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EXT IN PRE-ABELIAN CATEGORIES

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A natural definition for Ext is given in an arbitrary pre-abelian category. Ext is an additive bifunctor from the category to abelian groups, the five lemma and the 3×3 lemma hold, and Ext^n may be defined in the usual projectiveless way to yield the standard exact sequence of Ext's. Examples are given of what Ext is in various specific pre-abelian categories.

1. Semi-stable kernels and cokernels. Throughout this paper we shall be dealing with a *pre-abelian category*, that is, an additive category with kernels and cokernels. If $f: A \to B$ and $\xi: B' \to B$, then we can complete the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta} & B' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\xi} \\ A & \xrightarrow{f} & B \end{array}$$

by setting $P = \mathrm{Ker}\, \nabla(f \oplus (-\xi))$ where $\nabla: B \oplus B \to B$ is the codiagonal map. We say that β is the pullback of f along ξ. Dually, we can construct pushouts. In an abelian category pushouts and pullbacks of kernels (cokernels) are kernels (cokernels). In pre-abelian categories only half of this is true.

THEOREM 1. *Pullbacks of kernels are kernels. Specifically, if $f = \ker g$ and β is the pullback of f along ξ, then $\beta = \ker g\xi$. Dually, pushouts of cokernels are cokernels.*

Proof. Let α be the pullback of ξ along f so we have the following diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\beta} & \cdot \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\xi} \\ \cdot & \xrightarrow{f} & \cdot \end{array}$$

If $g\xi\lambda = 0$, then $\xi\lambda = f\phi$ so, by the pullback property, there is a map θ such that $\beta\theta = \lambda$ (and $\alpha\theta = \phi$). To show that θ is unique, suppose $\beta\delta = 0$. Then $\xi\beta\delta = f\alpha\delta = 0$ so $\alpha\delta = 0$ since f is a kernel and hence monic. But $\beta\delta = 0$ and $\alpha\delta = 0$ imply that $\delta = 0$ by the pullback property.

Pushouts of kernels need not be kernels, or even monic. In the category of abelian p-groups with no elements of infinite height, the kernels are the p-adically closed subgroups. If B is a direct sum of cyclic groups of order p^n for $n = 1, 2, 3, \cdots$ and $\bar{B}$ is the torsion subgroup of the corresponding product, and

$$G[p] = \{g \in G: pg = 0\} ,$$

then the diagram

$$\begin{array}{ccc} \bar{B}[p] & \longrightarrow & \bar{B} \\ \downarrow & & \downarrow \\ \bar{B}[p]/B[p] & \longrightarrow & \bar{B}/\bar{B}[p] \end{array}$$

is a pushout in the category. The top row is a kernel while the bottom row is the zero map.

DEFINITION. A kernel (cokernel) is said to be *semi-stable* if every pushout (pullback) is a kernel (cokernel).

Since a pushout of a pushout is a pushout, a pushout of a semi-stable kernel is a semi-stable kernel. The product of kernels need not be a kernel. In the example above $B[p] \to \bar{B}[p]$ is a kernel, and $\bar{B}[p] \to \bar{B}$ is a kernel, but the composite $B[p] \to \bar{B}$ is not a kernel. However we have the following.

THEOREM 2. *The product of semi-stable kernels* (cokernels) *is a semi-stable kernel* (cokernel).

Proof. Let $f: A \to B$ and $g: B \to C$ be semi-stable kernels. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & B/A \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \lambda} \\ A & \xrightarrow{gf} & C & \xrightarrow{(gf)'} & C/A \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \mu} \\ & & C/B & = & C/B \end{array}$$

where f', g', and $(gf)'$ are the cokernels of f, g, and gf. The upper right square is a pushout so λ is a kernel. Suppose $(gf)'\xi = 0$. Then $g'\xi = 0$ so $\xi = g\phi$. But $\lambda f'\phi = (gf)'g\phi = (gf)'\xi = 0$ and λ is monic (since it is a kernel) so $f'\phi = 0$ and hence $\phi = f\psi$ whereupon $\xi = gf\psi$. Since gf is monic ψ is unique whence gf is a kernel of

$(gf)'$. Thus the composite of semi-stable kernels is a kernel.

To show that gf is a semi-stable kernel consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ A' \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

where the two squares are pushouts. Both bottom arrows are semi-stable kernels, so their composite is a kernel. But the composite is a pushout of gf along α. Hence gf is a semi-stable kernel.

2. Stable exact sequences. *A sequence E* is a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ such that $gf = 0$. A morphism between two sequences E and E' is a triple (α, β, γ) of maps such that $f'\alpha = \beta f$ and $g'\beta = \gamma g$. We say that E is *left exact* if f is a kernel of g, *right exact* if g is a cokernel of f, and *exact* if it is both left exact and right exact.

