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PROJECTIVE IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

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An ideal in a ring Λ is said to be projective provided it is a projective Λ-module. This paper is concerned with the problem of topologically characterizing projectivity within the class of ideals of a ring of continuous functions. Since there are projective and nonprojective ideals having the same z-filter, the possibility of such a characterization appears remote. However, such a characterization is shown to exist for the projective z-ideals. Moreover, a relationship between projective z-ideals and arbitrary projective ideals is exhibited and used to show that, in some cases, every projective ideal is module isomorphic to a projective z-ideal.

1. **Preliminaries.** Let X be a completely regular, Hausdorff space and $C(X)$ be the ring of real-valued continuous functions on X. An ideal in $C(X)$ is said to be projective provided it is a projective $C(X)$-module. In [1], Bkouche has shown that if X is locally compact then $C_K(X)$, the ideal of functions with compact support, is projective if and only if X is paracompact. Actually, he has characterized projectivity within the class of pure submodules of $C(X)$ in terms of the topological properties of βX, the Stone-Čech compactification of X. Using the concept of a projective basis, Finney and Rotman [5] have presented a direct proof of Bkouche's result for locally compact spaces. This paper is concerned with the problem of topologically characterizing projectivity within the class of all ideals in $C(X)$.

The remaining paragraphs in this section introduce the terminology and notation which is used in the sequel. The reader is referred to [6] for additional background. In § 2 a characterization of projectivity in the class of ideals in $C(X)$ is given which is used to show the existence of projective and nonprojective ideals having the same z-filter. Such examples indicate that the topology of a space is not rich enough to distinguish between the projective and nonprojective ideals in the general setting. In § 3 projectivity within the class of z-ideals is topologically characterized and these results are shown to be a generalization of the work of Bkouche. In § 4 the general problem is again addressed. Here it is shown that any projective ideal I is closely associated with a projective z-ideal I_z. The relationship between I and I_z is studied and it is shown that often I is module isomorphic to I_z. Hence, in some cases, the projective ideals in $C(X)$ can be found, up to an isomorphism, by restricting attention to the class of z-ideals.

Let Λ be a commutative ring with identity and M be a Λ-module. A collection $\{m_\alpha\}_{\alpha \in A} \subseteq M$ combined with a set $\{\phi_\alpha\}_{\alpha \in A}$ of Λ-module homomorphisms from M into Λ is called a projective basis of M provided $m \in M$ implies $\phi_\alpha(m) = 0$ for almost all $\alpha \in A$ and $m = \sum_{\alpha \in A} \phi_\alpha(m) m_\alpha$. For the sake of conciseness, denote the projective basis above by $\{m_\alpha, \phi_\alpha\}_{\alpha \in A}$.

The characterization of projective Λ-modules in terms of a projective basis stated in part (a) of the following theorem is used extensively in the succeeding sections. Part (b) is a consequence of the proof of part (a) as given in [2, page 132]. It follows from part (b) that a finitely generated projective module has a finite projective basis.

THEOREM 1.1. *Let M be a Λ-module.*

(a) *M is projective if and only if M has a projective basis.*

(b) *If M is projective and $\{m_\alpha\}_{\alpha \in A} \subseteq M$ generates M, then M has a projective basis of the form $\{m_\alpha, \phi_\alpha\}_{\alpha \in A}$.*

The following notation relating to a function $f \in C(X)$ is adopted.

(f) = the ideal generated by f.

$\operatorname{pos} f = \{x \in X: f(x) > 0\}$.

$\operatorname{neg} f = \{x \in X: f(x) < 0\}$.

$Z(f) = \{x \in X: f(x) = 0\}$ = the zero-set of f.

$\operatorname{coz} f = \{x \in X: f(x) \neq 0\}$ = the cozero-set of f.

$\operatorname{supp} f = \operatorname{cl}(\operatorname{coz} f) = \overline{\operatorname{coz} f}$ = the support of f.

f^+ = the function mapping each $x \in X$ to $\max\{f(x), 0\}$.

$f^- = (-f)^+$.

$|f| = f^+ + f^-$.

If I is an ideal in $C(X)$, let $\operatorname{coz} I = \bigcup_{f \in I} \operatorname{coz} f$. Also let $Z[I] = \{Z(f): f \in I\}$ and $Z^{\leftarrow}[Z[I]] = \{f \in C(X): Z(f) \in Z[I]\}$. The ideal I is called a z-ideal if $Z^{\leftarrow}[Z[I]] = I$. It is said to be fixed if $\bigcap Z[I] \neq \emptyset$; otherwise it is free. This definition of a free ideal differs from the concept of a free $C(X)$-module (one that is isomorphic to a direct sum of copies of the ring) which is usually associated with the study of projective modules. In Example 2.6 (a) it is shown that the only proper ideals in $C(X)$ that are free $C(X)$-modules are fixed.

A collection of continuous functions $\{f_\alpha\}_{\alpha \in A}$ is said to be locally finite or star finite provided $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ is locally finite or star finite respectively. A collection of continuous functions $\{h_\alpha\}_{\alpha \in A}$ is said to be a partition of unity on $Y \subseteq X$ provided the collection $\{\operatorname{coz} h_\alpha\}_{\alpha \in A}$ is locally finite on Y, each h_α is nonnegative, and $\sum_{\alpha \in A} h_\alpha(x) = 1$ for each $x \in Y$. The collection $\{h_\alpha\}_{\alpha \in A}$ will be considered subordinate to a collection $\{S_\alpha\}_{\alpha\in A}$ of subsets of X provided $\operatorname{supp} h_\alpha \subseteqq S_\alpha$ for each $\alpha \in A$.

2. The fundamental theorem. This section is devoted to the development and immediate implications of Theorem 2.4.

LEMMA 2.1. *If I is a projective ideal in $C(X)$ with projective basis $\{f_\alpha, \Phi_\alpha\}_{\alpha\in A}$, then*

(a) $\operatorname{supp} \Phi_\alpha(f) \subseteqq \operatorname{supp} f$ *for each* $\alpha \in A$ *and* $f \in I$,

(b) $\{\operatorname{coz} \Phi_\alpha(f_\alpha)\}_{\alpha\in A}$ *is a star finite open cover of* $\bigcup_{f\in I} \operatorname{supp} f$, *and*

(c) $\bigcup_{\alpha\in A} \operatorname{coz} \Phi_\alpha(f_\alpha) = \bigcup_{\substack{\alpha\in A\\ f\in I}} \operatorname{supp} \Phi_\alpha(f) = \bigcup_{f\in I} \operatorname{supp} f$.

Proof. (a) Let $x \in X\backslash \operatorname{supp} f$. By the complete regularity of X, there is a $g \in C(X)$ such that $g(x) = 1$ and $\operatorname{supp} f \subseteqq Z(g)$. Thus, $gf = 0$ so $0 = \Phi_\alpha(gf) = g\Phi_\alpha(f)$. Therefore, $x \notin \operatorname{coz} \Phi_\alpha(f)$. Hence, $\operatorname{supp} \Phi_\alpha(f) \subseteqq \operatorname{supp} f$.

(b) Given $f \in I$ there is a finite set $B \subseteqq A$ such that $f = \sum_{\alpha\in B} \Phi_\alpha(f) f = \sum_{\alpha\in B} \Phi_\alpha(f_\alpha) f$ so $\sum_{\alpha\in B} \Phi_\alpha(f_\alpha) = 1$ on $\operatorname{supp} f$. Consequently, $\{\operatorname{coz} \Phi_\alpha(f_\alpha)\}_{\alpha\in A}$ covers $\bigcup_{f\in I} \operatorname{supp} f$.

If $\beta \in A$, then $\Phi_\alpha(f_\beta) = 0$ for almost all $\alpha \in A$. Thus, $\Phi_\beta(f_\beta)\Phi_\alpha(f_\alpha) = \Phi_\beta(\Phi_\alpha(f_\alpha, \phi_\alpha) f_\beta) = \Phi_\beta(\Phi_\alpha(f_\beta) f_\alpha) = 0$ for almost all $\alpha \in A$. Hence, $\operatorname{coz} \Phi_\beta(f_\beta)$ meets only a finite subset of $\{\operatorname{coz} \Phi_\alpha(f_\alpha)\}_{\alpha\in A}$. Consequently, $\{\operatorname{coz} \Phi_\alpha(f_\alpha)\}_{\alpha\in A}$ is star finite.

(c) This is a consequence of (a) and (b).

LEMMA 2.2. *If I is a projective ideal in $C(X)$ with projective basis $\{f_\alpha\}_{\alpha\in A}$ and $f_1, f_2, \cdots, f_n \in I$ then, for each $\beta \in A$, the function g_β defined by*

$$g_\beta = \begin{cases} \dfrac{\prod_{i=1}^{n} \Phi_\beta(f_i)}{\sum_{\alpha\in A} (\Phi_\alpha(f_\alpha))^2} & \text{on } \bigcup_{\alpha\in A} \operatorname{supp} f_\alpha \\ 0 & \text{otherwise} \end{cases}$$

is continuous.

Proof. By Lemma 2.1 (b), $\sum_{\alpha\in A} (\Phi_\alpha(f_\alpha))^2$ is continuous on $\bigcup_{\alpha\in A} \operatorname{supp} f_\alpha$ and, by Lemma 2.1 (a) and (c), $\operatorname{supp} \prod_{i=1}^{n} \Phi_\beta(f_i) \subseteqq \operatorname{coz} \sum_{\alpha\in A} (\Phi_\alpha(f_\alpha))^2$.

PROPOSITION 2.3. *If I is a projective ideal in $C(X)$, then I has a projective basis $\{f_\alpha, \Phi_\alpha\}_{\alpha\in A}$ such that*

(a) $\operatorname{supp} f_\alpha = \operatorname{supp} \Phi_\alpha(f_\alpha)$ *for each* $\alpha \in A$ *and*

(b) $g \in I$ *implies* $gf_\alpha = 0$ *for almost all* $\alpha \in A$.

