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STABLE ISOMORPHISM AND STRONG MORITA EQUIVALENCE OF C^*-ALGEBRAS

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We show that if A and B are C^*-algebras which possess countable approximate identities, then A and B are stably isomorphic if and only if they are strongly Morita equivalent. By considering Breuer ideals, we show that this may fail in the absence of countable approximate identities. Finally we discuss the Picard groups of C^*-algebras, especially for stable algebras.

0. **Introduction.** Theorem 2.8 of [4] states that if B is a full hereditary subalgebra of a C^*-algebra A, and if each of A and B have strictly positive elements (or, equivalently, countable approximate identities, by [1]), then B is stably isomorphic to A, that is, $B \otimes K$ is isomorphic to $A \otimes K$ where K is the algebra of compact operators on a separable infinite dimensional Hilbert space. The purpose of the present paper is to show how the above theorem implies that two C^*-algebras which are strongly Morita equivalent in the sense of having an imprimitivity bimodule [8, 9, 10], will be stably isomorphic if they possess strictly positive elements, and in particular if they are separable. (The converse is readily apparent.) On the other hand, in the second section, by considering Breuer ideals, we give examples of pairs of C^*-algebras which are strongly Morita equivalent but are not stably isomorphic, even if we allow tensor products with $K(H)$ for nonseparable H. (Of course, one of them will fail to have a strictly positive element.) Finally, in the last section we discuss the Picard groups of C^*-algebras. We show in particular that the Picard group of any C^*-algebra, B, which is stable (that is, $B \cong B \otimes K$) and has a strictly positive element, is isomorphic to the quotient of the automorphism group of B by the subgroup of generalized inner automorphisms of B.

1. **The main theorem.** Let A be a C^*-algebra, and let $M(A)$ denote the double centralizer algebra of A. By a corner of A we mean [4] a subalgebra of the form pAp where p is a projection in $M(A)$. A corner is said to be full if it is not contained in any proper two-sided ideal of A, that is, if ApA is dense in A. Two corners, pAp and qAq, are called *complementary* if $p + q = 1$. The device by which we relate strong Morita equivalence to the setting of [4] is:

THEOREM 1.1. *Let B and E be C^*-algebras. Then B and E are strongly Morita equivalent if and only if there is a C^*-algebra A with complementary full corners isomorphic to B and E respectively.*

Proof. Let A be a C^*-algebra and let B be a full corner, pAp, in A. Then, as in Example 6.7 of [8], Ap will be an $A-B$-imprimitivity bimodule (Definition 6.10 of [8]), so that A and B are strongly Morita equivalent. In particular, if B and E are full corners of A, then both are strongly Morita equivalent to A, and so to each other. (In fact, if $E = qAq$, $B = pAp$, then qAp will be an $E - B$-imprimitivity bimodule.) Note that essentially the same argument shows that a full hereditary subalgebra B of a C^*-algebra C will be strongly Morita equivalent to C. This should be compared to Theorem 2.8 of [4] which states that a full hereditary subalgebra B of a C^*-algebra C is stably isomorphic to C if they both possess strictly positive elements.

Conversely, suppose that X is an E-B-imprimitivity bimodule. Without loss of generality we can assume that the E- and B-valued inner-products are definite. We use X to construct in a canonical way a C^*-algebra A containing E and B as complementary full corners. We will call this algebra the *linking* algebra for X. Let $\tilde{X}$ denote the dual $B - E$-imprimitivity bimodule as defined in 6.17 of [8], so that if $x \in X$, then $\tilde{x}$ denotes x viewed as an element of $\tilde{X}$. Let A_0 denote the collection of 2×2 matrices

$$\begin{pmatrix} e & x \\ \tilde{y} & b \end{pmatrix} \quad e \in E\,, \quad x, y \in X\,, \quad b \in B\,,$$

with addition and scalar multiplication defined in the evident way. The product of two such matrices is defined by

$$\begin{pmatrix} e & x \\ \tilde{y} & b \end{pmatrix}\begin{pmatrix} e_1 & x_1 \\ \tilde{y}_1 & b_1 \end{pmatrix} = \begin{pmatrix} ee_1 + \langle x, y_1\rangle_E & ex_1 + xb_1 \\ \tilde{y}e_1 + b\tilde{y}_1 & \langle y, x_1\rangle_B + bb_1 \end{pmatrix}$$

and the adjoint of such a matrix is defined by

$$\begin{pmatrix} e & x \\ \tilde{y} & b \end{pmatrix}^* = \begin{pmatrix} e^* & y \\ \tilde{x} & b^* \end{pmatrix}.$$

It is readily verified that with these definitions A_0 is a $*$-algebra.