Suppose E is a sequence and $\alpha: A \to A'$ and $\beta: C' \to C$. Then we can construct sequences αE and $E\beta$ as follows. To construct αE pushout f along α.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \\ A' \arrow[r, "f'"] & P \arrow[ur, "\phi"'] & \end{tikzcd}$$

The map ϕ from P to C such that $\phi\beta = g$ and $\phi f' = 0$ exists and is unique by the property of pushouts. The sequence $A' \to P \to C$ is denoted by αE. We construct $E\beta$ dually.

LEMMA 3. *The pushout diagram $E \to \alpha E$ is characterized by the property that any morphism $(\theta\alpha, \cdot, \cdot)$ from E to F factors through a unique morphism $(\theta, \cdot, \cdot)$ from αE to F. Dually, the pullback diagram $E\beta \to E$ is characterized by the property that any morphism $(\cdot, \cdot, \beta\theta)$ from F to E factors through a unique morphism $(\cdot, \cdot, \theta)$ from F to $E\beta$.*

Proof. Follows immediately from the universal properties of pushouts and pullbacks.

THEOREM 4. *If E is a sequence, then*

$$(\alpha_1\alpha_2)E = \alpha_1(\alpha_2 E)$$
$$(E\beta_1)\beta_2 = E(\beta_1\beta_2)$$
$$(\alpha E)\beta = \alpha(E\beta)$$

Proof. All three equalities follow easily by diagram chasing using the characterizations of the preceding lemma.

THEOREM 5. *If E is right exact, then αE is right exact. If E is left exact, then $E\beta$ is left exact. Hence if αE is left exact and $E\beta$ is right exact, then $(\alpha E)\beta = \alpha(E\beta)$ is exact.*

Proof. Let the situation $E \to \alpha E$ be written out as

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\lambda} & & \| \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C . \end{array}$$

Suppose $\phi f' = 0$. Then $\phi f'\alpha = 0$ so $\phi\lambda f = 0$. Hence $\phi\lambda = \theta g$ for a unique θ. We shall show that $\phi = \theta g'$. Now $\phi f' = 0 = \theta g' f'$ and $\phi\lambda = \theta g = \theta g'\lambda$, so $\phi - \theta g'$ kills both λ and f'. But since B' is a pushout, we have $\phi - \theta g' = 0$. The second statement is proved dually. The third is immediate from the first two and Theorem 4.

Our objective is to define Ext (C, A). An element of Ext (C, A) should be an exact sequence. Moreover we would like Ext (C, A) to be a functor. In abelian categories if $\alpha: A \to A'$ we get a map from Ext (C, A) to Ext (C, A') by taking E to αE. In pre-abelian categories αE need not be exact even when E is, as we saw in the category of abelian p-groups with no elements of infinite height. Accordingly we restrict our attention to those exact sequence that remain exact upon composition with maps.

DEFINITION. An exact sequence E is said to be a *stable* if αE and $E\beta$ are exact for all maps α and β. A map is said to be a *stable kernel* (cokernel) if it is the kernel (cokernel) map in a stable exact sequence.

If E is exact, then αE is right exact by Theorem 5. Hence if the kernel of E is semi-stable, then αE is exact. Dually, if the cokernel of E is semi-stable, then $E\beta$ is exact. So if the kernel and cokernel of E are semi-stable, then E is stable. Conversely if E is stable, then its kernel and cokernel are clearly semi-stable. By Theorems 4 and 5 if E is stable, then so are αE and $E\beta$.

3. The five lemma. The proof of the five lemma in an easy diagram chase if you have elements. In a pre-abelian category the notion of a stable cokernel allows this diagram chase to be carried out using pullbacks.

THEOREM 6. *Let E be an exact sequence with a semi-stable* *cokernel, and let E' be a left exact sequence. If $(1, \phi, 1): E \to E'$, then ϕ is an isomorphism.*

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\xi"] \arrow[d, equal] & X \arrow[r, "\sigma"] \arrow[d, "\lambda"] & B \arrow[d, "g'"] \\ A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\phi"] & C \arrow[d, equal] \\ A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \end{tikzcd}$$

where the rows are Eg', E and E'. Then $g'(\phi\lambda - \sigma) = 0$ so $\phi\lambda - \sigma = f'\rho$ for some $\rho: X \to A$ since $f' = \ker g'$. We have $f'\rho\xi = \phi\lambda\xi = f'$ so $\rho\xi = 1$ since f' is monic. Thus $(1 - \xi\rho)\xi = 0$ so there is a $\theta: B' \to X$ such that $\theta\sigma = 1 - \xi\rho$ since g is a semi-stable cokernel and hence $\sigma = \operatorname{coker} \xi$. In particular $\rho\theta\sigma = 0$ and $\sigma\theta\sigma = \sigma$ so, since σ is epic, we have $\rho\theta = 0$ and $\sigma\theta = 1$. Then $\phi\lambda\theta = (f'\rho + \sigma)\theta = 1$. Moreover ϕ is monic for if $\phi\delta = 0$ then $g\delta = g'\phi\delta = 0$ so $\delta = f\varepsilon$, since $f = \ker g$, and we have $0 = \phi\delta = \phi f\varepsilon = f'\varepsilon$ so ε, and hence δ, is 0. Thus $\phi\lambda\theta\phi = \phi$ so $\lambda\theta\phi = 1$ and $\lambda\theta = \phi^{-1}$.