Proof. Let $\{k_\alpha, \psi_\alpha\}_{\alpha \in A}$ be a projective basis for I. For each $\beta \in A$ and each $g \in I$ define the function g_β by

$$g_\beta = \begin{cases} \dfrac{\psi_\beta(g)}{\sum_{\alpha \in A} (\psi_\alpha(k_\alpha))^2} & \text{on } \bigcup_{\alpha \in A} \operatorname{supp} k_\alpha \\ 0 & \text{otherwise} . \end{cases}$$

Note g_β is continuous by Lemma 2.2. Moreover, if g is fixed then $g_\alpha = 0$ for almost all $\alpha \in A$. Hence, for each $\alpha \in A$, the module homomorphism $\Phi_\alpha: I \rightarrow C(X)$ defined by $\Phi_\alpha(g) = g_\alpha$ has the property that $\Phi_\alpha(g) = 0$ for almost all $\alpha \in A$.

For each $\alpha \in A$, define $f_\alpha = \psi_\alpha(k_\alpha)k_\alpha \in I$. Then $\{f_\alpha\}_{\alpha \in A}$ generates I. Indeed, if $g \in I$, one can write $g = \sum_{\alpha \in A} g_\alpha \psi_\alpha(k_\alpha)k_\alpha = \sum_{\alpha \in A} g_\alpha f_\alpha = \sum_{\alpha \in A} \Phi_\alpha(g) f_\alpha$. Thus, $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ is a projective basis for I.

By Lemma 2.1, $\operatorname{supp} f_\alpha = \operatorname{supp} \psi_\alpha(k_\alpha)k_\alpha = \operatorname{supp} \psi_\alpha(k_\alpha) = \operatorname{supp} \Phi_\alpha(f_\alpha)$ so $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ satisfies condition (a) of the proposition. To obtain condition (b) recall that for each $\alpha \in A$, $\operatorname{coz} f_\alpha \subseteq \operatorname{coz} \psi_\alpha(k_\alpha)$ so, by Lemma 2.1 (b), $\{f_\alpha\}_{\alpha \in A}$ is star finite. Thus, $\{f_\alpha\}_{\alpha \in A}$ is both a generating set for I and star finite. This implies condition (b).

THEOREM 2.4. *An ideal in $C(X)$ is projective if and only if*

(a) *it is generated by a star finite set $\{f_\alpha\}_{\alpha \in A}$ such that*

(b) *there is a star finite partition of unity $\{h_\alpha\}_{\alpha \in A}$ on $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$ subordinate to $\{\operatorname{supp} f_\alpha\}_{\alpha \in A}$ and*

(c) *$f_\beta \in \{f_\alpha\}_{\alpha \in A}$ implies $f_\beta h_\alpha \in (f_\alpha)$ for all $\alpha \in A$.*

Moreover, given a projective ideal, either the functions can be chosen such that $\operatorname{supp} h_\alpha = \operatorname{supp} f_\alpha$ for each $\alpha \in A$, or if a star finite generating set is unknown then a corresponding partition of unity exists.

Proof. Suppose an ideal in $C(X)$ is projective. Then, by Theorem 2.3 (b), condition (a) is satisfied.

Given a star finite generating set $\{f_\alpha\}_{\alpha \in A}$, Theorem 1.1 (b) provides a projective basis of the form $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$. (If the choice of the generating set is not restricted, then, by Theorem 2.3, one could pick a projective basis $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ such that $\operatorname{supp} f_\alpha = \operatorname{supp} \Phi_\alpha(f_\alpha)$ for each $\alpha \in A$.) For each $\beta \in A$ define

$$h_\beta = \begin{cases} \dfrac{(\Phi_\beta(f_\beta))^2}{\sum_{\alpha \in A} (\Phi_\alpha(f_\alpha))^2} & \text{on } \bigcup_{\alpha \in A} \operatorname{supp} f_\alpha \\ 0 & \text{otherwise} . \end{cases}$$

The continuity of each h_β, the star finiteness of $\{h_\alpha\}_{\alpha \in A}$, and the fact that $\{h_\alpha\}_{\alpha \in A}$ is subordinate to $\{\operatorname{supp} f_\alpha\}_{\alpha \in A}$ all follow from Lemmas 2.1 and 2.2. Note also that $\{h_\alpha\}_{\alpha \in A}$ is a partition of unity on

$\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$. Moreover, if $f_\beta \in \{f_\alpha\}_{\alpha \in A}$, then $g_\beta h_\alpha = g_\alpha f_\alpha$ for each $\alpha \in A$ where $g_\alpha \in C(X)$ is defined by

$$g_\alpha = \begin{cases} \dfrac{\Phi_\alpha(f_\alpha)\Phi_\alpha(f_\beta)}{\sum_{\gamma \in A} (\Phi_\gamma(f_\gamma))^2} & \text{on } \bigcup_{\gamma \in A} \operatorname{supp} f_\gamma \\ 0 & \text{otherwise} . \end{cases}$$

Note that if $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ is chosen to satisfy the conditions of Proposition 2.3 then $\operatorname{supp} h_\alpha = \operatorname{supp} \Phi_\alpha(f_\alpha) = \operatorname{supp} f_\alpha$ for each $\alpha \in A$.

Conversely, suppose there are sets $\{f_\alpha\}_{\alpha \in A}$ and $\{h_\alpha\}_{\alpha \in A}$ satisfying conditions (a), (b), and (c). Then the following argument shows that there are homomorphisms which, combined with the set of functions $\{f_\alpha\}_{\alpha \in A}$, yield a projective basis for I.

Since $\{h_\alpha\}_{\alpha \in A}$ is star finite, the function $\sum_{\alpha \in A} h_\alpha^2$ is continuous and strictly positive on $\bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$. But, by (b), $\operatorname{supp} h_\beta \subseteqq \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$ for each $\beta \in A$, so the function h'_β defined by

$$h'_\beta = \begin{cases} \dfrac{h_\beta}{\sum_{\alpha \in A} h_\alpha^2} & \text{on } \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha \\ 0 & \text{otherwise} \end{cases}$$

is in $C(X)$ for each $\beta \in A$.

For each $\beta \in A$, define $\Phi_\beta: I \to C(X)$ by $\Phi_\beta(g) = h'_\beta k$ where $k \in C(X)$ is such that $gh_\beta = kf_\beta$. Such a k exists by condition (c). Moreover, if k' is another function such that $gh_\beta = k'f_\beta$, then $k' = k$ on $\operatorname{coz} f_\beta$ and thus on $\operatorname{supp} f_\beta$. Now since $\operatorname{supp} h_\beta \subseteqq \operatorname{supp} f_\beta$, it follows that $h'_\beta k = h'_\beta k'$, i.e., $\Phi_\beta(g)$ is independent of the choice of k.

To see that Φ_β is a module homomorphism, suppose that $g, h \in I$ with $k_1 f_\beta = gh_\beta$ and $k_2 f_\beta = hh_\beta$. Then

$$\Phi_\beta(g) + \Phi_\beta(h) = h'_\beta k_1 + h'_\beta k_2 = h'_\beta(k_1 + k_2) = \Phi_\beta(g + h) .$$

In addition, if $f \in C(X)$, then

$$\Phi_\beta(fg) = h'_\beta f k_1 = f h'_\beta k_1 = f\Phi_\beta(g) .$$

Finally, it must be shown that if $g \in I$ then $\Phi_\alpha(g) = 0$ for almost all $\alpha \in A$ and $g = \sum_{\alpha \in A} \Phi_\alpha(g) f_\alpha$. To this end pick $\alpha \in A$ and let k_α be chosen such that $gh_\alpha = k_\alpha f_\alpha$. If $gf_\alpha = 0$ then $gh_\alpha = 0$, since $\operatorname{supp} h_\alpha \subseteqq \operatorname{supp} f_\alpha$, and thus $k_\alpha f_\alpha = 0$. But if $k_\alpha f_\alpha = 0$ then $k_\alpha h_\alpha = 0$ since $\operatorname{supp} h_\alpha \subseteqq \operatorname{supp} f_\alpha$. Consequently, if $gf_\alpha = 0$ then $k_\alpha h_\alpha = 0$. But, by condition (a), $gf_\alpha = 0$ for almost all $\alpha \in A$. Thus, $\Phi_\alpha(g) = k_\alpha h'_\alpha = 0$ for almost all $\alpha \in A$. Furthermore, if B is the finite set $\{\alpha \in A: gh'_\alpha \neq 0\}$, then $\sum_{\beta \in B} h'_\beta h_\beta = 1$ on $\operatorname{supp} g$ and thus

$$g = g \sum_{\beta \in B} h'_\beta h_\beta = \sum_{\beta \in B} h'_\beta h_\beta g = \sum_{\beta \in B} h'_\beta k_\beta f_\beta = \sum_{\alpha \in A} \Phi_\alpha(g) f_\alpha .$$

Therefore, $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ is a projective basis for I.

COROLLARY 2.5. *The principal ideal* (f) *is projective if and only if* $\operatorname{supp} f$ *is open.*

Proof. If (f) is projective, the collection $\{f_\alpha\}_{\alpha \in A}$ in Theorem 2.4 must be finite. Thus, the partition of unity $\{h_\alpha\}_{\alpha \in A}$ of Theorem 2.4 is finite. Consequently, $\sum_{\alpha \in A} h_\alpha \in C(X)$ and is the characteristic function of $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha = \operatorname{supp} f$.

If $\operatorname{supp} f$ is open then (f) is module isomorphic to a direct summand of $C(X)$, namely, the ideal generated by the characteristic function of $\operatorname{supp} f$. Consequently, (f) is projective.

The following examples are provided for future reference and to clarify the technicalities of Theorem 2.4.

EXAMPLES 2.6.