Let $M = X \oplus B$, viewed as a B-rigged space with the evident right action of B, and with B-valued inner-product defined by

$$\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle_B = \langle x, y\rangle_B + b^*c\,.$$

Then A_0 acts on M by

$$\begin{pmatrix} e & x \\ \tilde{y} & b \end{pmatrix}\begin{pmatrix} z \\ c \end{pmatrix} = \begin{pmatrix} ez + xc \\ \langle y, z\rangle_B + bc \end{pmatrix}.$$

It is easily seen that this gives a $*$-representation of A_0 with respect to the B-valued inner-product defined just above. Furthermore it is a routine matter to show that this representation is by bounded operators as defined in 2.3 of [8]. (This is most easily done by checking separately the four cases in which all but one entry of the matrix is zero.) Thus we can equip A_0 with the corresponding operator norm so that it becomes a pre- C^*-algebra. We define A, the linking algebra for X, to be the completion of A_0. From the fact that M contains B as a summand, it quickly follows that the evident map of B into A is injective and so isometric. Similarly the evident map of E into A is injective and so isometric, since the map of E into operators on X is injective (for if $eX = 0$, then $0 = \langle eX, X\rangle_E = e\langle X, X\rangle_E$, so that $eE = 0$, since $\langle X, X\rangle_E$ is assumed to span a dense subspace of E).

Let

$$p = \begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}$$

where id_X and id_B denote the identity maps of X and B into themselves. It is clear that p and q may be viewed as self-adjoint projections in the algebra, $L(M)$, of all bounded operators on M, and that left and right multiplication by them map A_0, and so A, into itself. Thus p and q may be viewed as projections in the multiplier algebra, $M(A)$, of A, so that pAp and qAq are corners of A. But it is clear that $pA_0p = E$ and $qA_0q = B$, so that $pAp = E$ and $qAq = B$ since E and B are complete. It is also clear that $p + q$ is the identity element of $M(A)$, so that E and B are complementary corners of A.

Finally, we must check that B and E are full corners of A. But this follows from routine matrix calculations together with the fact that EX is dense in X for the norm $\|\langle x, x\rangle_B\|^{1/2}$ (see 6.14 of [8]), and a similar fact for XB.

THEOREM 1.2. *Let B and E be C^*-algebras. If B and E are stably isomorphic, then they are strongly Morita equivalent. Conversely, if B and E are strongly Morita equivalent and if they both possess strictly positive elements, then they are stably isomorphic.*

Proof. Let K denote the algebra of compact operators on some Hilbert space, and let p be a rank-one projection in K. Then B is isomorphic to $B \otimes p$, which is a full corner of $B \otimes K$, and so B is strongly Morita equivalent to $B \otimes K$ as mentioned earlier. Similarly E is strongly Morita equivalent to $E \otimes K$. Thus if $B \otimes K$ is isomorphic to $E \otimes K$, then B and E are strongly Morita equivalent.

Conversely, if B and E are strongly Morita equivalent, view them as complementary corners of the linking algebra, A, for some $E - B$-imprimitivity bimodule, as in the previous theorem. A routine argument shows that the sum of a strictly positive element for B with one for E will be a strictly positive element for A. Thus we can conclude from Theorem 2.8 of [4] that B and E are both stably isomorphic to A, and so to each other.

We mention now an intriguing example. Let G be a locally compact group and H a closed subgroup. Let $C^*(H)$ be the group C^*-algebra of H, and let $C^*(G, G/H)$ be the transformation group C^*-algebra of G acting on G/H. Then Mackey's imprimitivity theorem as formulated in [8] says that $C^*(G, G/H)$ and $C^*(H)$ are strongly Morita equivalent (and specifies a canonical imprimitivity bimodule). We conclude from Theorem 2 above that these algebras are stably isomorphic, at least if they are separable. Now if we let H act by left translation on itself, then it is well known that the transformation group C^*-algebra $C^*(H, H)$ is isomorphic to the algebra of compact operators on $L^2(H)$ (see e.g., [7]), which is separable and infinite if H is. Similarly, for $C^*(G, G)$. Thus, if G and H are separable and infinite, we conclude from the above theorem that

$$C^*(G, G/H) \otimes C^*(H, H) \cong C^*(H) \otimes C^*(G, G) .$$

If we view $C^*(H)$ as the transformation group algebra for H acting on the one-point space $\{p\}$, then it is easily seen that this implies that

$$C^*(G \times H, G/H \times H) \cong C^*(G \times H, G \times \{p\}) .$$

It is not presently clear to us how one might reach this conclusion from other considerations. Note that if $G = \boldsymbol{R}$, $H = \boldsymbol{Z}$, so $G/H = \boldsymbol{T}$, then

$$C^*(\boldsymbol{R} \times \boldsymbol{Z}, \boldsymbol{T} \times \boldsymbol{Z}) \cong C^*(\boldsymbol{R} \times \boldsymbol{Z}, \boldsymbol{R} \times \{p\}) .$$

