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CONSTRUCTIVE FOUNDATIONS OF POTENTIAL THEORY

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Classical potential theory is studied in the constructive framework. Green's functions are constructed for a large family of open regions.

0. Introduction. A constructive (as opposed to idealistic) theory is one in which every theorem is an explicit or implicit assertion about some computations. For example, an existence theorem gives a (general but not necessarily efficient) routine for the construction.

From this point of view, certain classical theorems become less meaningful, as they assert the existence of mathematical objects without providing means of their construction. An example in potential theory is the Perron-Wiener-Brelot method of obtaining Dirichlet solutions. Solutions for the Dirichlet problem are proved to exist as the infimum of a certain family of superharmonic functions associated to the boundary function. As the infimum of an infinite family cannot in general be calculated in finitely many steps, Perron-Wiener-Brelot's theorem, that the infimum (when certain conditions are satisfied) is the solution, has a less interesting constructive interpretation, namely if we can construct the solution, then it is the infimum of a certain family of functions—the wrong direction as far as computation of the solution is concerned.

In this article we attempt to examine classical potential theory from the constructive standpoint. The first step, also the harder one, is to give precise computational meaning to the basic notions. For instance, what kind of computations do we perform with a superharmonic function? We believe the answer to be its averages on balls rather than its evaluation at all points in an open set. Accordingly, a superharmonic function is defined as an integrable function (with certain properties) without the requirement that it be everywhere defined. (The reader familiar with the literature in constructive mathematics realizes, of course, that *everywhere* defined functions on E^d which are not also continuous have not yet been encountered). Thus the measure-theoretic approach is adopted to replace the usual pointwise approach. For example, convergence for superharmonic functions is always L_1-convergence, rather than the usual pointwise convergence. L_1-convergence also furnishes convenient numerical measures of rates of convergence. Thus we are able to talk about convergence in a constructive sense.

The main objective of this article, beyond a constructive formulation of potential theory, is the construction of Green functions for a large class of regions. The generalized Dirichlet solutions can thus be obtained for these regions.

Although the definitions and the statements of theorems will be explained in detail, many proofs are left out because, with the outline provided, the patient reader will find it easy to obtain constructive proofs by modifying the usual classical proofs. We will give only the proofs which are substantially different from their classical counterparts. Moreover, classical theorems whose constructive intent is clear, such as Herglotz's theorem, (see Helms 1969), will be left out entirely. The reader can amuse himself in finding the more or less routine constructive proofs. Also, there are many theorems which are already constructive, e.g., Harnack's inequality for harmonic functions. We will use them without hesitation, and refer the reader to classical references. The constructive measure theory of Bishop (Bishop 1967, Bishop and Cheng 1972) will also be used without further comment.

1. **Superharmonic functions.** Let R be a nonempty open subset of $E^d (d \geqq 2)$, equipped with the Lebesgue measure. A measurable function u on R is said to be locally integrable if it is integrable on every ball well contained in R. (A subset K is well contained in R if some metric neighborhood of K is contained in R. In symbols $K \Subset R$). Local L_1-convergence will mean L_1-convergence on all such balls.

Write B_{xr} for the closed ball in E^d with center x and radius r $(r > 0)$, and write R_r for $\{x \in R: B_{xr} \Subset R\}$. If u is locally integrable on R, write u_r for the continuous function on R_r whose value at x is the average of u over B_{xr}. Our attention will be centered upon those properties of u which are related to these averages. For this reason, we assume in the following that every locally integrable function u has been "regularized" by redefining $u(x) = \lim_{r \to 0} u_r(x)$ with the domain of u being those x where the limit exists in $[-\infty, \infty]$. For $x \in R_r$, the average of u on ∂B_{xr} may not be defined. When it is defined, we denote it by $u^r(x)$.

DEFINITION 1.1. A locally integrable function u on R is said to be superharmonic if for all $r > 0$ we have $u \geqq u_r$ a.e. on R_r; harmonic if $\geqq$ is replaced by $=$. A measurable function u on R is said to be lower semi-continuous if it is the a.e. limit of an increasing sequence of continuous functions on R.

We have defined lower semi-continuity as a measure-theoretic property rather than a topological one. The term lower semi-continuity is retained only becanse of the lack of a better name.

It can trivially be verified that a superharmonic function is lower semi-continuous, and that sums, minimums, positive constant multiples, as well as local L_1-limits of superharmonic functions are again superharmonic. Later we will prove that the a.e. limit of nonnegative superharmonic functions is also superharmonic. In case u_n is the nth partial sum of a series of nonnegative superharmonic functions, the a.e. limit will also be the local L_1-limit, hence obviously superharmonic.