(a) Let $f \in C(X)$ be such that $\operatorname{supp} f$ is open in X and $\operatorname{coz} f \neq \operatorname{supp} f$. Then the principal ideal (f) is projective. The characteristic function of $\operatorname{supp} f$ serves as the partition of unity in Theorem 2.4.

In particular, if $X = R$, the real numbers, and f is defined by $f(x) = x$, then both (f) and $(|f|)$ are projective. The second case shows that a projective ideal may not be convex because the function g defined by $g(x) = |x \sin 1/x|$ is bounded above by $|f|$ but is not in $(|f|)$.

Also, if the ideal I is module isomorphic to $C(X)$, then it must be principal with annihilator ideal equal to $\{0\}$. It follows that an ideal I in $C(X)$ is a free $C(X)$-module if and only if $I = (f)$ where $\operatorname{supp} f = X$.

(b) Let A_1 and A_2 be copies of N, the discrete space of positive integers, and $a \notin A_1 \cup A_2$. Make the set $A_1 \cup \{a\} \cup A_2$ a topological space by defining the subspaces $A_1 \cup \{a\}$ and $\{a\} \cup A_2$ to be homeomorphic to N^*, the one point compactification of N. Now to each $i \in A_1$ attach a copy of N, designated N_i, such that $N_i \cup \{i\}$ is homeomorphic to N^*. Let X be the topological space so defined on $(\bigcup_{i \in A_1} N_i) \cup A_1 \cup \{a\} \cup A_2$. For each $i \in A$, define $f_i \in C(X)$ by

$$f_i(x) = \begin{cases} \dfrac{1}{x} & \text{if } \quad x \in N_i \\ 0 & \text{otherwise} \end{cases}$$

and define $h_i \in C(X)$ to be the characteristic function of $N_i \cup \{i\}$. By identifying the collections $\{f_i\}_{i \in A_1}$ and $\{h_i\}_{i \in A_1}$ with those of Theorem 2.4 (a) and (b) respectively it is seen that the ideal I generated by $\{f_i\}_{i \in A_1}$ is projective.

Note that $\operatorname{coz} I = \bigcup_{i \in A_1} N_i$ is not closed, $\bigcup_{i \in A_1} \operatorname{supp} f_i = (\bigcup_{i \in A_1} N_i) \cup A_1$ is not closed, $\overline{\operatorname{coz} I} = X \backslash A_2$ is not open and all three sets are different and properly contained in X. Moreover, $\sum_{i \in A_1} h_i$ is not continuous at a.

(c) Let $X = \{x \in R: x \geqq 0\}$ and for each positive integer i define $f_i \in C(X)$ by

$$f_i(x) = \begin{cases} 0 & \text{if } x \leqq i - 1 \\ x - (i - 1) & \text{if } i - 1 < x \leqq i \\ i + 1 - x & \text{if } i < x \leqq i + 1 \\ 0 & \text{if } i + 1 < x . \end{cases}$$

Define $h_i = f_i$ for $i > 1$ and

$$h_1(x) = \begin{cases} 1 & \text{if } 0 \leqq x < 1 \\ f_1(x) & \text{if } 1 \leqq x . \end{cases}$$

The identification of the collections $\{f_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ with those of Theorem 2.4 (a) and (b) respectively shows that $\{f_i\}_{i=1}^{\infty}$ generates a projective ideal. Note that the members of $\{\operatorname{coz} f_i\}_{i=1}^{\infty}$ are not mutually disjoint and that $\{h_i\}_{i=1}^{\infty}$ is not a set of characteristic functions.

The following proposition yields a useful example of a nonprojective ideal in $C(X)$.

PROPOSITION 2.7. *If $f \in C(X)$ and $(f, |f|)$ is projective, then* $\operatorname{pos} f$ *and* $\operatorname{neg} f$ *are completely separated and $(f, |f|) = (f)$.*

Proof. Note that $(f, |f|) = (f^+, f^-)$. If $f_1 = f^+$ and $f_2 = f^-$, then by Theorem 1.1 (b) there is a projective basis for $(f, |f|)$ of the form $\{f_i, \Phi_i\}_{i=1,2}$. Following the proof of Theorem 2.4, one can construct a partition of unity $\{h_1, h_2\}$ on $\operatorname{supp} f_1 \cup \operatorname{supp} f_2$ such that conditions (a), (b), and (c) of Theorem 2.4 are satisfied. But, since $\operatorname{coz} f_1 \cap \operatorname{coz} f_2 = \varnothing$, each h_i must be the characteristic function of the corresponding $\operatorname{supp} f_i$. Consequently, $h_1 - h_2$ is a continuous function that is 1 on $\operatorname{pos} f$ and -1 on $\operatorname{neg} f$. Thus, the two sets are completely separated and $(f, |f|) = (f)$.

If one now defines $f \in C(R)$ by $f(x) = x$, then, by [6, 2H] and Proposition 2.7, $(f, |f|)$ is not projective, whereas, in Example 2.6 (a), it was shown that (f) is projective. Hence, there are projective and nonprojective ideals that have the same z-filter. Consequently, the chances seem remote that, in the general setting, the topological properties of X alone are rich enough to distinguish between the projective and nonprojective ideals in $C(X)$. Indeed, any such characterization must include a topological counterpart to the divisibility statement of Theorem 2.4 (c).

The following application of Theorem 2.4 illustrates the abundance of projective ideals in the ring $C(X)$.

PROPOSITION 2.8. *If p and q are distinct elements of βX, then* 0^p (see [6, 7.12]) *contains a projective ideal that is not contained in* 0^q.

Proof. It may be assumed that there is no open and closed neighborhood of p that does not contain q for otherwise the characteristic function of its complement would generate the desired projective ideal. Under this assumption define a sequence of open neighborhoods of p as follows. Let U_1 be an open neighborhood of p that does not contain q. By the complete regularity of βX there is a sequence $\{U_i\}_{i=1}^{\infty}$ of open neighborhoods of p such that $U_{i-1} \supset \bar{U}_i \supset U_i$ for each $i > 1$ and, by the assumption, each containment is proper.

Now, by the normality of βX, there is a sequence $\{f_i\}_{i=1}^{\infty}$ of nonnegative functions in $C(\beta X)$ such that f_1 is 1 on $\beta X \backslash U_2$ and 0 on $\bar{U}_3$ and, for $i > 1$, f_i is 1 on $\bar{U}_i \backslash U_{i+1}$ and 0 on $\bar{U}_{i+2} \cup (\beta X \backslash U_{i-1})$.

For each $i \geqq 1$ define h_i by

$$h_i = \begin{cases} \dfrac{f_i}{\sum_{j=1}^{\infty} f_j} & \text{on } \operatorname{supp} f_i \\ 0 & \text{otherwise}. \end{cases}$$

By identifying the restrictions to X of the functions $\{f_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ with the functions in Theorem 2.4, it is seen that the restrictions of $\{f_i\}_{i=1}^{\infty}$ generate a projective ideal in 0^p, but since $f_1(q) = 1$, this ideal is not contained in 0^q.

3. **Projective z-ideals.** The following proposition is one of the many indications of the importance of the class of z-ideals in the study of the projective ideals in $C(X)$.

PROPOSITION 3.1. *A projective free ideal is a z-ideal.*

Proof. If I is a projective free ideal and $\{f_\alpha\}_{\alpha \in A}$ is as in Theorem 2.4, then, for each $\beta \in A$, the function $f'_\beta = f_\beta / \sum_{\alpha \in A} f_\alpha^2$ is in $C(X)$. (Since I is free, the denominator is strictly positive on X.) Moreover, $\{f'_\alpha f_\alpha\}_{\alpha \in A}$ is a star finite partition of unity on X contained in I.

If $g \in C(X)$ is such that $Z(g) = Z(f)$ for some $f \in I$, then $B = \{\alpha \in A : g f_\alpha \neq 0\} = \{\alpha \in A : f f_\alpha \neq 0\}$ is finite. Hence, $g = g \sum_{\alpha \in B} f'_\alpha f_\alpha \in I$.

Since the z-ideals are closely related to the topology of the space, it is not too surprising that the projective z-ideals can be characterized in topological terms. This is the goal of the following lemma.

LEMMA 3.2. *If the ideal I in $C(X)$ has a projective basis $\{f_\alpha, \Phi_\alpha\}_{\alpha\in A}$ such that $f_\alpha \geqq 0$ and $f_\alpha^{1/2} \in I$ for each $\alpha \in A$, then* $\operatorname{supp} f \subseteqq \operatorname{coz} I$ *for each $f \in I$.*

Proof. It suffices to show that $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for each $\alpha \in A$. Therefore, suppose there is an $\alpha_0 \in A$ with $x \in \operatorname{supp} f_{\alpha_0} \backslash \operatorname{coz} I$. Let m be the number of elements $\alpha \in A$ for which $\Phi_\alpha(f_{\alpha_0}) \neq 0$. (Since $\{f_\alpha, \Phi_\alpha\}_{\alpha\in A}$ is a projective basis, m must be finite.) The following induction shows that there are at least $m+1$ elements $\alpha \in A$ for which $\Phi_\alpha(f_{\alpha_0}) \neq 0$. This contradiction proves that $\operatorname{supp} f_{\alpha_0} \subseteqq \operatorname{coz} I$.

From the fact that $x \in \operatorname{supp} f_{\alpha_0} \backslash \operatorname{coz} f_{\alpha_0}$ it follows that

(a) $f_{\alpha_0}^{1/2}$ is not a multiple of f_{α_0} on the intersection of any neighborhood of x with $\{x\} \cup \operatorname{coz} f_{\alpha_0}$.

