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FIXED POINT THEOREMS FOR MAPPINGS WITH A CONTRACTIVE ITERATE

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Several fixed point theorems are proved for metric-space mappings which satisfy a contractive condition involving an iterate of the mapping, where the iterate depends on the point in the space.

Let (X, d) denote a complete metric space. In [3] the second author has established fixed point theorems for mappings which satisfy a variety of contractive conditions. The common property of the mappings discussed in [3] is that the fixed point is unique, and can be found by using repeated iteration, beginning with some initial choice $x_0 \in X$.

The first result in this direction is that of V. M. Sehgal [5] who proved the following.

THEOREM S1. *Let (X, d) be a complete metric space, T a continuous self-mapping of X which satisfies the condition that there exists a real number k, $0 < k < 1$ such that, for each $x \in X$ there exists a positive integer $n(x)$ such that, for each $y \in X$,*

$$(1) \qquad d(T^{n(x)}(x)) \,, \quad T^{n(x)}(y) \leqq kd(x, y) \,.$$

Then T has a unique fixed point in X.

L. F. Guseman, Jr. [1], extended Sehgal's result by removing the condition of continuity of T and weakening (1) to hold on some subset B of X such that $T(B) \subset B$, where, for some $x_0 \in B$, B contains the closure of the iterates of x_0. Further extensions for a single mapping appear in [2] and in [4].

We shall be concerned with a pair of mappings which satisfy the following contractive condition.

Let T_1, T_2 be self-mappings of X such that there exists a constant k, $0 < k < 1$ such that there exist positive integers $n(x)$, $m(y)$ such that for each $x, y \in X$,

$$(2) \qquad \begin{aligned} d(T_1^{n(x)}(x) \,, \quad T_2^{m(y)}(y)) \leqq k \max \{d(x, y) \,, \quad d(x, T_1^{n(x)}(x)) \,, \\ d(y, T_2^{m(y)}(y)) \,, \quad [d(x, T_2^{m(y)}(y)) + d(y, T_1^{n(x)}(x))]/2\} \,. \end{aligned}$$

THEOREM 1. *Let T_1, T_2 be self-mappings of a complete metric space (X, d) which satisfy (2). Then T_1 and T_2 have a unique common fixed point.*

Proof. Let $x_0 \in X$, and define the sequence $\{x_n\}$ by $x_1 = T_1^{n(x_0)}(x_0)$, $x_2 = T_2^{m(x_1)}(x_1), \cdots, x_{2n+1} = T_1^{n(x_{2n})}(x_{2n})$, $x_{2n+2} = T_2^{m(x_{2n+1})}(x_{2n+1}), \cdots$. Using (2) and assuming $x_m \neq x_n$ for each $m \neq n$,

$$(3) \qquad d(x_{2n+1}, x_{2n+2}) \leqq k \max \{d(x_{2n}, x_{2n+1}) , \\ d(x_{2n+1}, x_{2n+2}) , \quad d(x_{2n}, x_{2n+2})/2\} .$$

If the maximum of the right-hand side of (3) is $d(x_{2n}, x_{2n+2})/2$ then we obtain the contradiction $d(x_{2n+1}, x_{2n+2}) \leqq kd(x_{2n+1}, x_{2n+2})$. Therefore, $d(x_{2n+1}, x_{2n+2}) \leqq kd(x_{2n}, x_{2n+1})$. Similarly $d(x_{2n}, x_{2n+1}) \leqq kd(x_{2n-1}, x_{2n})$, so that $d(x_{2n+1}, x_{2n+2}) \leqq k^{2n}d(x_1, x_2)$ and $d(x_{2n}, x_{2n+1}) \leqq k^{2n}d(x_0, x_1)$. With $r(x_0) = \max \{d(x_0, x_1), d(x_1, x_2)\}$, for any $m > n$,

$$d(x_m, x_n) \leqq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqq k^{2n}r(x_0)/(1 - k^2) .$$

Thus $\{x_n\}$ is Cauchy and hence convergent. Call the limit p.