We next give the constructive version of the classical minimal principle. Suppose u is superharmonic on R and suppose K is a compact set well contained in R. Classically the infimum of u on K exists because of idealistic considerations, and is attained because of lower semi-continuity, the latter also coupled with idealistic arguments. However, there is in general no way to compute this infimum. To see this, construct a superharmonic function which is discontinuous at one point. More specifically, let x_n be a sequence in E^3 converging to 0, but each unequal to 0. We will see later that the functions $|x - x_n|^{-1}$ are superharmonic on E^3. Borrowing this fact, let c_n be a sequence of positive numbers such that $\sum c_n |x_n|^{-1}$ converges. $\sum c_n |x - x_n|^{-1}$ converges a.e. and is therefore superharmonic in x. Now let y be a point in E^3 for which we are unable to decide whether (a) $y \neq 0$ or (b) $y \neq x_n$ for all n. Let $K = \{y\}$. If we could compute the infimum of $u = \sum c_n |\cdot - x_n|^{-1}$ on K, then either the infimum would be $< \sum c_n |x_n|^{-1} + 1$, in which case we would have $y \neq x_n$ for all n, or the infimum would be $> \sum c_n |x_n|^{-1}$ in which case we would know $y \neq 0$. Such a counter-example in the style of Brouwer can be modified to show that the infimum can also exist on K without being attained. For these reasons, superharmonic functions will be characterized by a lower bound principle, rather than the classical minimal principle.

Let u be a lower semi-continuous function on R. We say that u satisfies the lower bound principle on R if, for every positive number a, for every harmonic function h on R, and for every compact set K well contained in R, there exists a positive-measured subset A of $R - K$ such that $(u - h)(A) - a \leqq (u - h)(K)$ a.e. (i.e., for almost every $x \in K$ and a.e. $y \in A$ we have $u(y) - h(y) - a \leqq u(x) - h(x)$). In the cases of interest, u is locally integrable, and the lower bound principle is satisfied on R iff, given a, h, and K, there exists $y \in R - K$ such that $u(y) - h(y) - a \leqq (u - h)(K)$. (Recall that u is regularized.)

THEOREM 1.2. *Let u be a locally integrable, lower semi-continuous function on R such that $u(x) < u_r(x)$ for some $x \in R_r (r > 0)$. Then there exists an open set R', compact set K, harmonic function*

h on R', such that $x \in K \Subset R' \subset B_{xr}$ and such that for some $a > 0$ we have $u \wedge h - h - a > u(x) - h(x)$ on $R' - K$.

The integrability condition will later be dropped. Thus a lower semi-continuous function u on R is superharmonic if it satisfies the lower bound principle on every open subset of R. The converse follows (although not immediately) from the next theorem.

THEOREM 1.3. *Let u be a locally integrable function on R and let K be a compact subset well contained in R. Suppose $u(x) < u - a$ on $R - K$ for some $x \in K$ and $a > 0$. Then there exists an arbitrarily small $r > 0$ and $z \in R_r$ such that $u(z) < u_r(z)$.*

The characterization of superharmonic functions by the lower bound principle leads, as in the classical theory, to a third definition of superharmonic functions: a locally integrable function u on R is superharmonic iff $u_r \geqq u_s$ on R_s whenever $0 < r < s$.

We next note that, although a superharmonic function is defined only a.e. on R, the averages $u^r(x)$ are defined and continuous on R_r. First a real variable lemma which most likely is known although we fail to locate it in the literature.

LEEMA 1.4. *Let φ and ψ be concave functions on the interval (a, b) with finite L_1-norms* (i.e., $||\varphi||_1 \equiv \int_a^b |\varphi|$ *exists and similarly for* ψ). *Then on any proper subinterval $[a + h, b - h]$ we have*

$$\max_{[a+h,b-h]} |\varphi - \psi| \leqq 5h^{-1}(||\varphi||_1 + ||\psi||_1)^{1/2}||\varphi - \psi||_1^{1/2} .$$

Using this lemma, we are able to prove the following theorem, parts (i) and (ii) of which are of course well known.

THEOREM 1.5. *Let u and v be superharmonic functions on R and let $y \in R_s$. Then the following hold.*

(i) *For every $t \in (0, s)$ we have $u_r \uparrow u$ in $L_1(\partial B_{yt})$ as $r \downarrow 0$; in particular $u^t(y)$ is defined.*

(ii) *$u^t(y)$ is a concave function of t^{-d+2}* (if the dimension d is >2) *or of $-\log t$* (if $d = 2$).

(iii) *Suppose $[r + \alpha, s - \alpha]$ is a proper subinterval of (r, s) where $0 < r < s$. Then for $t \in [r + \alpha, s - \alpha]$, we have*

$$|u - v|^t(y(\leqq c_d r^{-2d+2} s^{d-1} \alpha^{-1} \left(\int_{B_{ys}} |u| + \int_{B_{ys}} |v| \right)^{1/2} \left(\int_{B_{ys} - B_{yr}} |u - v| \right)^{1/2}$$

where c_d is a constant depending only on d.

Following Helms (1969) we let $PI(\mu, B)$ denote the Poisson integral

$$\frac{1}{\sigma_d r}\int_{\partial B}\frac{r^2 - |y - x|^2}{|z - x|^d}d\mu(z)$$

where μ is a measure on the boundary ∂B of a ball $B = B_{yr}$, and where σ_d is the total surface area of the unit sphere B_{y1}. If μ has a density f relative to the surface area measure, we also write $PI(f, B)$ for $PI(\mu, B)$. The reader is referred to Helms (1969) for basic facts about $PI(\mu, B)$. The previous theorem shows that if u is superharmonic on R and if $B_{yr} \Subset R$, then $PI(u, B)$ is defined. Using Theorem 1.2, it can be shown that if v is defined to be u on $R - B$, to be $PI(u, B)$ on B, then v is also superharmonic and $u \geqq v$.