However, since $f_{\alpha_0}^{1/2} \in I$, there is a finite subset $B \subseteqq A$ such that $f_{\alpha_0}^{1/2} = \sum_{\alpha\in B} \Phi_\alpha(f_{\alpha_0}^{1/2}) f_\alpha$. Hence, $\{\operatorname{coz} \Phi_\alpha(f_{\alpha_0}^{1/2}) f_\alpha\}_{\alpha\in B}$ is a finite cover of $\operatorname{coz} f_{\alpha_0}^{1/2}$ so there is a $\beta \in B$ such that

(b) $x \in \operatorname{supp} \Phi_\beta(f_{\alpha_0}^{1/2}) f_\beta = \operatorname{supp} \Phi_\beta(f_\beta) f_{\alpha_0}^{1/2} = \operatorname{supp} \Phi_\beta(f_\beta) f_{\alpha_0} \subseteqq \operatorname{supp} f_{\alpha_0} f_\beta$.

Furthermore, of all the $\beta \in B$ satisfying (b) there must be at least one such that

(c) f_β is not a multiple of f_{α_0} on any neighborhood of x intersected with $\{x\} \cup \operatorname{coz} f_{\alpha_0}$.

(Otherwise, (c) would fail for every $\beta \in B$ satisfying (b), so there would be a neighborhood U of x such that, on $U \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0})$,

$$f_{\alpha_0}^{1/2} = \sum_{\alpha\in B} \Phi_\alpha(f_{\alpha_0}^{1/2}) f_\alpha = \sum_{\alpha\in B} \Phi_\alpha(f_{\alpha_0}^{1/2}) f_{\alpha_0} g_\alpha = [\sum_{\alpha\in B} \Phi_\alpha(f_{\alpha_0}^{1/2}) g_\alpha] f_{\alpha_0}$$

where each g_α is in $C(U \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0}))$; this contradicts (a).)

Let α_1 be a member of B for which (b) and (c) are satisfied. Then, by (b), $x \in \operatorname{supp} f_{\alpha_0} f_{\alpha_1}$ and $\Phi_{\alpha_1}(f_{\alpha_1}) f_{\alpha_0} \neq 0$. Moreover, by (c), $\alpha_1 \neq \alpha_0$ and f_{α_1} is not a multiple of f_{α_0} on the intersection of any neighborhood of x with $\{x\} \cup \operatorname{coz} f_{\alpha_0}$.

Suppose the subset $\{f_{\alpha_1}, \cdots, f_{\alpha_n}\} \subseteqq \{f_\alpha\}_{\alpha\in A}$ has been selected such that $x \in \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$, $\Phi_{\alpha_i}(f_{\alpha_i}) f_{\alpha_0} \neq 0$ for $1 \leqq i \leqq n$, and f_{α_n} is not a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_{n-1}}\}$ on the intersection of any neighborhood of x with $\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_{n-1}}$. Then,

(a′) $f_{\alpha_n}^{1/2}$ is not a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ on the intersection of any neighborhood of x with $\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$.

(Otherwise, there would exist a neighborhood V of x and functions $g_i \in C(V \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}))$ such that, on $V \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$,

$f_{\alpha_n}^{1/2} = \sum_{i=0}^{n} g_i f_{\alpha_i}$ and thus $f_{\alpha_n}(1 - g_n f_{\alpha_n}^{1/2}) = \sum_{i=0}^{n-1} (g_i f_{\alpha_n}^{1/2}) f_{\alpha_i}$. Now, since g_n is continuous on $V \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$, it can not equal $f_{\alpha_n}^{-1/2}$ arbitrarily close to x. Hence, there is a neighborhood $V' \subseteqq V$ of x for which $(1 - g_n f_{\alpha_n}^{1/2})^{-1} \in C(V' \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}))$. This implies that $f_{\alpha_n} = (1 - g_n f_{\alpha_n}^{1/2})^{-1} \sum_{i=0}^{n-1} (g_i f_{\alpha_n}^{1/2}) f_{\alpha_i}$ on $V' \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$ which contradicts one of the induction hypotheses.)

Since $f_{\alpha_n}^{1/2} \in I$, there is a finite subset $C \subseteqq A$ such that $f_{\alpha_n}^{1/2} = \sum_{\alpha \in C} \Phi_\alpha(f_{\alpha_n}) f_\alpha$. Hence, $\{\operatorname{coz} \Phi_\alpha(f_{\alpha_n}^{1/2}) f_\alpha\}_{\alpha \in C}$ is a finite cover of $\operatorname{coz} f_{\alpha_n}^{1/2} = \operatorname{coz} f_{\alpha_n}$ and thus a finite cover of $\operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$. Therefore, since $x \in \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$, there must be at least one $\beta \in C$ such that

(b′) $x \in \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n} \Phi_\beta(f_{\alpha_n}^{1/2}) f_\beta = \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n} \Phi_\beta(f_\beta) \subseteqq \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n} f_\beta$.

Let D be the subset of C containing those indices that satisfy (b′). There is a neighborhood U of x such that $f_{\alpha_n}^{1/2} = \sum_{\beta \in D} \Phi_\beta(f_{\alpha_n}^{1/2}) f_\beta$ on $U \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$. Therefore, if for each $\beta \in D$ there was a neighborhood U_β of x such that f_β is a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ on $U_\beta \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$, then $f_{\alpha_n}^{1/2}$ would be a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ on $U \cap (\bigcap_{\beta \in D} U_\beta) \cap (\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n})$ which contradicts (a′). Consequently, there is a $\beta \in D \subseteqq C$ such that

(c′) f_β is not a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ on any neighborhood of x intersected with $\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$.

Let α_{n+1} be a member of C for which both (b′) and (c′) are satisfied. Then, $\{f_{\alpha_1}, \cdots, f_{\alpha_{n+1}}\} \subseteqq \{f_\alpha\}_{\alpha \in A}$. Moreover, by (b′), $x \in \operatorname{supp} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_{n+1}}$ and $\Phi_{\alpha_i}(f_{\alpha_i}) f_{\alpha_0} \neq 0$ for $1 \leqq i \leqq n + 1$, and (c′), $\alpha_{n+1} \neq \alpha_i$ for $1 \leqq i \leqq n$ and $f_{\alpha_{n+1}}$ is not a linear combination of $\{f_{\alpha_0}, f_{\alpha_1}, \cdots, f_{\alpha_n}\}$ on any neighborhood of x intersected with $\{x\} \cup \operatorname{coz} f_{\alpha_0} f_{\alpha_1} \cdots f_{\alpha_n}$.

Thus, by induction, one can find $m + 1$ elements $\alpha \in A$ with $\Phi_\alpha(f_\alpha) f_{\alpha_0} \neq 0$. But $\Phi_\alpha(f_\alpha) f_{\alpha_0} = \Phi_\alpha(f_{\alpha_0}) f_\alpha$, so there must be $m + 1$ elements $\alpha \in A$ such that $\Phi_\alpha(f_{\alpha_0}) \neq 0$.

DEFINITION 3.3. If $\{f_\alpha\}_{\alpha \in A} \subseteqq C(X)$ is a star finite family such that $\operatorname{supp} f_\beta \subseteqq \bigcup_{\alpha \in A} \operatorname{coz} f_\alpha = \operatorname{coz} \sum_{\alpha \in A} f_\alpha^2$ for each $\beta \in A$, let $\bar{f}_\beta$ be the function defined by

$$\bar{f}_\beta = \begin{cases} \dfrac{f_\beta}{\sum_{\alpha \in A} f_\alpha^2} & \text{on } \bigcup_{\alpha \in A} \operatorname{coz} f_\alpha \\ 0 & \text{otherwise} \end{cases}$$

for each $\beta \in A$. Note that $\bar{f}_\beta \in C(X)$.

PROPOSITION 3.4. *If I is a projective ideal, then I is a z-ideal if and only if* $\operatorname{supp} f \subseteqq \operatorname{coz} I$ *for each* $f \in I$ (or equivalently $\operatorname{supp} f \subseteqq \operatorname{coz} I$ for each f in a generating set of I).

Proof. Suppose I is a projective z-ideal with projective basis $\{g_\beta, \psi_\beta\}_{\beta \in B}$. Since I is a z-ideal, both g_β^+ and g_β^- are in I for each $\beta \in B$. Let $A = B \times \{1, 2\}$ and define $f_{(\beta,1)} = g_\beta^+$, $f_{(\beta,2)} = g_\beta^-$, and $\Phi_{(\beta,1)} = -\Phi_{(\beta,2)} = \psi_\beta$ for each $\beta \in B$. Then $\{f_\alpha, \Phi_\alpha\}_{\alpha \in A}$ is a projective basis with the properties of Lemma 3.2. Therefore, $\operatorname{supp} f \subseteqq \operatorname{coz} I$ for each $f \in I$.

Conversely, suppose I is a projective ideal for which $\operatorname{supp} f \subseteqq \operatorname{coz} I$ for each $f \in I$. Let $\{f_\alpha\}_{\alpha \in A}$ be a star finite generating set for I as in Theorem 2.4 (a). Suppose $g \in C(X)$ is such that $Z(g) \in Z[I]$. Then $gf_\alpha \bar{f}_\alpha = 0$ for almost all $\alpha \in A$, and if $B = \{\alpha \in A : gf_\alpha \bar{f}_\alpha \neq 0\}$ then $g = g \sum_{\alpha \in B} f_\alpha \bar{f}_\alpha \in I$. Therefore, I is a z-ideal.

(The equivalence of the statements "$\operatorname{supp} f \subseteqq \operatorname{coz} I$ for each $f \in I$" and "$\operatorname{supp} f \subseteqq \operatorname{coz} I$ for each f in a generating set of I" follows from the fact that the support of any member of I is contained in a finite union of supports of members of a generating set.)

Proposition 3.4 has several significant consequences. One is the existence of projective ideals that are not z-ideals. Indeed, using Proposition 3.4, it is easy to see that Examples 2.6 (a), (b), and (c) are not z-ideals. Also, it follows from Proposition 3.4 that a finitely generated z-ideal is projective if and only if it is generated by an idempotent. This is a result obtained by DeMarco in [3]. The following corollary is related to the problem addressed in § 4.