2. **Counterexamples with Breuer ideals.** We now describe some examples of C^*-algebras which are strongly Morita equivalent but are not stably isomorphic. Let M be a type II_∞ factor [5]. Then the Breuer ideal, $B(M)$, of M is the sub-C^*-algebra of M generated by all the finite projections in M. Recall [5] that if N is a type II_1 factor, and if $L(H)$ denotes the algebra of bounded operators

on an infinite dimensional Hilbert space H, then $N \otimes L(H)$ (the von Neumann algebra tensor product) is a type II_∞ factor.

PROPOSITION 2.1. *Let N be a type II_1 factor, and let $M = N \otimes L(H)$. Then $B(M)$ is strongly Morita equivalent to N.*

Proof. Let p be a rank-one projection in $L(H)$. Then $N \cong N \otimes p$, which is a corner of $B(M)$. Furthermore, this corner is full, since the Breuer ideal of a type II_∞ factor is simple. But we have seen that a full corner of an algebra is strongly Morita equivalent to the original algebra.

LEMMA 2.2. *Let M be a type II_∞ factor. Then $B(M)$ does not contain a strictly positive element.*

Proof. Suppose that $B(M)$ contains a strictly positive element. Then, using its spectral resolution, we can obtain a strictly positive element m of the form $m = \Sigma 2^{-k} e_k$ where the e_k are orthogonal finite projections. This sum can not be finite, for otherwise we could find a finite projection which is strictly positive, which is clearly impossible. For each k let f_k be a projection smaller than e_k whose trace is smaller than 2^{-k}, so that $n = \Sigma f_k$ is a finite projection in M, and so in $B(M)$. Let A be the commutative C^*-algebra generated by the e_k, f_k, and n, and let C be the sub-C^*-algebra generated by the e_k and f_k. Then it is clear that C is a proper ideal in A. Thus there is a state on A which vanishes on C, and so on m. This state extends to a state on $B(M)$ which vanishes on m, contradicting the assumption that m is strictly positive.

After we had written up these results Chuck Akemann pointed out to us that the above lemma also appears as Proposition 4.5 of [2] with a quite similar proof.

PROPOSITION 2.3. *If C is a C^*-algebra which has a strictly positive element and if D is a corner of C, then D has a strictly positive element.*

Proof. If m is a strictly positive element of C, and if $D = eCe$ for e a projection in $M(C)$, then eme is a strictly positive element in D, for if φ is a state of D which vanishes on eme, then $c \mapsto \varphi(ece)$ is a state of C which vanishes on m.

We remark that the above result is false if D is only assumed to be a hereditary subalgebra of C, as can be seen by taking D to be any C^*-algebra without strictly positive element and letting C be the algebra obtained by adjoining a unit element to D.

PROPOSITION 2.4. *Let C be any C^*-algebra, and let H be a separable Hilbert space. Then $C \otimes K(H)$ has a strictly positive element if and only if C does.*

Proof. Let e be a finite projection in $K(H)$. Then C is isomorphic to $C \otimes e$, which is a corner of $C \otimes K(H)$. Thus if $C \otimes K(H)$ has a strictly positive element so does C by the previous proposition. Conversely, $K(H)$ has a strictly positive element, say k, since $K(H)$ is separable. Then if m is a strictly positive element in C, it is easily seen that $m \otimes k$ is a strictly positive element in $C \otimes K(H)$.

COROLLARY 2.5. *If B and C are C^*-algebras which are stably isomorphic, then B has a strictly positive element if and only if C does.*

COROLLARY 2.6. *If N is any type II_1 factor and M is any type II_∞ factor, then N is not stably isomorphic to $B(M)$.*

One can suspect that the difficulty comes from using separable Hilbert spaces in defining stable isomorphism, but we now show that this is not the case.

THEOREM 2.7. *Let N be a type II_1 factor and M be a type II_∞ factor, but now let H be a possibly nonseparable Hilbert space. Then $N \otimes K(H)$ is not isomorphic to $B(M) \otimes K(H)$.*

Proof. Suppose the two algebras are isomorphic. As before, $B(M)$ is a corner of $B(M) \otimes K(H)$, so that $B(M)$ is a corner of $N \otimes K(H)$. Let e be an infinite projection in M which is the supremum of a countable number of finite projections in M. Then eMe is a σ-finite II_∞ factor and so has a faithful state. Also, $B(eMe) = eB(M)e$ which is a corner of $B(M)$, and so of $N \otimes K(H)$. Thus we need only show that the Breuer ideal of a σ-finite II_∞ factor can not be a corner of $N \otimes K(H)$. Consequently, we assume from now on that M is σ-finite, so that $B(M)$ has a faithful state, say φ.