2. Green functions. With the lower bound principle characterization of superharmonic functions, it is easy to show that for $x \in E^d$, the function $u_x(y) = |y - x|^{-d+2}$ if $d \geqq 3$, $u_x(y) = -\log|y - x|$ if $d = 2$, is superharmonic an E^d. We will sometimes write $U(x, y)$ for $u_x(y)$.

DEFINITION 2.1. A harmonic function h on R is called the greatest harmonic minorant of a superharmonic function u on R if $h \leqq u$ on R and if for every $\varepsilon > 0$ and compact set $K \Subset R$ there exist an open set R' and compact set K' such that $K \Subset R' \subset K' \Subset R$ and such that for every harmonic function v on R' with $v \leqq u$ on R' we have $v \leqq h + \varepsilon$ on K.

Note the $\varepsilon - \delta$ form of our definition. Note also that h is unique.

DIFINITION 2.2. Suppose $U(x, \cdot)$ has a greatest harmonic minorant $H(x, \cdot)$ on R for every $x \in R$. Then $G(x, y) = U(x, y) - H(x, y)$ is called the Green function for R. We also write h_x for $H(x, \cdot)$.

By showing first that the families $\{H(\cdot, y)\}_{y \in K}$ and $\{H(x, \cdot)\}_{x \in K}$ are equicontinuous on any compact set $K \Subset R$, it is easy to prove that H is continuous on R if it exists. In particular $G: R \times R \to [0, \infty]$ is continuous in the extended sense.

It is well known that for the interior B of a ball B_{zr} in $E^d (d \geqq 3)$, the function $G_B(x, y) = U(x, y) - r^{d-2}|x - z|^{-d+2}|y - x^*|^{-d+2}$ is the Green function for B. Here x^* is the inverse of x relative to ∂B. The condition in 2.1 can be verified by using the fact that for a fixed $x \in B$ we have $G_B(x, y) \to 0$ as y approaches ∂B. Likewise $G_B(x, y) = \log|z - x||y - x^*|/r|y - x|$ is the Green function for B in E^d. Note that $G_B(x, y) = G_B(y, x)$.

Let R be an open set which has a Green function G. Let μ and

ν be measures on R and let m_B denote the uniform measure on a ball with positive radius, i.e., m_B is the Lebesgue measure on B divided by the volume $|B|$ of B. We adopt the convention of writing μf or $f\mu$ for $\int f(x)d\mu(x)$, writing μF for $\int F(x, \cdot)d\mu(x)$, writing $F\mu$ for $\int F(\cdot, y)d\mu(y)$, and $\mu F\nu$ for $\int F(x, y)d\mu \otimes \nu(x, y)$, provided the integrals exist.

Let μ be a measure on R such that for all $B \subset R$, the Green function G is integrable relative to $\mu \otimes m_B$. (This is always the case if μ has compact support.) We will call μG the potential of μ. The potential of μ is superharmonic on R, and is harmonic away from the support of μ. The Riesz decomposition theorem says that every superharmonic function can be decomposed on a ball B into a potential and a harmonic function. The constructive proof depends on the following continuity theorem.

THEOREM 2.3. *Let R be an open set with Green function G. Let f be a continuous function on R with compact support well contained in some integrable open set S of R. Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that for any measures μ_1, μ_2 whose potentials are defined, and for any harmonic function on R with $\int_S |\mu_1 G - \mu_2 G - h| < \delta$, we have $|\mu_1 f - \mu_2 f| < \varepsilon$.*

THEOREM 2.4 (Riesz decomposition). *Let u be a superharmonic function on the open set R. Let B be the interior of some ball B_{xr} well contained in R. Then there exists a measure μ on B such that $u = G\mu + PI(u, B)$ on B, where G is the Green function for B. Moreover μ is unique.*

In case u has continuous second partial derivatives, the above theorem follows at once from Green's identity

$$u = -\sigma_d^{-1}(d-2)^{-1}\int_B G(\cdot, z)\Delta u(z)dz + PI(u, B) ,$$

(say $d \geqq 3$). In general u can be approximated by such smooth superharmonic functions u_n. Theorem 2.3 (rather than the usual compactness argument) then helps the passage to the limit in $u_n = G\mu_n + PI(u_n, B)$.

The Riesz decomposition can be used to show that a family of nonnegative superharmonic functions bounded at the center of a ball is uniformly integrable on the ball. To be precise, suppose $B_{xr} \subset R$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all Lebesgue measurable subset A of B_{xr} with Lebesgue measure $m(A) < \delta$ and for all nonnegative superharmonic function u on R with $u(x) \leqq 1$ we have $\int_A u(y)dy < \varepsilon$. As a corollary, we see that the a.e. convergence of positive superharmonic functions implies L_1-convergence on balls, and so the limit must also be superharmonic. Theorem 1.2 can now be strengthened by dropping the local integrability requirement, for every lower semi-continuous function u is the a.e. limit of $u \wedge n (n \to \infty)$, the latter being locally integrable.

A subset Z of R is called polar if there exists a superharmonic function v on R such that $v(z) = +\infty$ for every $z \in Z$. (Recall that v is regularized so that this means $\lim_{r \to 0} v_r(z) = +\infty$ for every $z \in Z$.) Since v is locally integrable, a polar set is of zero Lebesgue measure. The next theorem, which is well known, says that polar sets are ignorable in a stronger sense. A constructive proof is presented here as it differs substantially from the classical compactness proof.