COROLLARY 7. *If E and F are exact, either E or F is stable, and $(\alpha, \cdot, \beta): E \to F$, then $\alpha E = F\beta$ in the sense that there is a map $(1, \phi, 1): \alpha E \to F\beta$ such that ϕ is an isomorphism.*

Proof. By duality it suffices to consider the case where E is stable. By Lemma 3 the map $(\alpha, \cdot, \beta): E \to F$ factors through a unique map $(1, \cdot, \beta): \alpha E \to F$ which factors through a unique map $(1, \cdot, 1): \alpha E \to F\beta$. Since αE is stable exact and $F\beta$ is left exact the conclusion follows from Theorem 6.

The five lemma need not hold for exact sequences that are not stable. We will give an example where it fails, in the category of valuated abelian groups, at the end of the paper.

4. The functor Ext. We define $\operatorname{Ext}(C, A)$ to be the set of stable exact sequences $A \to \cdot \to C$ where two sequences are considered equal if there is a map $(1_A, \phi, 1_C)$ between them with ϕ an isomorphism. Then Theorems 4 and 5 imply that $\operatorname{Ext}(C, A)$ is a bifunctor — covariant in A and contravariant in C. To turn $\operatorname{Ext}(C, A)$ into an abelian group we define the usual Baer sum of two stable exact sequences E_1 and E_2 by

$$E_1 + E_2 = \nabla(E_1 \oplus E_2)\Delta$$

where $\Delta: C \to C \oplus C$ is the diagonal map and $\nabla: A \oplus A \to A$ is the codiagonal map. The proof that this turns Ext (C, A) into an abelian group follows MacLane [3, pp. 70–71]. The only difficulty that is peculiar to the present situation is the stability of $E_1 \oplus E_2$.

THEOREM 8. *If E_1 and E_2 are stable exact sequences, then $E_1 \oplus E_2$ is a stable exact sequence.*

Proof. Let E_i be $A_i \to B_i \to C_i$. The map $A_1 \oplus A_2 \to B_1 \oplus A_2$ is a pushout of $A_1 \to B_1$ and so is a semi-stable kernel. The map $B_1 \oplus A_2 \to B_1 \oplus B_2$ is a pushout of $A_2 \to B_2$ and so is a semi-stable kernel. By Theorem 2 the map $A_1 \oplus A_2 \to B_1 \oplus B_2$ is a semi-stable kernel. Dually the map $B_1 \oplus B_2 \to C_1 \oplus C_2$ is a semi-stable cokernel. Thus $E_1 \oplus E_2$ is stable exact.

We now have all we need to follow the proof in [3, pp. 70–71] that Ext (C, A) is an abelian group, and in fact an additive bifunctor.

5. **Products of stable maps.** As in the case of proper monics in relative homological algebra in abelian categories, the key to proving exactness of the sequence of Ext's at one point is that the product of stable kernels (cokernels) is a stable kernel (cokernel).

THEOREM 9. *If f and g are stable kernels, then gf is a stable kernel.*

Proof. Let $A \xrightarrow{f} B \xrightarrow{f'} B/A$ and $B \xrightarrow{g} C \xrightarrow{g'} C/B$ be stable exact sequences. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & B/A \arrow[d, "\lambda"] \\
A \arrow[r, "gf"] \arrow[d, equal] & C \arrow[r, "(gf)'"] & C/A \\
A \arrow[r, "\mu"] & G \arrow[r, "\nu"] \arrow[u, "\rho"] & M \arrow[u, "\gamma"]
\end{tikzcd}$$

where $(gf)'$ is a cokernel of gf and the lower right hand square is a pullback. Since gf is a semi-stable kernel it suffices to show that ν is the cokernel of μ. Consider the diagram

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[dd, equal] & Y \arrow[r, "h'"] \arrow[dd, "\phi"] & M \arrow[d, "\gamma"] \\
 & & C/A \arrow[d, "\lambda'"] \\
B \arrow[r, "g"] & C \arrow[r, "g'"] & C/B
\end{tikzcd}$$

where g' and λ' are cokernels of g and λ, and the right hand square is a pullback, so the top row is exact. There is a unique map $\varepsilon: G \to Y$ such that $\phi\varepsilon = \rho$ and $h'\varepsilon = \nu$ since Y is a pullback and $\lambda'\gamma\nu = \lambda'(gf)'\rho = g'\rho$. Consider the map from Y to C/A given by

$$\gamma h' - (gf)'\phi .$$

Then $\lambda'\gamma h' - \lambda'(gf)'\phi = g'\phi - g'\phi = 0$ so $\gamma h' - (gf)'\phi = \lambda\theta$ because λ is a kernel since g is a stable kernel. Consider the pullback diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & Z & \xrightarrow{\alpha'} & Y \\ \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\theta} \\ A & \xrightarrow{f} & B & \xrightarrow{f'} & B/A . \end{array}$$