Let f be the projection in the double centralizer algebra of $N \otimes K(H)$ such that $B(M) = f(N \otimes K(H))f$. Let $\{e_\alpha\}$ be the family of rank-one projections in $K(H)$ corresponding to some orthonormal basis for H. Then the $f(1 \otimes e_\alpha)f$ are in $B(M)$, and for any finite set, F, of indices we have

$$\|\sum_{\alpha \in F} f(1 \otimes e_\alpha)f\| \leqq 1 .$$

It follows that

$$\sum_{\alpha \in F} \varphi(f(1 \otimes e_\alpha)f) \leqq 1 .$$

This implies that $f(1 \otimes e_\alpha) = 0$ except for a countable number of α's. Let e be the supremum in $L(H)$ of the e_α for which $f(1 \otimes e_\alpha) \neq 0$. Then $f \leqq 1 \otimes e$. It follows that $B(M)$ is a corner of $N \otimes K(eH)$, which contradicts Corollary 2.6.

We remark that with some more effort one can show that the Breuer ideal of II_∞ factor can not even be a hereditary subalgebra of $N \otimes K(H)$ where N is a type II_1 factor.

We would like to thank Bruce Blackadar for having shown us the fact that, with notation as in Proposition 2.1, $B(M) \neq N \otimes K(H)$. It was this remark which led us to find the counterexamples described in this section.

3. **Picard groups.** We now discuss the relationship between automorphisms of C^*-algebras and strong Morita equivalence, especially for stable C^*-algebras. Let X be an $E-B$-imprimitivity bimodule. By its Hausdorff completion we mean the $E-B$-imprimitivity bimodule obtained by completing X with respect to the norm $\|\langle x, x\rangle_B\|^{1/2}$, as discussed in Proposition 2.10 of [8] and Proposition 3.1 of [11]. If X and Y are $E-B$-imprimitivity bimodules, we say that X and Y are *equivalent* if their Hausdorff completions are isomorphic as $E-B$-imprimitivity bimodules. In this case X and Y determine equivalent equivalences between the category of Hermitian E-modules and the category of Hermitian B-modules, as can be seen from Proposition 5.8 of [8]. We can then form the category whose objects are C^*-algebras and whose morphisms are equivalence classes of imprimitivity bimodules. The composition of an $E-B$-imprimitivity bimodule X with a $B-C$-imprimitivity bimodule Y is taken to be $X \otimes Y$ as defined in Theorem 5.9 and Proposition 6.21 of [8] (and then extended to equivalence classes). This category is a category with inverses, as is seen from Lemma 6.22 of [8]. In particular, if B is a fixed C^*-algebra, then the equivalence classes of $B-B$-imprimitivity bimodules will form a group, which we will call the *Picard group* of B and denote by $\mathrm{Pic}(B)$, in analogy with the definitions in Chapter 2, §5 of [3].

If E and B are C^*-algebras and if θ is an isomorphism from E to B, then it is clear that θ should determine an $E-B$-imprimitivity bimodule, X_θ. As in [3] there are several ways to do this. We choose the following conventions. Let X_θ be the vector space E with the obvious left action of E on X_θ and the obvious E-valued inner-product, but define the right action of B on X_θ by $e \cdot b = e\theta^{-1}(b)$

and the B-valued inner-product by $\langle e_1, e_2\rangle_B = \theta(e_1^* e_2)$. It is then easily seen that if φ is an isomorphism from B to a C^*-algebra C, then $X_\theta \otimes_B X_\varphi$ is equivalent to $X_{\varphi\circ\theta}$, so that we obtain in this way a contravariant functor from the category of C^*-algebras with isomorphisms as morphisms, into the category of C^*-algebras with equivalence classes of imprimitivity bimodules as morphisms, and this functor is injective on objects. In particular, we obtain an anti-homomorphism from the group of automorphisms, $\mathrm{Aut}(B)$, of a C^*-algebra B, into $\mathrm{Pic}(B)$.

Let u be a unitary element of $M(B)$, and let Ad_u denote the automorphism of B defined by $Ad_u(b) = ubu^{-1}$. We will call Ad_u a generalized inner automorphism of B, and we will denote the group of all generalized inner automorphisms of B by $\mathrm{Gin}(B)$. It is easily seen that $\mathrm{Gin}(B)$ is a normal subgroup of $\mathrm{Aut}(B)$.