THEOREM 2.5. *Let Z be a compact polar set in R. Let u be a superharmonic function on the open set $R - Z$. If u is locally bounded from below, then u is also superharmonic on R.*

Proof. Replacing u by $u \wedge n$ if necessary, we may assume that u is bounded from above. Since u is locally bounded from below, it is locally integrable on R. Let v be any superharmonic function on R such that $v(z) = +\infty$ for every $z \in Z$. We will show that $v + u$ is superharmonic. Suppose $(v + u)(x_0) < (v + u)_{r_0}(x_0)$ for some $x_0 \in R_{r_0}$. Applying Theorem 1.2, we can find an open set R', compact set K, bounded harmonic function h_1 on R' such that $x_0 \in K \Subset R' \subset B_{x_0 r_0}$ and such that for some $a > 0$ we have

$$(v + u) \wedge h_1 - h_1 - a > (v + u)(x_0) - h_1(x_0) \geqq (v + u) \wedge h_1(x_0) - h_1(x_0)$$

a.e. on $R' - K$. So by Theorem 1.3, there exists $r_1 < r_0/2$ such that $((v + u) \wedge h_1)(x_1) < ((v + u) \wedge h_1)_{r_1}(x_1)$ for some $x_1 \in R'_{r_1}$. We may even assume that $((v + u) \wedge h_1)(x_1) = (v(x_1) + u(x_1)) \wedge h_1(x_1)$. (One should be careful here because all functions are assumed to be regularized.) Repeating the argument, we have for each $k = 1, 2, \cdots$

$$(v(x_k) + u(x_k)) \wedge h_1(x_k) \wedge \cdots \wedge h_k(x_k) < ((v + u) \wedge h_1 \wedge \cdots \wedge h_k)_{r_k}(x_i)$$

where each h_k is a bounded harmonic function on $R^{(k)} \subset B_{x_{k-1} r_{k-1}}$, where $x_k \in R^{(k)}$, and where $r_k < r_{k-1}/2$. Since the right hand side of the last displayed inequality is bounded by $(h_j)_{r_k}(x_k) = h_j(x_k)$, ($j = 1, \cdots, k$), the left hand side must equal $v(x_k) + u(x_k)$. In particular $v(x_k) + u(x_k) < h_1(x_k)$. Since both u and h_1 are bounded on $B_{x_0 r_0}$ there exists $M > 0$ such that $v(x_k) \leqq M$ for every k. Now $|x_k - x_{k-1}| < r_{k-1}$

and $\sum r_k < \infty$. Hence $x_k \to x$ for some x. Moreover, the last displayed inequality implies $(v + u)(x_k) < (v + u)_{r_k}(x_k)$ which in turn implies $d(x_k, Z) \leqq r_k$ since $v + u$ is superharmonic on $R - Z$. Consequently $x \in Z$, and $v(x) = +\infty$. Thus there exists $s > 0$ so small that $v_s(x) > M$. But $v_s(x_k) \leqq v(x_k) \leqq M$ and so by the continuity of v_s we have $v_s(x) \leqq M$, a contradiction. Hence $(v + u)(x_0) \geqq (v + u)_{r_0}(x_0)$ for all $x_0 \in R_{r_0}$, namely $v + u$ is superharmonic on R. Similarly $\varepsilon v + u$ is superharmonic for all $\varepsilon > 0$. But $\varepsilon v + u$ converges to u (locally L_1) as $\varepsilon \to 0$. Therefore u is also superharmonic.

3. Existence of Green functions.

In this section let u stand for a positive superharmonic function on an open set S in E^d. Let R be a bounded open subset of S. $\{B^k\}$ will denote a sequence of open balls in R with the following properties. (i) Each B^k is well contained in S. (ii) Each B^k appears infinitely often in the sequence $\{B^k\}$. (iii) For every compact set K well contained in R we can find an integer n so large that $K \subset \bigcup_{k=1}^{n} B^k$.

For each ball $B \subset S$ define $\Phi(u, B)$ to be the superharmonic function which is equal to $PI(u, B)$ on B, and equal to u off B. Write u^0 for u, and write u^k for $\Phi(u^{k-1}, B^k)$. Thus $u^{k-1} \geqq u^k$. The sequence u^k is said to be obtained by the sweeping process (of Poincaré) for u in R relative to $\{B^k\}$. We loosely say that the sweeping process converges for u if the a.e. limit of u^k exists on R. (Classically the limit always exists, the sequence being positive and decreasing.) The limit u^∞ is then harmonic on R since, for fixed k, it is the limit of the subsequence $\{u^j\}$ where j runs through the indices for which $B^j = B^k$ and where u^j is evidently harmonic on B^k. This section studies the constructive convergence of u^k: whether it is possible to find an integer k so large that u^k is arbitrarily close to being harmonic, in a sense to be made precise. First a lemma about the continuity of the map Φ.

LEMMA 3.1. *Let B, C denote balls well contained in the ball B'. Let u, w be superharmonic functions on B'. If $C \to B$* (with respect to the Euclidean metric) *and if $w \to u$* (in $L_1(B')$), *then $\Phi(w, C) \to \Phi(u, B)$* (in $L_1(B')$).