COROLLARY 3.5. *If I is a projective z-ideal, then it is the only projective ideal whose z-filter is $Z[I]$.*

Proof. Suppose J is a projective ideal with $Z[J] = Z[I]$. Then $J \subseteqq I$, so Proposition 3.4 implies that, for each $f \in J$, $\operatorname{supp} f \subseteqq \operatorname{coz} I$. But, $\operatorname{coz} I = \operatorname{coz} J$. Thus, J is a z-ideal by Proposition 3.4, and consequently must be equal to I.

THEOREM 3.6. *If I is an ideal in $C(X)$, then the following are equivalent.*

(a) *I is a projective z-ideal.*

(b) *There is a generating set $\{f_\alpha\}_{\alpha \in A} \subseteqq I$ such that $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ is a star finite cover of $\operatorname{coz} I$ and $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for all $\alpha \in A$.*

(c) *There is a set $\{f_\alpha\}_{\alpha \in A} \subseteqq I$ such that $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ covers $\operatorname{coz} I$, $I = \{g \in C(X) : gf_\alpha = 0$ for almost all $\alpha \in A$ and $\operatorname{coz} g \subseteqq \operatorname{coz} I\}$, and $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for all $\alpha \in A$.*

(d) *There is a set $\{f_\alpha\}_{\alpha \in A} \subseteqq I$ such that $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ covers $\operatorname{coz} I$, $g \in I$ implies $gf_\alpha = 0$ for almost all $\alpha \in A$, and $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for all $\alpha \in A$.*

(e) *I is generated by a star finite partition of unity $\{h_\alpha\}_{\alpha \in A}$ on $\operatorname{coz} I$.*

Proof. (a) implies (b). This follows from Theorem 2.4 and Proposition 3.4.

(b) implies (c). Since $\{f_\alpha\}_{\alpha \in A}$ generates I and $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ is star finite, it follows that if $g \in I$, then $gf_\alpha = 0$ for almost all $\alpha \in A$ and $\operatorname{coz} g \subseteqq \operatorname{coz} I$. On the other hand, if $g \in C(X)$ has these properties, let $B = \{\alpha \in A: gf_\alpha \neq 0\}$. Then, $g = g \sum_{\beta \in B} f_\beta \bar{f}_\beta \in I$.

(c) implies (d). Clear.

(d) implies (e). The functions $\{h_\alpha\}_{\alpha \in A}$ defined by $h_\alpha = f_\alpha \bar{f}_\alpha$ for each $\alpha \in A$ are in I and have desired properties.

(e) implies (a). Let $\{h_\alpha\}_{\alpha \in A}$ be a star finite partition of unity on $\operatorname{coz} I$ generating I. If $\beta \in A$ and B is the finite set $\{\alpha \in A: h_\alpha h_\beta \neq 0\}$, then $\sum_{\alpha \in B} h_\alpha$ is continuous on X and equal to 1 on $\operatorname{coz} h_\beta$. Hence, $\sum_{\alpha \in B} h_\alpha$ must equal 1 on $\operatorname{supp} h_\beta$ so $\operatorname{supp} h_\beta \subseteqq \operatorname{coz} \sum_{\alpha \in B} h_\alpha \subseteqq \operatorname{coz} I$. Now, I is seen to be projective by identifying $\{h_\alpha\}_{\alpha \in A}$ with the functions in both (a) and (b) of Theorem 2.4 and it must be a z-ideal by Proposition 3.4.

The following theorem, which is essentially a restatement of Theorem 3.6, characterizes the projective z-ideals in topological terms.

THEOREM 3.7. *The ideal I is a projective z-ideal if and only if*

(a) $\bigcap Z[I] \subseteqq \operatorname{int} Z$ *for each* $Z \in Z[I]$ *and*

(b) *there is a collection* $S \subseteqq Z[I]$ *such that* $\bigcap S = \bigcap Z[I]$ *and if* $Z_0 \in Z[I]$, *then* $Z_0 \cup Z = X$ *for almost all* $Z \in S$.

Proof. If I is a projective z-ideal, let $\{f_\alpha\}_{\alpha \in A}$ be a generating set as described in Theorem 3.6 (d). As noted in Proposition 3.4, the fact that $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for each $\alpha \in A$ implies $\operatorname{supp} f \subseteqq \operatorname{coz} I$ for all $f \in I$. This implies condition (a) above. The collection S of condition (b) above can be taken as $\{Z(f_\alpha)\}_{\alpha \in A}$.

If, on the other hand, an ideal I satisfies conditions (a) and (b) above, then for each $Z \in S$ pick $f_Z \in I$ with $Z(f_Z) = Z$. By condition (b), the collection $\{\operatorname{coz} f_Z\}_{Z \in S}$ covers $\operatorname{coz} I$, and if $g \in I$, then $gf_Z = 0$ for almost all $Z \in S$. Furthermore, $\operatorname{supp} f_Z \subseteqq \operatorname{coz} I$ for each $Z \in S$ by condition (a). Therefore, $\{f_Z\}_{Z \in S}$ satisfies the conditions of Theorem 3.6 (d), so I is a projective z-ideal.

Due to condition (e) of Theorem 3.6, one might conjecture that only condition (b) of Theorem 3.7 is required to characterize the projective z-ideals. This, however, is not the case. Indeed, every fixed maximal ideal in $C(R)$ satisfies condition (b) of Theorem 3.7 but not condition (a). It is true that condition (b) of Theorem 3.7 alone characterizes projectivity within the class of free ideals since, in this case, condition (a) is obviously superfluous.

As stated, Theorems 3.6 and 3.7 characterize the projective

z-ideals within the class of ideals in $C(X)$. It should be noted that both theorems can be restated as a characterization of projectivity within the class of z-ideals with only superficial changes in their proofs. Thus, conditions (a) and (b) of Theorem 3.7 characterize projectivity within the class of z-ideals. Although weaker, the theorems in this form are quite useful.

In [1], Bkouche has shown that if X is locally compact then $C_K(X)$, the ideal of functions with compact support, is projective if and only if X is paracompact. Although this is the only statement he makes in terms of $C(X)$, his results actually characterize projectivity within the class of pure ideals in $C(X)$ in terms of the topological properties of the Stone-Čech compactification of X. The next two theorems point out the relationship between the work of Bkouche and this paper.

THEOREM 3.8. *A pure ideal in $C(X)$ is a z-ideal.*

Proof. Let I be a pure ideal in $C(X)$. There is an open $U \subseteqq \beta X$ such that $I = \{f \in C(X): \operatorname{supp} f^* \subseteqq U\}$ where f^* is the continuous extension of f to a function from βX into the two point compactification of the reals (see [1] and [7]). Suppose $g \in C(X)$ with $Z(g) = Z(f)$ for some $f \in I$. By the normality of βX, there is a non-negative function k that is 2 on $\operatorname{supp} f^*$ and 0 on $\beta X \backslash U$. Let h be the restriction to X of $(k-1) \vee 0$. Then, $\operatorname{supp} h^* \subseteqq U$ so $h \in I$. Moreover, h is 1 on $\operatorname{supp} f = \operatorname{supp} g$. Thus, $g = gh \in I$. Hence, I is a z-ideal.

THEOREM 3.9. *A projective z-ideal in $C(X)$ is a pure ideal of $C(X)$.*

Proof. Let I be a projective z-ideal generated by the star finite partition of unity $\{h_\alpha\}_{\alpha \in A}$ on $\operatorname{coz} I$ as in Theorem 3.6 (e). Suppose f and g are in $C(X)$ with $fg \in I$. Let B be the finite set defined by $\{\alpha \in A: h_\alpha fg \neq 0\}$. Then $\sum_{\alpha \in B} h_\alpha \in I$ is 1 on $\operatorname{coz} fg$, so $fg = fg \sum_{\alpha \in B} h_\alpha = f(g \sum_{\alpha \in B} h_\alpha)$ where $g \sum_{\alpha \in B} h_\alpha \in I$. Therefore, I is a pure ideal of $C(X)$.

Since there are projective ideals that are not z-ideals, it follows from Theorem 3.8 that there are projective ideals that are not pure. Conversely, there are pure ideals that are not projective. For example, let X be a locally compact, nonparacompact space. Then, by the results of Bkouche, $C_K(X)$ is not projective. However, if the product gh is in $C_K(X)$, the normality of βX provides a $k \in C^*(X)$ such that k^β is 1 on $\operatorname{supp} gh$ and 0 on a closed neighborhood of $\beta X \backslash X$. Therefore, $k \in C_K(X)$ and $gh = (gh)k = g(hk)$ where $hk \in C_K(X)$.

Thus, $C_K(X)$ is pure.

Although the class of z-ideals properly contains the class studied by Bkouche, the preceding theorem (which depends on Theorem 3.6) shows that no additional projective ideals can be found by considering the larger class. Thus, the value of Theorems 3.6 and 3.7 when considered as characterizing projectivity within the class of z-ideals lies in the fact that they rule out the existence of projective z-ideals that are not pure and that they provide a test for projectivity within the space X as opposed to βX.

The following application of Theorem 3.6 generalizes the result of Bkouche for locally compact spaces.

THEOREM 3.10. *Let I be a z-ideal contained in $C_K(X)$. Then I is projective if and only if* $\operatorname{coz} I$ *is paracompact and contains* $\operatorname{supp} f$ *for each* $f \in I$.

Proof. If I is a projective z-ideal, then, by Proposition 3.4, $\operatorname{supp} f \subseteqq \operatorname{coz} I$ for all $f \in I$. By Theorem 3.6 (b), there is a set $\{f_\alpha\}_{\alpha \in A} \subseteqq I$ such that $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ is a star finite cover of $\operatorname{coz} I$ and $\operatorname{supp} f_\alpha \subseteqq \operatorname{coz} I$ for each $\alpha \in A$. Hence, $\{\operatorname{supp} f_\alpha\}_{\alpha \in A}$ is a locally finite cover of $\operatorname{coz} I$ consisting of compact sets. If U is an open cover of $\operatorname{coz} I$, let $\{U_{\alpha,i}\}_{i=1}^{n_\alpha}$ be a finite subset of U covering $\operatorname{supp} f_\alpha$ for each $\alpha \in A$. The cover $\{U_{\alpha,i} \cap \operatorname{coz} f_\alpha : \alpha \in A, 1 \leqq i \leqq n_\alpha\}$ is a locally finite refinement of U. Hence, $\operatorname{coz} I$ is paracompact.