Define $\sigma: Z \to G$ by $\nu\sigma = h'\alpha'$ and $\rho\sigma = g\beta + \phi\alpha'$. This is OK because G is a pullback and

$$(gf)'(g\beta + \phi\alpha') = \lambda f'\beta + (gf)'\phi\alpha' = \lambda\theta\alpha' + (gf)'\phi\alpha' = \gamma h'\alpha' .$$

Define $\sigma^*: G \to Z$ by $\alpha'\sigma^* = \varepsilon$ and $\beta\sigma^* = 0$. This is ok because Z is a pullback and $\lambda\theta\varepsilon = (\gamma h' - (gf)'\phi)\varepsilon = \gamma\nu - (gf)'\rho = 0$ and λ is a kernel so $\theta\varepsilon = 0$. Since $\rho\sigma\sigma^* = (g\beta + \phi\alpha')\sigma^* = \phi\varepsilon = \rho$ and $\nu\sigma\sigma^* = h'\alpha'\sigma^* = h'\varepsilon = \nu$, we have $\sigma\sigma^* = 1$.

Suppose $\xi: G \to W$ and $\xi\mu = 0$. Then $\nu\sigma\alpha = h'\alpha'\alpha = 0$ and $\rho\sigma\alpha = g\beta\alpha = gf$, so $\sigma\alpha = \mu$ so $\xi\sigma\alpha = 0$ so $\xi\sigma = \tau\alpha'$ for some $\tau: Y \to W$. Define $\beta^*: B \to Z$ by $\alpha'\beta^* = -h$ and $\beta\beta^* = 1$. This is ok because $-\theta h = f'$ since $\lambda f' = (gf)'g = -\lambda\theta h$ and λ is a kernel. So $\tau h = -\tau\alpha'\beta^* = -\xi\sigma\beta^*$. Now $\nu\sigma\beta^* = h'\alpha'\beta^* = 0$ and $\rho\sigma\beta^* = g - \phi h = 0$, so $\sigma\beta^* = 0$ so $\tau h = 0$ so $\tau = \delta h'$ for some $\delta: M \to W$. It remains to show that $\delta\nu = \xi$. But $\delta\nu\sigma = \delta h'\alpha' = \tau\alpha' = \xi\sigma$, so $\delta\nu = \delta\nu\sigma\sigma^* = \xi\sigma\sigma^* = \xi$. The map δ is unique because ν is epic since $\nu\sigma = h'\alpha'$.

We have the following converse to Theorem 9.

THEOREM 10. *If fg is a* (stable) *kernel, and f is a semi-stable kernel, then g is a* (stable) *kernel.*

Proof. Consider the following diagram where f', g', and $(fg)'$ are the cokernels of f, g, and fg.

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{g'} & B/A \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\lambda} \\ A & \xrightarrow{fg} & C & \xrightarrow{(fg)'} & C/A \\ & & \downarrow{\scriptstyle f'} & \swarrow{\scriptstyle\mu} & \\ & & C/B . & & \end{array}$$

We shall show that the upper right square is a pullback, which proves the theorem by Theorems 4 and 5. Suppose $\lambda\phi = (fg)'\psi$. Then $0 = \mu\lambda\phi = \mu(fg)'\psi = f'\psi$, so $\psi = f\theta$ for a unique θ. We must show that $g'\theta = \phi$. But $\lambda g'\theta = (fg)'f\theta = (fg)'\psi = \lambda\phi$ and λ is a kernel since f is a semi-stable kernel. Thus $g'\theta = \phi$ and B is a pullback.

6. The 3×3 lemma. The proof of the 3×3 lemma is another exercise in translating element chases to pullbacks.

THEOREM 11. *If in the commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & B/A \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mu} \\ C & \xrightarrow{g} & D & \xrightarrow{g'} & D/C \\ \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\lambda} \\ C/A & \xrightarrow{h} & D/B & \xrightarrow{h'} & G \end{array}$$

the rows and the first two columns are stable exact, then the last column is stable exact.

Proof. We first show that $\lambda = \operatorname{coker} \mu$. Now $\lambda g' = h'\beta'$ is epic, so λ is epic. Suppose $\xi\mu = 0$. Then $\xi\mu f' = \xi g'\beta = 0$ so $\xi g' = \phi\beta'$ for some ϕ since $\beta' = \operatorname{coker} \beta$. Then $\phi h\alpha' = \phi\beta' g = \xi g' g = 0$ so $\phi h = 0$ so $\phi = \psi h'$ for some ψ since h' is the cokernel of h. We must show that $\psi\lambda = \xi$. But $\psi\lambda g' = \psi h'\beta' = \phi\beta' = \xi g'$ and g' is epic. So $\lambda = \operatorname{coker} \mu$.