THEOREM 3.2. *If the limit u^∞ exists for the sweeping process, then it is the greatest harmonic minorant of u on R in the sense of Definition 2.1. In particular u^∞ is independent of $\{B^k\}$. Conversely, if u has a greatest harmonic minorant h, then $u^k \downarrow h$* a.e.

Proof. Suppose u^∞ exists. Let K be a compact set well contained in R. Let p be an integer so large that $K \subset \bigcup_1^p B^k$. For each k, let $C^k \subset B^k \subset B'^k$ be concentric balls well contained in S, such that $K \subset \bigcup_1^p C^k$ also. Let n be so large that $u^n - u^\infty < \varepsilon$ on $\bigcup_1^p C^k$, where ε is an arbitrary positive number. (n exists because u^n converges to u^∞ uniformly on compact subsets well contained in R, thanks to Harnack's inequality.) Let $w^0 = u$ and $w^k = \Phi(w^{k-1}, C^k)$. According to the previous lemma, we can make the integral of $|w^n - u^n|$ on $\bigcup_1^p B'^k$ arbitrarily small, if we choose C^k close enough to $B^k(k = 1, \cdots, n)$. It follows from Theorem 1.5 that we can make $|PI(u^n, C^k) - PI(w^n, C^k)| < \varepsilon$ on $\bigcup_1^p C^k_\varepsilon$. Now let v be any harmonic function on $\bigcup_1^p C^k$ dominated by u. Then $v \leqq \Phi(u, C^1)$ and inductively $v \leqq w^n$. Hence for every $k = 1, \cdots, p$ (again using the lemma) we have $v \leqq PI(w^n, C^k) < PI(u^n, C^k) + \varepsilon \leqq u^n + \varepsilon \leqq u^\infty + 2\varepsilon$ on C^k_ε. The condition in Definition 2.1 is thus satisfied with $R' = \bigcup_1^p C^k$, provided ε is so small that $K \subset R'_\varepsilon$.

Conversely, assume that h is a greatest harmonic minorant of u on R. Let K be a compact set well contained in R and let $\varepsilon > 0$ be arbitrary. Let K' and R' satisfy the condition in Definition 2.1. Let p be so large that $K' \subset \bigcup_1^p B^k$. Let w^k be obtained from the sweeping process for u in the region $\bigcup_1^p B^k$ relative to $\{B^{k \bmod p}\}$. We will later show that w^k converges on $\bigcup_1^p B^k$. The limit w^∞ is harmonic and dominated by u on $R' \subset \bigcup_1^p B^k$. Hecce, by the definition of R', we have $w^\infty \leqq h + \varepsilon$ on K. Choosing k large enough, we have $w^k \leqq h + 2\varepsilon$ on K. But the sequence $\{B^{k \bmod p}\}$ is a subsequence of $\{B^k\}$. Hence $w^k \geqq u^n$ if n is chosen large enough. Combining, we see that $u^n \leqq h + 2\varepsilon$ on K. As $u^n \geqq h$ also, $\{u^n\}$ converges uniformly on compact subsets of R to h, as asserted.

For the remainder of this section assume that S is an open ball well containing R, and denote its Green function by G_S. From the above theorem we see that R has a Green function if the sweeping process converges for all $u_x (x \in R)$, the choice of B^k and S being immaterial. Our next task is to show that the sweeping process does converge for a large family of open regions R. First we introduce two assumptions on R which are classically trivial, but which spell out necessary numerical data about R in our computations. Assumptions: (i) R is bilocated, i.e., we are able to compute the distance from any point in E^d to R and to $-R$, (ii) R is strongly Lebesgue measurable, i.e., given any $\varepsilon < 0$ we can find a compact set $K \subset R$ such that any Lebesgue measurable set contained in $R - K$ has measure at most ε. With these two assumptions, it is easy to see that if u_x^k converges a.e. on R then it also converges a.e. on E^d. Thus u_x^∞ may be regarded as a superharmonic function on E^d. By Theorem 2.4, we can write $u_x^\infty = \mu_x G_S + h$ on S, where μ_x is a unique measure independent of S, and h is harmonic on S. The measure μ_x is supported by ∂R since u_x^∞ is harmonic on R as well as off R. Averaging over x in a ball $B_{yr} \subset R$ and using the uniqueness of the decomposition, we have $|B_{yr}|^{-1}\int_{B_{ry}} \mu_x dx = \mu_y$. So μ_x is called the harmonic measure relative to R and x, (see e.g., Helms 1969). In particular, for every continuous function f on ∂R, we see that $\mu_x f$ is harmonic in x on R. Thus, if we can construct the Green function for R, or equivalently show that u_x^∞ exists for all $x \in R$, then $\mu_x f$ is the generalized Dirichlet solution for the boundary function f.

In the following we consider the sweeping process as an operation on measures on E^d with compact support. More precisely, let μ be such a measure relative to which the open ball B is measurable. Define $\Psi(\mu, B)$ to be the measure whose value at a continuous function f on E^d is

$$\Psi(\mu, B)f = \int_{-B} f d\mu + (r\sigma_d)^{-1}\int_{x \in B}\int_{z \in \partial B} f(z)(r^2 - |x-y|^2)|z-x|^{-d} d\sigma(z) d\mu(x)$$

where we have let B_{yr} be B, and σ the surface area measure on ∂B. We will write μ^n or $\Psi(\mu, B^1, \cdots, B^n)$ for $\Psi(\Psi(\cdots(\mu, B^1)\cdots), B^n)$. The next lemma states some well known properties of Ψ.