Conversely, if $\operatorname{coz} I$ satisfies the stated condition, then the cover $\{\operatorname{coz} f\}_{f \in I}$ has a partition of unity $\{h'_\alpha\}_{\alpha \in A} \subseteqq C(\operatorname{coz} I)$ subordinate to it. Now, $\operatorname{coz} h'_\alpha \subseteqq \operatorname{supp} f$ for some $f \in I \subseteqq C_K(X)$, so $\operatorname{supp} h'_\alpha$ is compact and inside $\operatorname{coz} I$ for all $\alpha \in A$. Hence, each h'_α can be extended to $h_\alpha \in C(X)$ by setting it equal to 0 on $X \backslash \operatorname{coz} I$. Since each $\operatorname{coz} h_\alpha \subseteqq \operatorname{coz} f$ for some $f \in I$, the zero-set $Z(h_\alpha)$ is in $Z[I]$ and thus $\{h_\alpha\}_{\alpha \in A} \subseteqq I$.

The proof is completed by showing that $\{h_\alpha\}_{\alpha \in A} \subseteqq I$ satisfies the conditions of Theorem 3.6 (e). It is already known that $\{h_\alpha\}_{\alpha \in A}$ is a partition of unity on $\operatorname{coz} I$, and each $\operatorname{supp} h_\alpha \subseteqq \operatorname{coz} I$. Thus, for $\beta \in A$ and $x \in \operatorname{supp} h_\beta$ there is a neighborhood U_x such that $U_x \cap \operatorname{coz} h_\alpha = \varnothing$ for almost all $\alpha \in A$. Now, the cover $\{U_x\}_{x \in \operatorname{supp} h_\beta}$ of the compact set $\operatorname{supp} h_\beta$ has a finite subcover and, for $\alpha \in A$, $\operatorname{coz} h_\alpha$ meets $\operatorname{coz} h_\beta$ only if $\operatorname{coz} h_\alpha$ meets one of the sets in this subcover. Hence, $\{h_\alpha\}_{\alpha \in A}$ is star finite. If $g \in I$, a similar argument shows that $B = \{\alpha \in A : \operatorname{coz} h_\alpha \cap \operatorname{coz} g \neq \varnothing\}$ is finite, so $g = g \sum_{\alpha \in B} h_\alpha = \sum_{\alpha \in B} g h_\alpha$. Consequently, $\{h_\alpha\}_{\alpha \in A}$ generates I.

EXAMPLE 3.11. Let X be the subspace of the real line consisting of $\{r \in R : 0 \leqq r \leqq 1$ or r is rational$\}$. Then $C_K(X) = \{f \in C(X) : \operatorname{supp} f \subseteqq [0, 1]\}$. If $f \in C_K(X)$ is such that $\operatorname{coz} f = (0, 1)$, then $\operatorname{supp} f =$

$[0, 1] \nsubseteq (0, 1) = \operatorname{coz} C_K(X)$. Consequently, $C_K(X)$ is not projective by Theorem 3.10. Since $C_K(X)$ is not pure, the results of Bkouche are not applicable.

In [7], Vasconcelos discusses the pure and projective ideals in $C(I)$ where I is the closed unit interval. In particular, he shows that every pure and every projective ideal in $C(I)$ is countably generated. Theorem 3.6 shows that this is not true in general. For example, if X is the space of ordinals less than or equal to the first uncountable ordinal, then the characteristic functions of the non-limit ordinals generate a projective z-ideal [Theorem 3.6 (a) and (e)] (and hence a pure ideal by Theorem 3.9) that is not countably generated.

4. The role of z-ideals. The characterizations of projective z-ideals given in Proposition 3.4 and Theorem 3.6 suggest two ways in which these ideals can be considered abundant. First, Proposition 3.4 shows that the projective ideal generated in the proof of Proposition 2.8 is a z-ideal. Hence, if X has cardinality greater than 1, then $C(X)$ contains a proper projective z-ideal. Next, Theorem 3.6 (e) shows that the collection of functions $\{h_\alpha\}_{\alpha \in A}$ in Theorem 2.4 generates a (possibly nonproper) projective z-ideal. Consequently, each projective ideal is closely related to a projective z-ideal. The present section deals with this relationship. First on the agenda is to show that the projective z-ideal obtained above using Theorem 2.4 is independent of the choice of functions. The following lemma will be used for this and other purposes.

LEMMA 4.1. *If $\{f_\alpha\}_{\alpha \in A}$ and $\{h_\alpha\}_{\alpha \in A}$ are families of functions as in Theorem 2.4 and U is an open subset of X that is covered by finitely many members of $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$, then*

(a) $B = \{\alpha \in A : \bar{U} \cap \operatorname{coz} h_\alpha \neq \varnothing\}$ *is finite,*

(b) $\{\operatorname{coz} h_\alpha\}_{\alpha \in B}$ *covers* $\bar{U}$,

(c) $\sum_{\alpha \in B} h_\alpha = 1$ *on* $\bar{U}$, *and*

(d) $\bar{U}$ *and* $X \backslash \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$ *are completely separated.*

Proof. (a) If an open set U is covered by finitely many members of $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$, then the star finiteness of $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ dictates that $U \cap \operatorname{coz} f_\alpha = \varnothing$ for almost all $\alpha \in A$. Now, if $\bar{U} \cap \operatorname{coz} h_\alpha \neq \varnothing$, then $U \cap \operatorname{coz} h_\alpha \neq \varnothing$; and since $\operatorname{coz} h_\alpha \subseteq \operatorname{supp} f_\alpha$, $U \cap \operatorname{coz} h_\alpha \neq \varnothing$ implies $U \cap \operatorname{supp} f_\alpha \neq \varnothing$ which in turn implies $U \cap \operatorname{coz} f_\alpha \neq \varnothing$. But, this was shown to hold for only finitely many $\alpha \in A$ so $B = \{\alpha \in A : \bar{U} \cap \operatorname{coz} h_\alpha \neq \varnothing\}$ is finite.

(b) Since U is covered by finitely many members of $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$,

$\bar{U} \subseteq \bigcup_{\alpha \in A} \operatorname{supp} f_\alpha = \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$. Therefore, those members of $\{\operatorname{coz} h_\alpha\}_{\alpha \in A}$ that meet $\bar{U}$ must form a cover of $\bar{U}$.

(c) This is a consequence of (a), (b), the fact that $\{h_\alpha\}_{\alpha \in A}$ is a partition of unity on $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$, and the fact that $\bar{U} \subseteq \bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$.

(d) For each $\beta \in A$, $\operatorname{supp} h_\beta \subseteq \operatorname{supp} f_\beta \subseteq \bigcup_{\alpha \in A} \operatorname{supp} f_\alpha = \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$. Therefore, since B is finite, $\operatorname{supp} \sum_{\alpha \in B} h_\alpha \subseteq \sum_{\alpha \in B} \operatorname{supp} h_\alpha \subseteq \sum_{\alpha \in A} \operatorname{coz} h_\alpha$. Combining this with (c) completes the proof.

PROPOSITION 4.2. *The projective z-ideal obtained from a given projective ideal using Theorem* 2.4 *as above is independent of the choice of functions.*

Proof. Let $\{f_\alpha\}_{\alpha \in A}$ and $\{h_\alpha\}_{\alpha \in A}$ be families of functions satisfying conditions (a), (b), and (c) of Theorem 2.4. Suppose $\{f'_\beta\}_{\beta \in A'}$ and $\{h'_\beta\}_{\beta \in A'}$ are another such pair. If $\beta \in A'$, then f'_β is a linear combination of finitely many members of $\{f_\alpha\}_{\alpha \in A}$ so $\operatorname{coz} f'_\beta$ is covered by finitely many members of $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$. Thus, by Lemma 4.1, there is a finite subset $B \subseteq A$ such that $\sum_{\alpha \in B} h_\alpha = 1$ on $\operatorname{supp} f'_\beta$. Therefore, since $\operatorname{supp} h'_\beta \subseteq \operatorname{supp} f'_\beta$, $h'_\beta = h'_\beta \sum_{\alpha \in B} h_\alpha$. Consequently, h'_β is in the ideal generated by $\{h_\alpha\}_{\alpha \in A}$. It follows by symmetry that the ideals generated by $\{h_\alpha\}_{\alpha \in A}$ and $\{h'_\beta\}_{\beta \in B}$ must be the same.

It will be convenient to have a notation that distinguishes between ideals that are z-ideals and ideals that may or may not be z-ideals. For this purpose a subscript z will be used to designate those ideals that are known to be z-ideals, e.g., I_z. Moreover, if I is a projective ideal, the unique projective z-ideal associated with it will be denoted by I_z. One should be cautioned not to assume that $Z[I] = Z[I_z]$. In fact, by Corollary 3.5, this is true only if $I = I_z$. The following theorem states the relationship that holds between $Z[I]$ and $Z[I_z]$ in general.

THEOREM 4.3. *If I is a projective ideal in $C(X)$, then* $I_z = \{g \in C(X)\colon \operatorname{coz} g \subseteq \operatorname{supp} f$ *for some* $f \in I\}$ *and* $Z[I_z] = \{Z \in Z(X)\colon \operatorname{int} Z(f) \subseteq Z$ *for some* $f \in I\}$.