PROPOSITION 3.1. *The kernel of the anti-homomorphism of* $\mathrm{Aut}(B)$ *into* $\mathrm{Pic}(B)$ *is exactly* $\mathrm{Gin}(B)$. *That is, we have an exact sequence*

$$1 \longrightarrow \mathrm{Gin}(B) \longrightarrow \mathrm{Aut}(B) \longrightarrow \mathrm{Pic}(B)\ .$$

Proof. The identity element of $\mathrm{Pic}(B)$ is (represented by) B viewed in the usual way as a $B-B$-imprimitivity bimodule. If u is a unitary element of $M(B)$, then it is easily seen that the map $b \rightarrow bu$ is an imprimitivity bimodule isomorphism of B with X_{Ad_u}. Then X_{Ad_u} also represents the identity element of $\mathrm{Pic}(B)$, so that Ad_u is in the kernel of the homomorphism from $\mathrm{Aut}(B)$ to $\mathrm{Pic}(B)$. Conversely, let $\theta \in \mathrm{Aut}(B)$ and suppose that X_θ represents the identity element of $\mathrm{Pic}(B)$. Since B and X_θ are both complete, this means that there is an isomorphism, f, from B to X_θ. In particular, f is a linear map from B to B having the properties that

$$f(bc) = bf(c)\ , \quad f(cb) = f(c)\theta^{-1}(b)\ , \quad f(b)f(c)^* = bc^*$$

for all $b, c \in B$. But it is easily seen from this that the pair $u = (f\circ\theta, f)$ is an element of $M(B)$. The third equation above can then be rewritten as

$$buu^*c^* = bc^*\ , \quad \text{or} \quad b(1-uu^*)c^* = 0\ ,$$

for all $b, c \in B$, so that $uu^* = 1$. Now the fact that f preserves the rightsided inner-product says that

$$\theta(f(b)^*f(c)) = b^*c\ , \quad \text{or} \quad \theta(u^*b^*cu) = b^*c$$

for $b, c \in B$. Then for any $d \in B$ we have

$$(b^*c)d = (\theta(u^*(b^*c)u))d = \theta(u^*(b^*c)(u\theta^{-1}(d)))$$

which, if we let b^*c run through an approximate identity for B, gives

$$d = \theta(u^*u\theta^{-1}(d)) , \quad \text{or} \quad \theta^{-1}(d) = u^*u\theta^{-1}(d) .$$

It follows that $u^*u = 1$, so that u is unitary. Thus we can rewrite the equation several lines above as

$$\theta(Ad_u^{-1}(b^*c)) = b^*c ,$$

from which it is clear that $\theta = Ad_u$.

COROLLARY 3.2. *Let E and B be C^*-algebras, and let θ and φ be isomorphisms from E to B. If X_θ and X_φ are equivalent, then there is a unitary element, u, of $M(B)$ such that $\varphi = Ad_u \circ \theta$.*

Proof. From our earlier comments it is clear that the "inverse" of X_θ is $X_{\theta^{-1}}$, which will also represent the inverse of X_φ since X_φ and X_θ are equivalent. Thus $X_{\theta^{-1}} \otimes X_\varphi = X_{\varphi\circ\theta^{-1}}$ represents the identity element of Pic (B), so that by the above proposition there is a unitary element, u, of $M(B)$ such that $\varphi \circ \theta^{-1} = Ad_u$.

To give an indication of the kind of information which the Picard group gives about a C^*-algebra, we mention that if B is a finite dimensional C^*-algebra, then Pic (B) will be isomorphic to the group of permutations of the spectrum of B, while Aut (B)/Gin (B) will be smaller than this if there are minimal two-sided ideals in B which have different dimensions. In another direction, if T is a compact Hausdorff space and $B = C(T)$, then Pic (B) will include the multiplicative group of line bundles over T, and in fact will be the semidirect product of this group with the group of homeomorphisms of T.

We now wish to show that for stable algebras with strictly positive elements, every imprimitivity bimodule comes from an isomorphism. Our main tool for doing this is:

LEMMA 3.3. *Let E and B be C^*-algebras and let X be an $E-B$-imprimitivity bimodule. Let A be the linking algebra of X. Then X is equivalent to X_θ for some isomorphism θ from E to B if and only if there is a partial isometry, v, in $M(A)$ such that*

$$v^*v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad vv^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

In this case θ is defined by $\theta(e) = vev^*$.

