [5]). *If f is univalent on $\{|z| \leqq 1\}$ and f' is in H^1, then f generates l^1.*

THEOREM (Hedberg [3]). *If the function $f(z) = \sum_0^\infty a_n z^n$ is univalent on $\{|z| \leqq 1\}$ and $\sum_2^\infty n(\log n)^\alpha |a_n|^2 < \infty$ for some $\alpha > 1$, then f generates l^1.*

Hedberg also showed, by examples, that the conditions $f' \in H^1$ and $\Sigma n(\log n)^\alpha |a_n|^2 < \infty$ are independent even when f is univalent [4]. In light of these two results and Hedberg's examples, one wonders whether every univalent function in l^1 generates l^1. No answer is known.

In this paper, we equip l^1 with the weak* topology and consider the functions f in l^1 that generate l^1 in the weak* topology. By using evaluation at points of $\{|z| < 1\}$, one can show that each weak* generator of l^1 must be univalent on the open unit disc $\{|z| < 1\}$. Because evaluations at points of the unit circle are *not* continuous in the weak* topology, this argument will not show that each weak* generator of l^1 must be univalent on the set $\{|z| \leqq 1\}$. However, the following corollary to Theorem 5 does show that a weak* generator of l^1 must be univalent on the closed unit disc.

COROLLARY 3. *If f is a weak* generator of l^1, then f is univalent on $\{|z| \leqq 1\}$.*

Proof. Suppose f is a weak* generator of l^1. We have already observed that f is univalent on $\{|z| < 1\}$. If f is not univalent on $\{|z| \leqq 1\}$, then there are two distinct points, $\mathfrak{a}$ and $\mathfrak{b}$, such that $f(\mathfrak{a}) = f(\mathfrak{b})$. If $|\mathfrak{a}| < 1$, then we must have $|\mathfrak{a}| = 1$ since f is known to be univalent on $\{|z| < 1\}$. Since an analytic function is an open mapping, the image $f(V)$ of the set

$$V = \{z \mid |z - \mathfrak{a}| < 1/2 \min(|\mathfrak{b} - \mathfrak{a}|, 1 - |\mathfrak{a}|)\}$$

is a neighborhood of $f(\mathfrak{a})$; hence of $f(\mathfrak{b})$. The function f is continuous on $\{|z| \leqq 1\}$, so there is a point $\mathfrak{c}$, with $|\mathfrak{c}| < 1$ and $\mathfrak{d} \notin V$, such that $f(\mathfrak{c}) \in f(V)$. But then there exists a point $\mathfrak{d} \in V$ with $f(\mathfrak{c}) = f(\mathfrak{d})$, contradicting the univalence of f on $\{|z| < 1\}$. Consequently, if $f(\mathfrak{a}) = f(\mathfrak{b})$, we must have $|\mathfrak{a}| = |\mathfrak{b}| = 1$. By the continuity of f,

$$f(\mathfrak{a}) = \lim_{r \to 1} f(r\mathfrak{a}) \quad \text{and} \quad f(\mathfrak{b}) = \lim_{r \to 1} f(r\mathfrak{b}) .$$

By Corollary 2, we know that f is a weak* generator of H^∞. By Theorem 5, $f(\mathfrak{a}) = f(\mathfrak{b})$ implies $\mathfrak{a} = \mathfrak{b}$, contrary to our assumption that $\mathfrak{a} \neq \mathfrak{b}$.

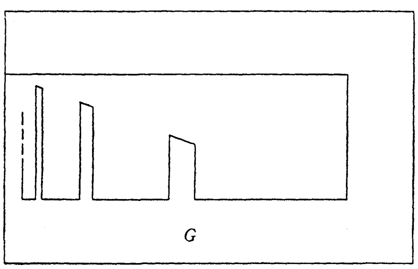
Our results suggest the following questions:

(1) Is every univalent function in l^1 a weak* generator of l^1?

(2) Is every weak* generator of l^1 a norm generator of l^1?

(3) Given a countable ordinal number σ, is there a weak* generator of l^1 of order σ? In particular, is there a weak* generator of l^1 of any order $\sigma \geqq 2$?

The first question is the analogue of a question due (according to the author's sources) to H. S. Shapiro: Does every univalent function in l^1 generate l^1 (in the *norm* topology)? By Corollary 3, a negative answer to question (1) or question (2) or an affirmative answer to question (3) will provide a negative answer to Shapiro's



question.

F. An example. To conclude the discussion about weak* generators of H^∞, we give an example to show that the converse of Theorem 5 is false. We describe an H^∞ function f which is univalent on U and has distinct radial limits (that is, $\lim_{r\to 1} f(re^{i\alpha}) = \lim_{r\to 1} f(re^{i\beta})$ implies $e^{i\alpha} = e^{i\beta}$), yet is not a weak* generator of H^∞.

The figure above suggests a simply connected domain G. Let f be a conformal map of U onto G. The boundary of G contains the entire boundary of the circumscribing rectangle. Consequently, G^* is the interior of the rectangle; and $G^1 = G^* \neq G$. We use a lemma due to Sarason to prove that f is not a weak* generator of H^∞. Sarason stated and proved the lemma for a disc, but we will state it for a rectangle; the proof is the same.

LEMMA ([7], Lemma 3). *Let the domain G be contained in a rectangle E. Then the E-hull of G is equal to G^*.*

We have already noted that $G^1 = G^*$, which is the whole rectangle. By Sarason's lemma, the G^1-hull of G is also G^*. Therefore, $G^2 = G^* \neq G$. By induction, $G^\sigma = G^* \neq G$ for each countable ordinal number σ. By Theorem 2, f cannot be a weak* generator of H^∞.

In order to verify that the radial limits of f are distinct, we will use the following theorem due to E. Collingwood and G. Piranian. We refer the reader to [2] for a more complete discussion of the material and the appropriate definitions.

THEOREM ([2], Theorem 2). *Let the function f map the unit disc conformally onto a simply connected domain G, let L be a Stolz path in the unit disc, and let $\{S_n\}$ be a side-chain of a prime end of G; then the set $f(L)$ meets at most finitely many of the crosscuts S_n.*

Roughly, the conclusion of the theorem says that a Stolz path (in particular, a radius) does not make infinitely many uniformly deep excursions into the sidepockets of the domain G. If we apply this theorem to G and f, we see that the radial limits of f must be distinct.

Thus, the function f is bounded, analytic, and univalent on U, has distinct radial limits, yet is not a weak* generator of H^∞.

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