From (2),

$$(4) \qquad d(x_{2n+1}, T_2^{m(p)}(p)) \leqq k \max \{d(x_{2n}, p) , \quad d(x_{2n}, x_{2n+1}) , \\ d(p, T_2^{m(p)}(p)) , \quad [d(x_{2n}, T_2^{m(p)}(p)) + d(p, x_{2n+1})]/2\} .$$

Taking the limit of (4) as $n \to \infty$ we obtain $d(p, T_2^{m(p)}(p)) \leqq k \max \{0, 0, d(p, T_2^{m(p)}(p)), d(p, T_2^{m(p)}(p))/2\}$, which implies $p = T_2^{m(p)}(p)$. Similarly $p = T_1^{n(p)}(p)$.

Suppose q is also a periodic point of T_1 and T_2; i.e., $q = T_1^{n(q)}(q) = T_2^{m(q)}(q)$. From (2),

$$d(p, q) = d(T_1^{n(p)}(p) , \quad T_2^{m(q)}(q) \leqq k \max \{d(p, q), 0, d(q, p)\} ,$$

which implies $p = q$. The condition $p = T_1^{n(p)}(p)$ implies $T_1(p) = T_1^{n(p)}(T_1(p))$, so that $T_1(p)$ is also a periodic point of T_1. From the uniqueness of p, $p = T_1(p)$. Similarly $T_2(p) = p$.

COROLLARY 1. *Let T be a self-mapping of X such that there exists a positive real number k, $0 < k < 1$ such that, for each x, $y \in X$ there exists a positive integer $n(x)$ such that*

$$d(T^{n(x)}(x) , \quad T^{n(y)}(y)) \leqq k \max \{d(x, y) , \quad d(x, T^{n(x)}(x)) , \\ d(y, T^{n(y)}(y)) , \quad [d(x, T^{n(y)}(y)) + d(y, T^{n(x)}(x)]/2\} .$$

Then T has a unique fixed point in X.

Proof. In Theorem 1 set $T_1 = T_2$, $m(y) = n(y)$.

COROLLARY 2. *Let $\{f_k\}$ be a sequence of self-mappings of X*

satisfying

$$d(f_i^{n(x)}(x)\,,\quad f_j^{n(y)}(y)) \leqq k \max \{d(x, y)\,,\quad d(x, f_i^{n(x)}(x))\,,\\ d(y, f_j^{n(y)}(y))\,,\quad [d(x, f_j^{n(y)}(y)) + d(y, f_j^{n(x)}(x))]\}$$

for each $x,\ y \in X$, *each* $i,\ j = 1, 2, \cdots$. *Then there exists a unique common fixed point.*

THEOREM 2. *Let* $\{f_k\}$ *be a sequence of continuous functions satisfying: there exists a positive constant* k, $0 < k < 1$ *such that for each* $x,\ y \in X$ *there exists a positive integer* $n(x)$ *such that*

$$(5)\qquad d(f_k^{n(x)}(x)\,,\quad f_k^{n(x)}(y)) \leqq k \max \{d(x, y)\,,\quad d(x, f_k^{n(x)}(x))\,,\\ d(y, f_k^{n(x)}(y))\,,\quad [d(x, f_k^{n(x)}(y)) + d(y, f_k^{n(x)}(x))]/2\}\,.$$

Suppose $\{f_k\}$ *tends pointwise to a continuous function* f. *Then* f *has a unique fixed point* p *and* $p_k \longrightarrow p$, *where the* p_k *are the unique fixed points of* f_k.

Proof. In (5) take the limit as $k \longrightarrow \infty$ and use the continuity of f, f_k, and d to obtain the result that f satisfies (5). From Corollary 1 f has a unique fixed point p. $d(p_k, p) = d(f_k^{n(p_k)}(p_k), f^{n(p_k)}(p)) \leqq d(f_k^{n(p_k)}(p_k), f_k^{n(p_k)}(p)) + d(f_k^{n(p_k)}(p), f^{n(p_k)}(p))$. From (5),

$$d(f_k^{n(p_k)}(p_k)\,,\quad f_k^{n(p_k)}(p)) \leqq h \max \{d(p_k, p)\,,\quad d(p, f_k^{n(p_k)}(p))\}\,.$$

Therefore, $d(p_k, p) \leqq (1 - h)^{-1} d(f_k^{n(p_k)}(p), p)$, which tends to zero as $k \longrightarrow \infty$.

REMARKS. 1. In each of the results of this paper one can obviously weaken the contractive condition by replacing X with a subset B which is invariant under the mappings involved and which contains the closure of all of the iterates of some $x_0 \in B$.

2. Corollary 1 is a generalization of [4], which in turn generalizes [1], [2] and [5].

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