LEMMA 3.3. (1) *Ψ is linear in μ, and $|\Psi(\mu, B)| = |\mu|$.*

(2) *$\Psi(\mu, B)$ is supported by $-B$, and by ∂B if $\mu(-B) = 0$.*

(3) *If μ is supported by B_s, then for every measurable subset C of ∂B*

$$(2r)^{-d}\sigma_d^{-1} s\sigma(C)|\mu| \leqq \Psi(\mu, B)(C) \leqq (2r)\sigma_d^{-1}s^{-d}\sigma(C)|\mu| .$$

(4) $\Phi(\mu U, B) = \Psi(\mu, B)U$.

LEMMA 3.4. *Let ε and a be positive real numbers. Then there exists $r = r(\varepsilon, a) > 0$ such that for every measure μ on E^d supported by the r-neighborhood C_r of some $(d-2)$-dimensional sphere C* (i.e., *C is the intersection of two spheres in E^d*) *having radius a, with $\mu U \leqq 1$ on E^d, we have $|\mu| \leqq \varepsilon$.*

The proof of this lemma is typical in classical potential theory, and is sketched as follows. Let ν be the measure on C with density 1. Elementary calculation then shows that $\nu U \geqq c_d \log(a/r)$ on ∂C_r, where c_d is a constant. Thus the harmonic functions $h = \mu U$ and $g = (c_d \log(a/r))^{-1}\nu U$ on $E_d - C_r$ obeys $h \leqq g$. In particular $h(x) \leqq g(x)$ where x is the center of C. But $g(x) = (c_d \log(a/r))^{-1}\nu_{d-1}$, where ν_{d-1} is the total surface area of the unit sphere in E^{d-1}. Similarly $h(x) \geqq (a+r)^{-d+2}|\mu|$. Combining, we see that $|\mu| \leqq c_d^{-1}\nu_{d-1}a^{d+2}/\log(a/r) \leqq$

ε, if we let $r = a \exp(-c_d^{-1}\nu_{d-1}a^{d-2}/\varepsilon)$.

LEMMA 3.5. *Let D, α, ε be positive real numbers and let n be a positive integer. Then there exists a positive number $a_n = a_n(D, \alpha, \varepsilon)$ with the following properties. Suppose R is the union of open balls $B^1, \cdots, B^n$ whose diameters are at most D and whose centers are at least α units apart, and suppose $\sigma(\partial B^k - B^1 - \cdots - B^{k-1}) \geqq \sigma(\partial B^j - B^1 - \cdots - \hat{B}^j - \cdots - B^k)$, $(1 \leqq j \leqq k \leqq n$* and $\wedge$ signifies omission). *Suppose μ is a measure supported by $R \cap (\partial B^1 \cup \cdots \cup \partial B^n)$ with total mass $|\mu| > \varepsilon$ and with $\mu U \leqq 1$. Then*

$$\Psi(\mu, \beta_n)(\partial R) \geqq a_n$$

where $\beta_1 = \{B^1\}$ and β_n is the finite sequence of balls obtained from β_{n-1} by adjoining $B^1, \cdots, B^{n-1}$ in the front and adjoining B^n at the end. (For example $\beta_4 = \{B^1, B^2, B^3, B^1, B^2, B^1, B^1, B^2, B^3, B^4\}$.)

Proof. The lemma is trivially true for $n = 1$. Let D, α, ε and $n > 1$ be given, and assume that a_{n-1} has been constructed for $D, \alpha, \varepsilon/3$. Let $r = r(D, \alpha, \varepsilon) > 0$ be so small that for every measure ν supported by the set $\bigcup_{i=1}^{n-1}(B^n - B^n_r) \cap \partial B^i$ (where $B^1, \cdots B^n$ are arbitrary balls as in the hypothesis) with $\nu U \leqq 1$, we have $|\nu| \leqq (\varepsilon \wedge a_{n-1})/6$. The number r exists because of the previous lemma and because the balls have diameters at most D and centers at least α apart. We will show that

$$a_n = (\varepsilon/3) \wedge D^{-d}\sigma_d^{-1}r(r^{d-1}\sigma_d/n)(\varepsilon \wedge a_{n-1})/6$$

has the desired properties. Thus let $B^1, \cdots, B^n$ and μ be as given in the hypothesis. Write R' for $B^1 \cup \cdots \cup B^{n-1}$. The

$$\begin{aligned}\Psi(\mu, B^1, \cdots, B^{n-1}) &= \Psi(\mu|R', B^1, \cdots, B^{n-1}) + \mu|\partial R' \cap B^n \\ &= \{\Psi(\mu|R', B^1, \cdots, B^{n-1})|R'\} \\ &\quad + \{\Psi(\mu|R', B^1, \cdots, B^{n-1})|(\partial R' - B^n)\} \\ &\quad + \{\Psi(\mu|R', B^1, \cdots, B^{n-1})|(\partial R' \cap B^n) + \mu|\partial R' \cap B^n\} \\ &= \nu_1 + \nu_2 + \nu_3, \text{ say .}\end{aligned}$$

Since $|\nu_1| + |\nu_2| + |\nu_3| = |\mu| > \varepsilon$, at least one of the following alternatives holds.