Proof. If X is equivalent to X_θ, then we can identify A with the linking algebra of X_θ. We can then set

$$v = \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix}, \quad v^* = \begin{pmatrix} 0 & \theta^{-1} \\ 0 & 0 \end{pmatrix}.$$

If these symbolic expressions are viewed as operators on $X_\theta \oplus B$ in the obvious way, then it is easily seen that they are bounded operators which normalize A, so that they can be viewed as elements of $M(A)$, which will clearly have the desired properties.

Conversely, suppose we are given $v \in M(A)$ with the given properties, and define θ by $\theta(e) = vev^*$. Then it is clear that θ is an isomorphism from E to B. We must show that X is equivalent to X_θ. If we identify E with its image in A, then we can define a map, f, from X to E by

$$f(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} v .$$

Then if we put on E in A the operations under which it becomes X_θ, routine calculations show that f is a bimodule homomorphism from X to X_θ which preserves both inner-products and has dense range.

THEOREM 3.4. *Let E and B be stable C^*-algebras with strictly positive elements. Then every E-B-imprimitivity bimodule is equivalent to one of the form X_θ for some isomorphism θ from E to B. Furthermore, θ is uniquely determined up to left multiplication by an element of* Gin (B).

Proof. The uniqueness statement follows immediately from Corollary 3.2. (One could instead use right multiplication by elements of Gin (E).) Now let X be an $E - B$-imprimitivity bimodule, and let A be the linking algebra of X. Since E and B are full corners of A, they are stably isomorphic to A, and in fact, by Corollary 2.6 of [4], there are partial isometries in $M(A \otimes K)$ which give isomorphisms of $E \otimes K$ and $B \otimes K$ with $A \otimes K$. When these partial isometries are composed, we obtain $w \in M(A \otimes K)$ such that

$$w^*w = \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad ww^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix}.$$

We show that because E and B are stable, we can drop this situation to A. Let p be a rank-one projection in K, so that $E \otimes p$ is a corner of $E \otimes K$ which is isomorphic to E. In this situation it is obvious by looking just at $p \otimes K$ in $K \otimes K$ that there is a partial isometry, v_E, in $M(E \otimes K \otimes K)$ such that $v_E^* v_E = 1 \otimes 1 \otimes 1$ while $v_E v_E^* = 1 \otimes p \otimes 1$. Since E is stable, we can identify E with $E \otimes 1 \otimes K$, so that $v_E \in M(E \otimes K)$, $v_E^* v_E = 1 \otimes 1$ and $v_E v_E^* = 1 \otimes p$. Similarly, since B is stable, there exists $v_B \in M(B \otimes K)$ such that $v_B^* v_B = 1 \otimes 1$ and $v_B v_B^* = 1 \otimes p$. Define $v \in M(A \otimes K)$ by

$$v = \begin{pmatrix} 0 & 0 \\ 0 & v_B \end{pmatrix} w \begin{pmatrix} v_E^* & 0 \\ 0 & 0 \end{pmatrix} .$$

Then simple calculations show that

$$v^* v = \begin{pmatrix} 1 \otimes p & 0 \\ 0 & 0 \end{pmatrix}, \quad vv^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes p \end{pmatrix}.$$

In particular, v commutes with $\begin{pmatrix} 1 \otimes p & 0 \\ 0 & 1 \otimes p \end{pmatrix}$, and so can be viewed as an element of $M(A)$ such that

$$v^* v = \begin{pmatrix} 1_E & 0 \\ 0 & 0 \end{pmatrix}, \quad vv^* = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

From Lemma 3.3 it follows that X is equivalent to X_θ for some isomorphism θ from E to B.

COROLLARY 3.5. *Let B be a stable C^*-algebra having a strictly positive element. Then every $B-B$-imprimitivity bimodule is equivalent to one determined by an automorphism of B. Thus we have an exact sequence*

$$1 \longrightarrow \mathrm{Gin}\,(B) \longrightarrow \mathrm{Aut}\,(B) \longrightarrow \mathrm{Pic}\,(B) \longrightarrow 1 .$$

We remark that stability is not necessary for the above conclusion. For example, it holds for any finite dimensional commutative C^*-algebra. We also remark that in Theorem 3.4 the condition that E and B have strictly positive elements can not be dropped, as follows a fortiori from §2.

We will now show that every C^*-algebra is strongly Morita equivalent to a C^*-algebra, B, whose Picard group is $\mathrm{Aut}\,(B)/\mathrm{Gin}\,(B)$. From Corollary 3.5 it is clear that this is true for a C^*-algebra which is strongly Morita equivalent to a stable C^*-algebra having a strictly positive element, and so, by Proposition 2.4, also for a C^*-algebra which is strongly Morita equivalent to any C^*-algebra having a strictly positive element (e.g., Breuer ideals of II_∞ factors). We begin by characterizing such C^*-algebras.