Next we show that μ is monic by showing that $\ker \mu = 0$. Consider the pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho} & K \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\ker \mu} \\ B & \xrightarrow{f'} & B/A \end{array} \quad .$$

Since ρ is epic, because f' is stable, it sufficies to show that $(\ker \mu)\rho = f'\sigma = 0$. Now $g'\beta\sigma = \mu f'\sigma = 0$ so $\beta\sigma = g\tau$ for some $\tau: X \to C$. Thus $h\alpha'\tau = \beta' g\tau = \beta'\beta\sigma = 0$ so $\alpha'\tau = 0$ so $\tau = \alpha\theta$ for some $\theta: X \to A$. Hence $\beta\sigma = g\tau = g\alpha\theta = \beta f\theta$ which implies, since β is monic, that $\sigma = f\theta$, whereupon $f'\sigma = f'f\theta = 0$ and we have shown that μ is monic.

To show that $\mu = \ker \lambda$, suppose $\xi\lambda = 0$ and consider the pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\sigma} \\ D & \xrightarrow{g'} & D/C\,. \end{array}$$

We have $h'\beta'\sigma = \lambda g'\sigma = \lambda\xi\rho = 0$ so $\beta'\sigma = h\delta$ for some $\delta\colon Y \to C/A$. Consider the pullback diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & Y \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\delta} \\ C & \xrightarrow{\alpha'} & C/A\,. \end{array}$$

We have $\beta' g\varepsilon = h\alpha'\varepsilon = h\delta\tau = \beta'\sigma\tau$, so $\beta'(\sigma\tau - g\varepsilon) = 0$ so $\sigma\tau - g\varepsilon = \beta\phi$ for some $\phi\colon Z \to B$. Now $\mu f'\phi = g'\beta\phi = g'\sigma\tau$ kills $\ker\tau$ and μ is monic, so $f'\phi = \nu\tau$ for some $\nu\colon Y \to B/A$ since τ is the pullback of the stable cokernel α'. Similarly $\mu\nu\tau = \mu f'\phi = g'\beta\phi = g'\sigma\tau$, so $\mu\nu = g'\sigma = \xi\rho$ kills $\ker\rho$, so $\nu = \eta\rho$ for some $\eta\colon X \to B/A$ since ρ is a pullback of the stable cokernel g'. But $\mu\eta\rho = \mu\nu = \xi\rho$ and ρ is epic, so $\mu\eta = \xi$, and since μ is monic, η is unique. This completes the proof that the last column is exact. Finally, $\lambda g' = h'\beta'$ is a stable cokernel by Theorem 9 (or rather its dual) and hence λ is a stable cokernel by the dual of Theorem 10.

7. The long exact Ext sequences. We can define the groups $\mathrm{Ext}^n(C, A)$ for $n = 0, 1$ by letting $\mathrm{Ext}^0(C, A) = \mathrm{Hom}(C, A)$ and $\mathrm{Ext}^1(C, A) = \mathrm{Ext}(C, A)$. The elements of $\mathrm{Ext}^n(C, A)$ for $n > 1$ are Yoneda composites

$$E_n \circ E_{n-1} \circ \cdots \circ E_1$$

where $E_i \in \mathrm{Ext}(X_i, X_{i+1})$ and $X_1 = C$ and $X_{n+1} = A$ subject to the usual equivalence relation (see [3, pp. 82–87]). There are no further problems in pursuing the usual treatment of "Ext without projectives." In particular the proof of Theorem 5.1 in [3] goes through giving us

THEOREM 12. *Let E be a stable exact sequence $A \to B \to C$ in a pre-abelian category, and let G be any object in that category. Then we get an exact sequence of abelian groups*

$$\mathrm{Ext}^{n-1}(B, G) \to \mathrm{Ext}^{n-1}(A, G) \to \mathrm{Ext}^n(C, G) \to \mathrm{Ext}^n(B, G) \to \mathrm{Ext}^n(A, G)$$

for $n = 1, 2, \cdots$.

8. Relative theories. Quite often in abelian categories we are more interested in a relative homological algebra than in the absolute one. For example we may wish to study purity in abelian groups. The same holds true in pre-abelian categories, as we shall see in the case of topological abelian groups and valuated abelian groups. Instead of considering all sequences in $\operatorname{Ext}(C, A)$, we select a subset $\operatorname{Pext}(C, A)$ of *proper* exact sequences such that

1. If $E \in \operatorname{Pext}(C, A)$ and $\alpha: A \to A'$, then $\alpha E \in \operatorname{Pext}(C, A')$.
2. If $E \in \operatorname{Pext}(C, A)$ and $\beta: C' \to C$, then $E\beta \in \operatorname{Pext}(C', A)$.
3. If $E_1 \in \operatorname{Pext}(C_1, A_1)$ and $E_2 \in \operatorname{Pext}(C_2, A_2)$, then $E_1 \oplus E_2$ is in $\operatorname{Pext}(C_1 \oplus C_2, A_1 \oplus A_2)$.
4. If $A \subseteqq B \to B/A$ and $B \subseteqq C \to C/B$ are proper exact, then $A \subseteqq C \to C/A$ is proper exact.
5. If $\operatorname{Ker} f \to A \xrightarrow{f} B$ and $\operatorname{Ker} g \to B \xrightarrow{g} C$ are proper exact, then $\operatorname{Ker} gf \to A \xrightarrow{gf} C$ is proper exact.