(i) $|\nu_2| > \varepsilon/3$. Then, since ν_2 is supported by $\partial R' - B^n \subset \partial R$, we have

$$\Psi(\mu, \beta_n)(\partial R) \geqq \Psi(\nu_2, \beta_{n-1}, B^n)(\partial R) = \nu_2(\partial R) = |\nu_2| > \varepsilon/3 \geqq a_n .$$

(ii) $|\nu_3| > \varepsilon/3$. Note that ν_3 is supported by $\partial R'$. Hence $\Psi(\nu_3, \beta_{n-1}) = \nu_3$. On the other hand, since $\nu_3 U \leqq \mu U \leqq 1$ and so

$\nu_3(\partial R' \cap (B^n - B^n_r)) \leqq \varepsilon/6$ by the choice of r, we have $\nu_3(B^n_r) > \varepsilon/6$. Therefore

$$\begin{aligned}\Psi(\mu, \beta_n)(\partial R) &\geqq \Psi(\nu_3, \beta_{n-1}, B^n)(\partial R)\\ &= \Psi(\nu_3, B^n)(\partial R)\\ &\geqq \Psi(\nu_3 | B^n_r, B^n)(\partial B^n - B^1 - \cdots - B^{n-1})\\ &\geqq D^{-d}\sigma_d^{-1} r(r^{d-1}\sigma_d/n)(\varepsilon/6)\\ &\geqq a_n .\end{aligned}$$

(iii) $|\nu_1| > \varepsilon/3$. Then, since $\nu_1 U \leqq 1$ and since ν_1 is supported by $R' \cap (\partial B^1 \cup \cdots \cup \partial B^{n-1})$, the induction hypothesis implies $\Psi(\nu_1, \beta_{n-1})(\partial R') > a_{n-1}$. Hence if we write ν_4, ν_5 for $\Psi(\nu_1, \beta_{n-1}) | (\partial R' - B^n)$ and $\Psi(\nu_1, B_{n-1}) | \partial R' \cap B^n$ respectively, then either $|\nu_4| > a_{n-1}/3$ or $|\nu_5| > a_{n-1}/3$. Thus the arguments in (i) or (ii) can be repeated with ν_4 or ν_5 respectively, ε being replaced by a_{n-1}, in either case yielding again $\Psi(\mu, \beta_n)(\partial R) \geqq a_n$.

THEOREM 3.6. *Let R be the union of the open balls $B^1, \cdots, B^n$ with distinct centers. Then the sweeping process for u_x $(x \in R)$ relative to $\{B^{k \bmod n}\}$ converges.*

Proof (given for $d \geqq 3$ only). Since $\Phi(u, B) \geqq u$ in general, it suffices to prove that the sweeping process converges for some subsequence of $B^k = B^{k \bmod n}$. We may assume that $x \in B^1$ so that $\delta_x U = u_x \leqq M$ for some $M > 0$ on $-B^1$. (Here δ_x is the unit mass at x.) In particular $\delta_x^1 U = u_x^1 \leqq M$ on E_d. Let $\varepsilon > 0$ be arbitrary. Let $\mu = \Psi(\delta_x, B^1, \cdots, B^n)$. Then clearly μ is supported by $\partial B^1 \cup \cdots \cup \partial B^n$. Let β_n and a_n be constructed for the balls $B^1, \cdots, B^n$ and for ε as in the previous lemma. Let γ_k stand for the sequence β_n repeated k times. Let N be an integer greater than a_n^{-1}. Suppose $\Psi(\mu, \gamma_N)(R) > \varepsilon$. Then $\Psi(\mu, \gamma_k)(R) > \varepsilon$ for all $k \leqq N$. Hence

$$\begin{aligned}\Psi(\mu, \gamma_N)(\partial R) &\geqq \Psi(\mu, \gamma_{N-1})(\partial R) + \Psi(\Psi(\mu, \gamma_{N-1}) | R, \beta_n)(\partial R)\\ &\geqq \Psi(\mu, \gamma_{N-1})(\partial R) + a_n\\ &\cdots\\ &\geqq N a_n > 1 ,\end{aligned}$$

a contradiction. Hence $\Psi(\mu, \gamma_N)(R) \leqq \varepsilon$. Therefore $\Psi(\mu, \gamma_k)(R) \downarrow 0$, and $\Psi(\mu, \gamma_k)$ converges (w^*) to some measure supported by ∂R. As a consequence, $\Phi(u_x, B^1, \cdots, B^n, \gamma_k) = \Psi(\mu, \gamma_k) U$ converges (locally L_1) to some u_x^∞.

In the proof of the next theorem we need a simple consequence of Theorem 3.1: if the sweeping process for a superharmonic function u converges on the open subsets R and R', to u^∞ and $u^{\infty\prime}$ respectively, then $u^\infty \geqq u^{\infty\prime}$ provided $R \subset R'$.