PROPOSITION 3.6. *A C^*-algebra B is strongly Morita equivalent to a C^*-algebra having a strictly positive element if and only if* Prim (B) *is σ-compact* (i.e., a countable union of quasi-compact sets).

Proof. If a C^*-algebra, C, has a strictly positive element, then it follows from 3.3.7 of [6] that $\mathrm{Prim}(C)$ is σ-compact. If B is strongly Morita equivalent to C, then $\mathrm{Prim}(B)$ is a homeomorphic to $\mathrm{Prim}(C)$ (3.8 of [11]), so that $\mathrm{Prim}(B)$ is σ-compact also. Conversely, if $\mathrm{Prim}(B)$ is σ-compact, then a simple compactness argument shows that we can find $b \in B^+$ which is contained in no proper 2-sided ideal of B. Then the hereditary subalgebra, C, of B generated by b will have b as a strictly positive element and will be full in B, and so Morita equivalent to B. (Take $X = Bb$ as a $B - C$-imprimitivity bimodule.)

For any C^*-algebra B let R_B denote the collection of equivalence classes under "isometric" isomorphism of right B-rigged spaces (2.8 of [8]) which are complete in the B-norm (2.10 of [8]), have no nonzero elements of length zero, and the ranges of whose B-valued inner-products have dense span in B. If Z is in R_B and if X is a $B - B$-imprimitivity bimodule, then the Hausdorff completion of $Z \otimes_B X$, defined as in 5.9 of [8], is again a right B-rigged space. It is easily seen that its equivalence class in R_B depends only on the equivalence class of Z in R_B and of X in $\mathrm{Pic}(B)$. In this way we see that the group $\mathrm{Pic}(B)$ acts on the right on the collection R_B. Our method for obtaining, for any C^*-algebra B, a C^*-algebra C which is strongly Morita equivalent to B and is such that $\mathrm{Pic}(C) = \mathrm{Aut}(C)/\mathrm{Gin}(C)$, is based on the following observation.

LEMMA 3.7. *Let Z belong to an equivalence class in R_B and let E denote the imprimitivity algebra of Z* (6.4 of [8]). *Then the element of R_B represented by Z is invariant under* $\mathrm{Pic}(B)$ *if and only if*

$$\mathrm{Pic}(B) \cong \mathrm{Pic}(E) \cong \mathrm{Aut}(E)/\mathrm{Gin}(E)\ .$$

Proof. First, $\mathrm{Pic}(B) \cong \mathrm{Pic}(E)$ because Z will be an $E - B$-imprimitivity bimodule, and "conjugating" elements of $\mathrm{Pic}(B)$ by Z will give an isomorphism. Suppose now that the element of R_B represented by Z is invariant under $\mathrm{Pic}(B)$, and let Y represent an element of $\mathrm{Pic}(E)$. Let $\tilde{Z}$ denote the dual of Z as in 6.17 of [8], which represents the "inverse" of Z in the category whose morphisms are equivalence classes of imprimitivity bimodules. Then $\tilde{Z} \otimes_E Y \otimes_E Z$ will represent an element of $\mathrm{Pic}(B)$, which by hypothesis will leave Z invariant. This means, after cancellation, that $Y \otimes_E Z$ is isomorphic to Z as right B-rigged spaces. Applying $\tilde{Z}$ on the right, we find that Y is isomorphic to E as right E-rigged spaces. Let φ denote such an isomorphism, mapping Y to E. Then φ establishes an isomorphism between the imprimitivity algebras of the right E-rigged spaces Y and E, and so determines an automorphism, θ, of E, defined by

$$\theta^{-1}(e)y = \varphi^{-1}(e\varphi(y)) \quad \text{for} \quad y \in Y, e \in E .$$

Let $\psi = \theta^{-1} \circ \varphi$. Then it is easily checked that ψ is an isomorphism of Y with X_θ as $E - E$-imprimitivity bimodules. Thus all elements of $\mathrm{Pic}\,(E)$ come from elements of $\mathrm{Aut}\,(E)$.

Conversely suppose that $\mathrm{Pic}\,(E) \cong \mathrm{Aut}\,(E)/\mathrm{Gin}\,(E)$, and let W represent an element of $\mathrm{Pic}\,(B)$. Then $Z \otimes_B W \otimes_B \tilde{Z}$ represents an element of $\mathrm{Pic}\,(E)$, and so is isomorphic to X_θ for some automorphism, θ, of E. Then $Z \otimes_B W \cong X_\theta \otimes_E Z$, from which it follows that $Z \otimes_B W \cong Z$ as elements of R_B. Thus Z is invariant under $\mathrm{Pic}\,(B)$.