A kernel (cokernel) is called *proper* if it is the kernel (cokernel) of a proper exact sequence. We could, equivalently, write down properties that must be satisfied by proper kernels (or proper cokernels). Note that 4 says that the product of proper kernels is a proper kernel, and 5 says that the product of proper cokernels is a proper cokernel. These five properties imply that $\operatorname{Pext}(C, A)$ is a subgroup of $\operatorname{Ext}(C, A)$. Actually, property 3 is redundant and can be derived from 1 and 4, or 2 and 5, as in the proof of Theorem 8. More useful is a remarkable theorem proved by Nunke [2] in abelian categories, that properties 4 and 5 are equivalent. The proof for pre-abelian categories follows Nunke's argument.

THEOREM 13. *If products of proper cokernels are proper, then products of proper kernels are proper.*

Proof. Suppose $A \xrightarrow{f} B \xrightarrow{f'} B/A$ and $B \xrightarrow{g} C \xrightarrow{g'} C/B$ are proper exact. We wish to show that the exact sequence $A \xrightarrow{gf} C \xrightarrow{(gf)'} C/A$ is proper. The map $h: B/A \to C/A$ is a (proper) kernel since it is a pushout of $g: B \to C$. Let ι_i and π_i be the injection and projection on the i^{th} factor of $B/A \oplus C$ and consider the diagram

$$\begin{array}{ccc} B/A \oplus C & \xrightarrow{\nabla(h \oplus (gf)')} & C/A \\ \Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle k} \\ C & \xrightarrow{\quad g' \quad} & C/B \end{array}$$

which commutes since

$$k\nabla(h \oplus (gf)') = \nabla(k \oplus k)(h \oplus (gf)') = \nabla(kh \oplus k(gf)') = \nabla(0 \oplus g') = g'\pi_2 .$$

We shall show that the diagram is a pullback. Suppose $g'\phi = k\psi$. If we want a map λ into $B/A \oplus C$ such that $\pi_2\lambda = \phi$ and $\nabla(h \oplus (gf)')\lambda = \psi$, then we must have

$$\psi = \nabla(h \oplus (gf)')(\iota_1\pi_1\lambda + \iota_2\phi) = h\pi_1\lambda + (gf)'\phi$$

so we need $h\pi_1\lambda = \psi - (gf)'\phi$. But $k(\psi - (gf)'\phi) = k\psi - g'\phi = 0$ so such a map $\pi_1\lambda$ exists and is unique. Conversely, the same calculations show that if we define λ by $\pi_2\lambda = \phi$ and $h\pi_1\lambda = \psi - (gf)'\phi$, then $\pi_2\lambda = \phi$ and $\nabla(h \oplus (gf)')\lambda = \psi$. Thus the diagram is a pullback.

The map $(f' \oplus 1): B \oplus C \to B/A \oplus C$ is a proper cokernel since it is a pullback of f'. Hence the map $B \oplus C \to C/A$ given by

$$\nabla(h \oplus (gf)')(f' \oplus 1) = \nabla(hf' \oplus (gf)') = (gf)'\nabla(g \oplus 1)$$

is a proper cokernel, so $(gf)'$ is a proper cokernel (it is an easy consequence of property 1 and the five lemma that if $\alpha\beta$ is a proper cokernel then so is α).

The same comments made in the last section apply here, and we get Theorem 12 with E a proper exact sequence and Ext^n replaced by Pext^n.

9. Pre-abelian categories from radicals. One way pre-abelian categories arise is as full subcategories of abelian categories. Recall that a *radical* R in an abelian category is a functorial subobject such that $R(A/R(A)) = 0$ for every object A.

THEOREM 14. *If R is a radical in an abelian category $\mathscr{A}$, then the full subcategory $\mathscr{S}$ of objects A such that $R(A) = 0$ is a pre-abelian category in which every cokernel is semi-stable.*

Proof. If f is a map in $\mathscr{S}$, then any kernel of f in $\mathscr{A}$ is in $\mathscr{S}$ and serves as a kernel there. If $g: B \to C$ is a cokernel of f in $\mathscr{A}$, then hg is a cokernel of f in $\mathscr{S}$ where h is the map $C \to C/R(C)$. To see that cokernels are semi-stable it suffices to observe that g is a cokernel in $\mathscr{S}$ if and only if g is a cokernel in $\mathscr{A}$ whose domain and range are in $\mathscr{S}$, and that pullbacks in $\mathscr{S}$ are pullbacks when viewed in $\mathscr{A}$.