THEOREM 3.7. *Let f be continuous function on E^d such that $f(x) \to +\infty$ as $|x| \to \infty$. Then except for a countable set of real numbers a, the open set $R(a) = \{x \in E^d : f(x) < a\}$ has a Green function.*

Proof (given for $d \geqq 3$ only). Let x be an arbitrary point in E^d. We will first show that the sweeping process for u_x converges for all but countably many a. It suffices to consider those a's in $[0, 1]$. Let S be a ball containing $R(1)$, and let p be any positive integer. Construct a step function g_p on $[0, 1]$ in the following manner. For each $k = 1, 2, \cdots, 2^p$ let R_k be a union of finitely open balls such that $R((k-1)2^{-p}) \subset R^k \Subset R(k2^{-p})$. This is possible since f is continuous. By the previous theorem, the sweeping process for u_x converges on R_k, say to the limit $u_x^{(k)}$. Define g_p to be the step function whose value on $((k-1)2^{-p}, k2^{-p}]$ is $\int_S u_x^{(k)}$. In view of the remark before this theorem, g_p is a decreasing function on $[0, 1]$. Moreover, if $n > p$, then the values of g_n on $((k-1)2^{-p}, k2^{-p}]$ lie in the interval $[g_p((k-1)2^{-p}, g_p(k+1)2^{-p})]$. Consequently g_p converges a.e. (Lebesgue) to a function g. The function g, being decreasing, is continuous except at countably many points. Suppose a is a point at which g is continuous. Let $\varepsilon > 0$ be arbitrary. Then there exist a', a'' such that (i) $a' < a < a''$, (ii) $g(a') - g(a'') < \varepsilon$, and (iii) $g_p \to g$ at a' and a''. Pick p so large that $g_p(a') - g_p(a'') < 3\varepsilon$, and that $a' < (k-1)2^{-p} < k2^{-p} < a < (j-1)2^{-p} < j2^{-p} < a''$ for some $k, j = 1, \cdots, 2^p$. Then we have $g_p(a') \geqq g_p(k2^{-p}) \geqq g_p(j2^{-p}) \geqq g_p(a'')$, whence $g_p(k2^{-p}) - g_p(j2^{-p}) < 3\varepsilon$. Equivalently $\int_S u_x^{(k)} - \int_S u_x^{(j)} < 3\varepsilon$. Now let $\{B^i\}$ be a sequence of open balls associated to $R(a)$ as in the beginning of this section. By the definition of $u_x^{(k)}$, there exists a sequence of balls $C^1, \cdots, C^q$ such that $\int_S \Phi(u_x, C^1, \cdots, C^q) - \int_S u_x^{(k)} < \varepsilon$ and $C^1 \cup \cdots \cup C^q \subset R_k \Subset R(k2^{-p}) \subset R(a)$. Let i be so large that

$$\int_S \Phi(u_x, B^1, \cdots, B^i) \leqq \int_S \Phi(u_x, C^1, \cdots, C^q) + \varepsilon .$$

Combining, we see that for any given ε, there exist i, j as above with $\int_S \Phi(u_x, B^1, \cdots, B^i) \leqq g_p(j2^{-p}) + 4\varepsilon \leqq \int_S \Phi(u_x, B^1, \cdots, B^i) + 4\varepsilon$. It follows that $\int_S \Phi(u_x, B^1, \cdots, B^i)$ converges as $i \to \infty$. The Monotone convergence theorem then implies that $\Phi(u_x, B^1, \cdots, B^i)$ converges a.e. as $i \to \infty$. In other words the sweeping process for u_x on $R(a)$ converges. Now let $\{x_{(n)}\}$ be a dense sequence in E^d. We already know that except a countable set of a's, the sweeping process converges for every $u_{x(n)}$ on $R(a)$. Now let a be a fixed number not in the exceptional set. Suppose $x \in R(a)$. By passing to a subsequence we may assume $x(n) \to x$. Let $\{B^k\}$ be a sequence of open balls associated to $R(a)$. There is no loss of generality in assuming that $x \in B^1$. Let $\varepsilon > 0$ be arbitrary. Then for some n large enough we have $\Phi(u_{x(n)}, B^1) + \varepsilon \geqq \Phi(u_x, B^1) \geqq \Phi(u_{x(n)}, B^1) - \varepsilon$, according to Lemma 3.1. Hence

$$\Phi(u_{x(n)}, B^1, \cdots, B^k) + \varepsilon \geqq \Phi(u_x, B^1, \cdots, B^k) \geqq \Phi(u_{x(n)}, B^1, \cdots, B^k) - \varepsilon .$$

Since $\int_S \Phi(u_{x(n)}, B^1, \cdots, B^k)$ converges, we see that $\int_S \Phi(u_x, B^1, \cdots, B^k) - \int_S \Phi(u_x, B^1, \cdots, B^{k+p}) \leqq 2\varepsilon$ if k is large enough. By the monotone convergence theorem we see that the sweeping process for u_x on $R(a)$ converges. Theorem 3.2 therefore implies that $R(a)$ has a Green function.

COROLLARY 3.8. *If an open set R is such that $R = \{x \in E^d$: $x = x_0 + rf(z)z$ for some $r \in [0, 1)$ and $z \in \partial B_{01}\}$ where x_0 is a fixed point in R and where f is a continuous function on the unit sphere ∂B_{01}, then R has a Green function. In particular a bounded convex open set has a Green function.*

Proof. We may assume that $x_0 = 0$. Define a continuous function F on E^d by $F(x) = r$ if $x = rf(z)z$. Then clearly $R = R(1) = \{x: F(x) < 1\}$. By the previous theorem, there exists $a > 0$ such that $R(a)$ has a Green function. The observation that $R(1) = a^{-1}R(a)$ together with a scaling argument yields the assertion.

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