LEMMA 3.8. *Let B be a C^*-algebra such that* $\mathrm{Prim}(B)$ *is σ-compact. Then there is a unique element of R_B having the property that if Y is any representative of this element, then the imprimitivity algebra of Y is stable and has strictly positive element. Furthermore, this unique element of R_B will be invariant under* $\mathrm{Pic}\,(B)$.

Proof. Let Y and Z represent elements of R_B, and assume that their imprimitivity algebras, E and F respectively, are stable and have strictly positive elements. (Such exist by Propositions 2.4 and 3.6.) Then according to Theorem 3.4 there is an isomorphism θ from E to F such that $Y \otimes_B \tilde{Z} \cong X_\theta$. This means that $Y \cong X_\theta \otimes_F Z$ as imprimitivity bimodules, which is easily seen to imply that $Y \cong Z$ as right B-rigged spaces, so that Y and Z represent the same element of R_B. The fact that this element is invariant under $\mathrm{Pic}\,(B)$ follows immediately from Lemma 3.7 and Corollary 3.5.

THEOREM 3.9. *Let B be any C^*-algebra. Then there is a C^*-algebra, E, strongly Morita equivalent to B such that*

$$\mathrm{Pic}\,(B) \cong \mathrm{Pic}\,(E) \cong \mathrm{Aut}\,(E)/\mathrm{Gin}\,(E) .$$

Proof. Let Q be the collection of σ-compact open subsets, U, of $\mathrm{Prim}\,(B)$, and for each $U \in Q$ let I_U be the two-sided ideal of B whose hull is the complement of U, so that $\mathrm{Prim}\,(I_U)$ is naturally identified with U. Let Y_U represent the unique element of R_{I_U} described in Lemma 3.8, so that the imprimitivity algebra of Y_U is stable and has strictly positive element. Then Y_U can be viewed as a right B-rigged space in the evident way (see 3.9 of [8]), though of course, now the range of the B-valued inner product need not span B. Let

$$Y = \bigoplus \{Y_U : U \in Q\} ,$$

which will represent an element of R_B. We claim that Y is invariant under Pic (B). Let X represent an element of Pic (B). Then X determines a homeomorphism, α, of Prim (B) as in 3.8 of [11] such that for each U we have $XI_{\alpha(U)} = I_U X$ and this space is an imprimitivity bimodule between I_U and $I_{\alpha(U)}$ as in 3.4 of [11]. It follows that $Y_U \otimes_B X$, which is equivalent to $Y_U \otimes_{I_U} XI_{\alpha(U)}$, represents an element of $R_{I_{\alpha(U)}}$, whose imprimitivity algebra is isomorphic to that of Y_U, and so still is stable with strictly positive element. Then according to Lemma 3.8 $Y_U \otimes_B X$ is isomorphic to $Y_{\alpha(U)}$ as $I_{\alpha(U)}$-rigged spaces, and so as B-rigged spaces. From this it follows that $Y \otimes_B X$ is isomorphic to Y as B-rigged spaces, so that Pic (B) leaves Y invariant. From Lemma 3.7 we conclude that the imprimitivity algebra of Y fulfils the requirements of the theorem.

We remark that if one restricts attention to the category of C^*-algebras having a strictly positive element, then there is actually a functor constructing algebras having the properties of Theorem 3.9, namely $B \mapsto B \otimes K$. But in the general case there does not seem to be any way to make this construction functorial.

Finally, we remark that the considerations of this section are of interest in the theory of extensions of C^*-algebras by K. Specifically, it follows immediately from Corollary 2.7 of [4] and Theorem 1.1 that any $E - B$-imprimitivity bimodule determines an isomorphism of Ext (E) with Ext (B), at least if E and B are separable, and this isomorphism depends only on the equivalence class of the bimodule. Of course this isomorphism is actually determined by an isomorphism of $E \otimes K$ to $B \otimes K$, but in practice the imprimitivity bimodule may be a natural object whereas it may be hard to describe the isomorphism of $E \otimes K$ with $B \otimes K$ explicitly. Note in particular that if B is separable, then we obtain a homomorphism from Pic (B) into the group of automorphisms of Ext (B).

Consider, for instance, the examples at the end of § 1. We obtain an isomorphism from Ext $(C^*(H))$ to Ext $(C^*(G, G/H))$. If G/H is compact, there is a natural homomorphism from $C^*(G)$ to $C^*(G, G/H)$ which gives us a map from Ext $(C^*(G, G/H))$ to Ext $(C^*(G))$. Thus, for G/H compact, we have an analogue for Ext of the inducing process for representation theory, namely a map from Ext $(C^*(H))$ to Ext $(C^*(G))$.

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