If we let R be the radical $R(A) = p^\omega A = \cap\, p^n A$ in the category of abelian groups $\mathscr{A}$ we get the pre-abelian category $\mathscr{S}$ of abelian groups with no elements of infinite p-height. The exact sequences $A \to B \to C$ in $\mathscr{S}$ are simply those sequences in $\mathscr{S}$ which are exact in $\mathscr{A}$. Since every cokernel is semi-stable, Ext will be determined if we characterize the semi-stable kernels.

THEOREM 15. *Let $\mathscr{S}$ be the category of abelian groups with no elements of infinite p-height. Let B be in $\mathscr{S}$ and let $A \subseteqq B$ be a subgroup. Then A is a semi-stable* (and hence stable) *kernel in $\mathscr{S}$ if and only if*

(i) *A is p-adically closed in B,* i.e., *B/A has no elements of infinite p-height, and*

(ii) *The p-adic topology on A is the relative topology induced by the p-adic topology on B,* i.e., *for each m there is an n such that $A \cap p^n B \subseteqq p^m A$.*

Proof. Suppose (i) and (ii) hold and consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteqq"] \arrow[d, "\alpha"'] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

in the category of abelian groups, where $A' \in \mathscr{S}$. It suffices to show that $B' \in \mathscr{S}$. Now

$$B' = (A' \oplus B)/K \quad \text{where} \quad K = \{(\alpha a, -a) : a \in A\} .$$

Suppose $(a', b) \in A' \oplus B$ represents an element of infinite p-height in B'. By (i) we may assume that b has maximum p-height in $b + A$. If $b \neq 0$, then the p-height of $(a' + \alpha a, b - a)$ is bounded, contradicting the assumption that (a', b) represents an element of infinite p-height in B'. Hence we may assume that $b = 0$. So there must exist elements $a_n \in A$ such that $(a' - \alpha a_n, a_n)$ has arbitrarily large height in $A' \oplus B$. Hence $a_n \to 0$ in B, and therefore also in A by (ii), while $\alpha a_n \to a'$ in A'. Thus $\alpha a_n \to 0$ and $\alpha a_n \to a'$, whence $a' = 0$.

Conversely, suppose A is a semi-stable kernel and $a_n \to 0$ in B. We must show that $a_n \to 0$ in A. Let $\alpha: A \to Z(p^m)$, where $Z(p^m)$ is the cyclic group of order p^m, and consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteqq"] \arrow[d, "\alpha"'] & B \arrow[d] \\ Z(p^m) \arrow[r] & B' . \end{tikzcd}$$

Then $\alpha(a_n) = 0$ eventually, for otherwise we would have $0 \neq t \in Z(p^m)$ such that $t = \alpha(a_n)$ for infinitely many n, giving t infinite height in B'. So a_n must be eventually in $p^m A$ (or we could find a bad α). Since this holds for arbitrary m, we have $a_n \to 0$ in A.

10. Topological abelian groups. Another way in which pre-abelian categories arise is by imposing additional structure on the objects of an abelian category. In the category of (Hausdorff) topological abelian groups the kernel of a map $f: A \to B$ is its group kernel, and the cokernel of f is gotten by dividing B by the closure of $f(A)$ and imposing the quotient topology. Thus kernels are precisely the closed subgroups and cokernels are the onto maps that are open (since the map onto a quotient group with the quotient topology is open).

THEOREM 16. *In the category of topological abelian groups every exact sequence is stable.*

Proof. Let $A \to B \to C$ be an exact sequence, let $\alpha: A \to A'$ and consider the diagram

$$\begin{array}{ccc} A & \subseteq & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A' & \longrightarrow & (A' \oplus B)/K \end{array}$$

where $K = \{(\alpha a, -a): a \in A\}$. To show that this is a pushout it suffices to prove that K is closed in $A' \oplus B$ and hence $(A' \oplus B)/K$, with the quotient topology from $A' \oplus B$, is Hausdorff. Suppose $(a', b) \notin K$. Then, if $b \notin A$, we have $(a', b) \in A' \oplus (B \backslash A)$ which is an open set disjoint from K. If $b \in A$, then $-\alpha b \neq a'$ so we can find disjoint open sets U' and V' in A' so that $a' \in U'$ and $-\alpha b \in V'$. Then $-\alpha^{-1}V' = A \cap V$ for some open set V in B. So (a', b) is in the open set $U' \oplus V$ which is disjoint from K for if $v \in V \cap A$ then $\alpha(-v) \in V'$ which is disjoint from U'.