Proof. Let I be a projective ideal with $\{f_\alpha\}_{\alpha \in A}$ and $\{h_\alpha\}_{\alpha \in A}$ satisfying conditions (a), (b), and (c) of Theorem 2.4. Recall that $\{h_\alpha\}_{\alpha \in A}$ generates I_z so if $g \in I_z$ then there is a finite subset $B \subseteq A$ such that $\operatorname{coz} g \subseteq \bigcup_{\alpha \in B} \operatorname{coz} h_\alpha \subseteq \bigcup_{\alpha \in B} \operatorname{supp} f_\alpha = \operatorname{supp} \sum_{\alpha \in B} f_\alpha^2$ where $\sum_{\alpha \in B} f_\alpha^2 \in I$.

On the other hand, if $g \in C(X)$ is such that $\operatorname{coz} g \subseteq \operatorname{supp} f$ for some $f \in I$, then $g = g \sum_{\alpha \in B} h_\alpha \in I_z$ where $B = \{\alpha \in A\colon \operatorname{supp} f \cap \operatorname{coz} h_\alpha \neq \emptyset\}$ is finite by Lemma 4.1 (a).

Thus, $I_z = \{g \in C(X)\colon \operatorname{coz} g \subseteqq \operatorname{supp} f$ for some $f \in I\}$ which, when translated to terms of zero-sets, yields $Z[I_z] = \{Z \in Z(X)\colon \operatorname{int} Z(f) \subseteqq Z$ for some $f \in I\}$.

THEOREM 4.4. *If I is a projective ideal in $C(X)$ then I_z is the smallest projective z-ideal containing I.*

Proof. Let J_z be a projective z-ideal containing the projective ideal I and suppose $Z \in Z[I_z]$. Then, by Theorem 4.3, there is an $f \in I$ such that $\operatorname{int} Z(f) \subseteqq Z$. But, $I \subseteqq J_z$ so $f \in J_z$. Therefore, if $\{h_\alpha\}_{\alpha \in A}$ is a star finite partition of unity generating J_z as in Theorem 3.6 (e), then $\operatorname{coz} f$ is covered by a finite subset of $\{\operatorname{coz} h_\alpha\}_{\alpha \in A}$. Thus, by identifying both families of functions in Lemma 4.1 with $\{h_\alpha\}_{\alpha \in A}$, Lemma 4.1 implies there is a finite subset $B \subseteqq A$ such that $\sum_{\alpha \in B} h_\alpha = 1$ on $\operatorname{supp} f$, i.e., $Z(\sum_{\alpha \in B} h_\alpha) \subseteqq \operatorname{int} Z(f)$. Thus, there is a zero-set in $Z[I_z]$, namely $Z(\sum_{\alpha \in B} h_\alpha)$, contained in Z. Consequently $I_z \subseteqq J_z$.

Since the projective z-ideals can be characterized in topological terms, a topological method of finding the projective ideals associated with a given projective z-ideal is desirable. However, this remains an unsolved problem. Its solution is complicated by two facts. The first is that many projective ideals may be associated with a single projective z-ideal. For example, all the principal ideals generated by functions whose cozero-sets are dense in X are associated with the (nonproper) z-ideal $C(X)$. The second, and more devastating, is the fact that any such process must provide a counterpart to the divisibility requirement of Theorem 2.4 (c).

Given the similarities between the generating sets of I and I_z, one is tempted to try to circumvent this problem with the conjecture that I and I_z are module isomorphic. This, however, is false as seen in the following example.

EXAMPLE 4.5. Let X be the space obtained by identifying the points -4 and 4 in the interval $[-4, 4]$. Define f_1 and f_2 in $C(X)$ as follows. Both f_1 and f_2 are linear in each interval $[i, i+1]$, $i = 0, 1, 2, 3$; $f_1(x) = -f_1(-x)$, $f_2(x) = f_2(-x)$, $f_1(0) = 0$, $f_1(1) = 1$, $f_1(2) = 1$, $f_1(3) = 0$, $f_1(4) = 0$, $f_2(0) = 0$, $f_2(1) = 0$, $f_2(2) = 0$, $f_2(3) = 1$, and $f_2(4) = 1$. Then the ideal I generated by f_1 and f_2 is seen to be projective by identifying $\{f_1, f_2\}$ with the functions in Theorem 2.4 (a) and defining $\{h_1, h_2\}$ in Theorem 2.4 (b) by $h_1 = f_2$ and $h_1 = 1 - h_2$.

Since $h_1 + h_2 = 1$, the ideal I_z is equal to $C(X)$. Therefore, if I and I_z were module isomorphic, I would be principal. Suppose $f \in C(X)$ with $(f) = I$. Since X is a closed loop, f cannot change sign, for otherwise $Z(f)$ would contain more than a single point which would contradict the fact that $\operatorname{coz} f = \operatorname{coz}(f) = \operatorname{coz} I = X \backslash \{0\}$. Since $f_1 \in (f)$, there is a $g \in C(X)$ such that $f_1 = gf$; and since $f \in (f_1, f_2)$ and f_2 is 0 on a neighborhood U of 0, there is an $h \in C(U)$ such that $f = hf_1$ on U. Thus, $f_1 = gf = ghf_1$ on U which implies $gf = 1$ on U. However, since f_1 changes sign at 0 and f does not, $g(0) = 0$. Thus, gf cannot be the identity on any neighborhood of 0. Consequently, I cannot be principal, so I and I_z are not isomorphic.

In Example 4.5 $\operatorname{coz} f_1 + f_2 \neq \operatorname{coz} I$, so the ratio $f_1/(f_1 + f_2)$ is not defined on all of $\operatorname{coz} I$. The next theorem shows that the existence of such ratios is fundamental to the existence of a module isomorphism between I and I_z.

THEOREM 4.6. *A projective ideal I is module isomorphic to I_z if and only if it is generated by a star finite family $\{f_\alpha\}_{\alpha \in A}$ such that for each $\beta \in A$ the function $f_\beta / \sum_{\alpha \in A} f_\alpha$ is in $C(\operatorname{coz} I)$ and can be extended to a function $g_\beta \in C(X)$ where g_β is 0 on $X \backslash \operatorname{supp} f_\beta$.*

Proof. Let $\phi: I_z \to I$ be a module isomorphism and $\{h_\alpha\}_{\alpha \in A}$ be a star finite partition of unity on $\operatorname{coz} I_z$ generating I_z. Let $\phi(h_\alpha) = f_\alpha$ for each $\alpha \in A$. Then, $\{f_\alpha\}_{\alpha \in A}$ is a generating set for I. Moreover, since ϕ is a module isomorphism, the annihilator ideals of h_α and f_α must be equal. Thus, $\operatorname{supp} h_\alpha = \operatorname{supp} f_\alpha$ for each $\alpha \in A$. Furthermore, if $\alpha, \beta \in A$ then $f_\alpha f_\beta = \phi(h_\alpha)\phi(h_\beta) = \phi(h_\alpha \phi(h_\beta)) = \phi(\phi(h_\alpha h_\beta))$ so the star finiteness of $\{h_\alpha\}_{\alpha \in A}$ implies that of $\{f_\alpha\}_{\alpha \in A}$.

By [4, 2.5], ϕ must be multiplication by an element $g \in C(\operatorname{coz} I_z)$. For $x \in \operatorname{coz} I_z$ let B be the finite set $\{\alpha \in A: h_\alpha(x) \neq 0\}$. Then, $\sum_{\alpha \in B} h_\alpha(x) = 1$ so $g(x) = g(x) \sum_{\alpha \in B} h_\alpha(x) = \phi(\sum_{\alpha \in B} h_\alpha)(x) = \sum_{\alpha \in B} f_\alpha(x)$. Thus, $g = \sum_{\alpha \in A} f_\alpha |_{\operatorname{coz} I_z}$. Now the inverse of ϕ must be multiplication by $1/g$ so, for each $\beta \in A$, $h_\beta = f_\beta / \sum_{\alpha \in A} f_\alpha$ on $\operatorname{coz} I_z$. But this is extendible to a function in $C(X)$, namely h_β, satisfying the conditions in the theorem.

Conversely, suppose $\{f_\alpha\}_{\alpha \in A}$ has the properties stated in the theorem and let $\{h_\alpha\}_{\alpha \in A}$ be chosen as in Theorem 2.4. If $B \in A$, then by Lemma 4.1 the set $B = \{\alpha \in A: \operatorname{supp} f_\beta \cap \operatorname{coz} h_\alpha \neq \varnothing\}$ is finite and $\sum_{\alpha \in B} h_\alpha$ is 1 on $\operatorname{supp} f_\beta = \operatorname{supp} g_\beta$. Thus, $g_\beta = g_\beta \sum_{\alpha \in B} h_\alpha \in I_z$ since each $h_\alpha \in I_z$. Consequently, $\{g_\alpha\}_{\alpha \in A} \subseteqq I_z$.

For $\beta \in A$, the set $B = \{\alpha \in A: f_\alpha f_\beta \neq 0\}$ is finite since $\{\operatorname{coz} f_\alpha\}_{\alpha \in A}$ is star finite; and since $\operatorname{coz} f_\alpha = \operatorname{coz} I \cap \operatorname{coz} g_\alpha$ for each $\alpha \in A$, $\sum_{\alpha \in B} g_\alpha$ is 1 on $\operatorname{coz} f_\beta$. But, $\operatorname{coz} f_\beta$ is dense in $\operatorname{coz} h_\beta$ so $\sum_{\alpha \in B} g_\alpha$ is 1 on a dense subset of $\operatorname{coz} h_\beta$ and therefore on $\operatorname{coz} h_\beta$ itself. Consequently, $h_\beta = h_\beta \sum_{\alpha \in B} g_\alpha$. It follows that $\{g_\alpha\}_{\alpha \in A}$ generates I_z since $\{h_\alpha\}_{\alpha \in A}$ does.

The desired isomorphism $\phi: I_z \to I$ is now obtained by defining $\phi(g_\alpha) = f_\alpha$ for each $\alpha \in A$.