To show that $A \subseteq B$ is a semi-stable kernel we must show that A' is imbedded homeomorphically as a closed subset of $(A' \oplus B)/K$. It is certainly mapped one-to-one and continuously into $(A' \oplus B)/K$. Moreover its image is the kernel of the induced map $(A' \oplus B)/K \to B/A$ so it is a closed subset. To show that the imbedding is a homeomorphism, suppose $U \subseteq A'$ is an open set containing zero. Choose an open set $V \subseteq A'$ such that $V + V \subseteq U$ and $0 \in V$. Then $\alpha^{-1}V$ is open is A and so $\alpha^{-1}V = A \cap W$ for some open set W in B. Now $(V \oplus W)/K$ is an open set in $(A' \oplus B)/K$ containing zero. We claim that $A' \cap ((V \oplus W)/K) \subseteq U$. In fact if $(v, w) + (\alpha a, -a) = (v + \alpha a, 0)$, then $w = a$ so $\alpha a \in V$ so $v + \alpha a \in U$.

To show that the cokernel $\theta: B \to C$ is semi-stable, let $\gamma: C' \to C$ and consider the pullback diagram

$$\begin{array}{ccc} B & \xrightarrow{\theta} & C \\ \uparrow & & \uparrow{\scriptstyle\gamma} \\ X & \xrightarrow[\phi]{} & C' \end{array}$$

where $X = \{(b, c') \in B \oplus C' : \theta b = \gamma c'\}$. To show that ϕ is an open map, suppose $U \subseteqq X$ is an open set containing zero. Then there exist open sets $V \subseteqq B$ and $W \subseteqq C'$ containing zero such that $(V \oplus W) \cap X \subseteqq U$. We may assume that $\gamma W \subseteqq \theta V$ since θV is open. Thus $W \subseteqq \phi U$.

The locally compact abelian groups form a full subcategory of the topological abelian groups that is closed under kernels, cokernels, and extensions. Hence the exact sequences in this category are also all stable and of the form $A \subseteqq B \rightarrow B/A$ where A is closed in B and $B \rightarrow B/A$ is an open map. Moskowitz [4] studied this set-up, determining the projectives and injectives. With these he defined Ext via resolutions for those groups that had them, which are very few. The theory we present defines $\operatorname{Ext}^n(C, A)$ in a natural way for any pair of locally compact abelian groups and any n. It is an easy consequence of the exact sequence of Ext's that this agrees with the resolution definition whenever resolutions exist.

Back in the category of topological abelian groups we obtain a relative theory if we consider the class of exact sequences $A \rightarrow B \rightarrow C$ which split as topological spaces in the sense that the map $B \rightarrow C$ admits a continuous cross-section. It is easily checked that these sequences form a proper class. This is the natural setting for free topological abelian groups which are projective here and allow one to define the relative Ext^n in terms of projective resolutions. Nummela [1] uses these resolutions to determine that the relative projective dimension of a compact abelian group is 1.

11. Valuated abelian p-groups. This was the motivating example for the theory presented in this paper. Let p be a fixed prime. Then a *valuation* v on an abelian p-group G assigns to each $x \in G$ an ordinal number vx, or the symbol ∞, such that

(1) $v(x + y) \geqq \min(vx, vy)$

(2) $vpx > vx$

(3) $vux = vx$ if p does not divide u.

Such objects arise naturally in abelian group theory as subgroups $G \subseteqq H$ where v is the height function on H restricted to G. The category of valuated abelian groups will be the subject of a forthcoming paper by the authors. We state here, without proof, some of the results of that paper.

(1) A map $A \rightarrow B$ is a stable kernel if and only if A is a subgroup of B with the induced valuation, and every coset of A contains an element of maximum value.

(2) A map $B \rightarrow C$ is a semi-stable cokernel if and only if B maps onto C and every element c of C comes from an element in B whose value is arbitrarily close to the value of c (in particular if

the value of c is ∞ or a nonlimit, then it must come from an element of the same value).

The category of valuated p-groups provides an example of the failure of the five lemma for exact sequences. Let G be a p-group such that $p^{\omega}G$ is not divisible, and let B be a basic subgroup of G. Let H be the valuated group gotten from G by setting vx equal to the height of x if $x \notin p^{\omega}G$ and $vx = \infty$ if $x \in p^{\omega}G$. Then we have the five lemma set-up

$$\begin{array}{ccccc} B & \longrightarrow & H & \longrightarrow & D \\ \| & & \uparrow & & \| \\ B & \longrightarrow & G & \longrightarrow & D \end{array}$$

where D is the divisible group G/B. But G and H are clearly not isomorphic as valuated groups.

References

1. Eric C. Nummela, *The progective dimension of a compact abelian group*, Proc. Amer. Math. Soc., **38** (1973), 452–456.
2. R. J. Nunke, *Purity and Subfunctors of the Identity, Topics in Abelian Groups*, Scott, Foresman and Co., 1963.
3. Saunders MacLane, *Homology*, Academic Press, 1963.
4. Martin Moskowitz, *Homological algebra in locally compact abelian groups*, Trans. Amer. Math. Soc., **127** (1967), 361–404.

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