COROLLARY 4.7. *If a projective ideal I can be generated by a star finite family $\{f_\alpha\}_{\alpha \in A}$ such that* $\operatorname{pos} f_\alpha \cap \operatorname{neg} f_\beta = \varnothing$ *for all $\alpha, \beta \in A$, then it is module isomorphic to I_z.*

Proof. Due to Theorem 4.6, one needs to show only that the function $f_\beta/\sum_{\alpha \in A} f_\alpha$ is in $C(\operatorname{coz} I)$ for each $\beta \in A$ and can be extended to a function in $C(X)$ which is 0 on $X \backslash \operatorname{supp} f_\beta$. But since $\operatorname{pos} f_\alpha \cap \operatorname{neg} f_\beta = \varnothing$ for all $\alpha, \beta \in A$, $\operatorname{coz} \sum_{\alpha \in A} f_\alpha = \operatorname{coz} I$. So, $f_\beta/\sum_{\alpha \in A} f_\alpha$ is in $C(\operatorname{coz} I)$ for each $\beta \in A$.

The required extension is now done in two stages; first to $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$, then to X. Corresponding to $\{f_\alpha\}_{\alpha \in A}$, Theorem 2.4 provides a star finite partition of unity $\{h_\alpha\}_{\alpha \in A}$ on $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$. Therefore, $x_0 \in (\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha) \backslash \operatorname{coz} I$ implies that $h_{\alpha_0}(x_0) > 0$ for some $\alpha_0 \in A$; and by Theorem 2.4 (c), $h_{\alpha_0} f_\alpha = g_\alpha f_{\alpha_0}$ for some $g_\alpha \in C(X)$. Hence, on the neighborhood U of x_0 defined by $U = \{x \in X: h_{\alpha_0}(x) > (1/2)\, h_{\alpha_0}(x_0)\}$, one can write $f_\alpha = k_\alpha f_{\alpha_0}$ where $k_\alpha = (g_\alpha / h_{\alpha_0}) \in C(U)$. Moreover, since $\operatorname{pos} f_{\alpha_0} \cap \operatorname{neg} f_\alpha = \varnothing$ and $\operatorname{neg} f_{\alpha_0} \cap \operatorname{pos} f_\alpha = \varnothing$ for all $\alpha \in A$, each k_α is nonnegative on U. Therefore, on $U \cap \operatorname{coz} I$,

$$\frac{f_\beta}{\sum\limits_{\alpha \in A} f_\alpha} = \frac{k_\beta f_{\alpha_0}}{(1 + \sum\limits_{\alpha \neq \alpha_0} k_\alpha) f_{\alpha_0}} = \frac{k_\beta}{1 + \sum\limits_{\alpha \neq \alpha_0} k_\alpha} .$$

But the right hand side is continuous at x_0 since the denominator is bounded away from 0 on U. Hence, $f_\beta/\sum_{\alpha \in A} f_\alpha$ can be extended continuously to $\{x_0\} \cup \operatorname{coz} I$. Thus, by [6, 6H], there exists a function $f'_\beta \in C(\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha)$ which is a continuous extension of $f_\beta/\sum_{\alpha \in A} f_\alpha$.

To obtain an extension of f'_β to X, note that f'_β is 0 on $(\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha) \backslash \operatorname{supp} f_\beta$ since it must be 0 on $\operatorname{coz} I \backslash \operatorname{supp} f_\beta$. Also, $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha = \bigcup_{\alpha \in A} \operatorname{coz} h_\alpha$ is open and contains $\operatorname{supp} f_\beta$. Therefore, f'_β can be extended continuously to X by defining it to be 0 on $X \backslash \operatorname{supp} f_\beta$.

If an ideal is absolutely convex, then any generating set $\{f_\alpha\}_{\alpha \in A}$ can be replaced by $\{f^+_\alpha, f^-_\alpha\}_{\alpha \in A}$. Hence, the class of ideals covered by Corollary 4.7 includes the absolutely convex ones. Recall that if X is an F-space [6, 14.25], then every ideal in $C(X)$ is absolutely convex [6, 14.26]. Hence, there are cases for which every projective ideal is module isomorphic to its associated projective z-ideal.

Also of interest is the fact that if a projective ideal is generated by the star finite family $\{f_\alpha\}_{\alpha \in A}$, then the ideal generated by $\{f^2_\alpha\}_{\alpha \in A}$ is projective. (If $\{h_\alpha\}_{\alpha \in A}$ is a partition of unity on $\bigcup_{\alpha \in A} \operatorname{supp} f_\alpha$ associated with $\{f_\alpha\}_{\alpha \in A}$ as in Theorem 2.4, then $\{h_\alpha \bar{h}_\alpha\}_{\alpha \in A}$ (recall Definition 3.3) is a star finite partition of unity on $\bigcup_{\alpha\in A}\operatorname{supp} f_\alpha^2$. Moreover, by Theorem 2.4 if $\alpha, \beta \in A$, there is a $g \in C(X)$ such that $f_\beta h_\alpha = g f_\alpha$; but by Lemma 4.1 (d), $\operatorname{supp} f_\alpha$ and $X\backslash\bigcup_{\gamma\in A}\operatorname{coz} h_\gamma$ are completely separated so g can be chosen such that $\operatorname{supp} g \subseteq \bigcup_{\gamma\in A}\operatorname{coz} h_\gamma$. Consequently, $f_\beta^2(h_\alpha\bar{h}_\alpha) = f_\alpha^2[g^2/\sum_{\gamma\in A} h_\gamma^2]$ where $g^2/\sum_{\gamma\in A} h_\gamma^2$ is extended continuously to X. Therefore, $\{f_\alpha^2\}_{\alpha\in A}$ and $\{h_\alpha\bar{h}_\alpha\}_{\alpha\in A}$ satisfy the conditions of Theorem 2.4.) By Corollary 4.7, this ideal is module isomorphic to its associated projective z-ideal. Furthermore, as will be seen in Proposition 4.8, the ideals generated by $\{f_\alpha\}_{\alpha\in A}$ and $\{f_\alpha^2\}_{\alpha\in A}$ are associated with the same projective z-ideal. Thus, a projective ideal I always contains a module isomorphic copy of I_z.

The following proposition characterizes the relationship between projective ideals which are associated with the same projective z-ideal.

PROPOSITION 4.8. *If I and J are projective ideals, then $I_z = J_z$ if and only if there are star finite generating sets $\{f_{1,\alpha}\}_{\alpha\in A}$ and $\{f_{2,\alpha}\}_{\alpha\in A}$ of I and J respectively such that* $\operatorname{supp} f_{1,\alpha} = \operatorname{supp} f_{2,\alpha}$ *for each $\alpha \in A$.*

Proof. Suppose $I_z = J_z$. Theorem 2.4 provides star finite generating sets $\{f'_\beta\}_{\beta\in B}$ and $\{f''_\gamma\}_{\gamma\in C}$ for I and J respectively with corresponding partitions of unity $\{h'_\beta\}_{\beta\in B}$ and $\{h''_\gamma\}_{\gamma\in C}$ such that $\operatorname{supp} f'_\beta = \operatorname{supp} h'_\beta$ for each $\beta \in B$ and $\operatorname{supp} f''_\gamma = \operatorname{supp} h''_\gamma$ for each $\gamma \in C$. Let $A = B \times C$ and, for each $(\beta, \gamma) = \alpha \in A$, define $f'_\alpha = f'_\beta h''_\gamma$ and $f''_\alpha = f''_\gamma h'_\beta$. From the star finiteness of the collections involved, it follows that both $\{f'_\alpha\}_{\alpha\in A}$ and $\{f''_\alpha\}_{\alpha\in A}$ are star finite. Moreover, since $I_z = J_z$, each h'_β must be a linear combination of a finite subset $\{h''_\gamma\}_{\gamma\in C_\beta}$ of $\{h''_\gamma\}_{\gamma\in C}$; and since $\{h''_\gamma\}_{\gamma\in C}$ is star finite, $\{\gamma \in C: h''_\gamma h''_\delta \neq 0$ for some $\delta \in C_\beta\} = C'_\beta$ is finite. Thus, $\sum_{\gamma\in C'_\beta} h''_\gamma = 1$ on $\operatorname{coz} h'_\beta$ and consequently on $\operatorname{supp} h'_\beta = \operatorname{supp} f'_\beta$ so $f'_\beta = f'_\beta \sum_{\gamma\in C'_\beta} h''_\gamma = \sum_{\gamma\in C'_\beta} f'_\beta h''_\gamma$. Therefore, $\{f'_\alpha\}_{\alpha\in A}$ generates I.

By symmetry, $\{f''_\alpha\}_{\alpha\in A}$ generates J. The observation that $\operatorname{supp} f'_\alpha = \operatorname{supp} h'_\beta h''_\gamma = \operatorname{supp} f''_\alpha$ for each $(\beta, \gamma) = \alpha \in A$ completes this part of the proof.

Conversely, suppose $\{f_{1,\alpha}\}_{\alpha\in A}$ and $\{f_{2,\alpha}\}_{\alpha\in A}$ satisfy the conditions of the proposition. Then, by Theorem 2.4, there are star finite partitions of unity $\{h_{1,\alpha}\}_{\alpha\in A}$ and $\{h_{2,\alpha}\}_{\alpha\in A}$ generating I_z and J_z respectively such that $\operatorname{supp} h_{1,\alpha} \subseteq \operatorname{supp} f_{1,\alpha} = \operatorname{supp} f_{2,\alpha}$ and $\operatorname{supp} h_{2,\alpha} \subseteq \operatorname{supp} f_{2,\alpha} = \operatorname{supp} f_{1,\alpha}$ for each $\alpha \in A$. It follows by Lemma 4.1 (c) that if $\alpha \in A$ then there is a finite $B \subseteq A$ such that $h_{1,\alpha} = h_{1,\alpha}(\sum_{\beta\in B} h_{2,\beta}) \in J_z$. (Recall that $\{h_{2,\alpha}\}_{\alpha\in A}$ generates J_z.) Thus, $I_z \subseteq J_z$. The rest follows by symmetry.

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