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A CHARACTERIZATION FOR COMPACT CENTRAL DOUBLE CENTRALIZERS ON C^*-ALGEBRAS

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The purpose of this note is to give a characterization for compact central double centralizers on any C^*-algebra A in view of the Dixmier's representation theorem of central double centralizers on A. The proof makes use of the Urysohn's lemma for spectra of C^*-algebras and algebraic properties of a central double centralizer.

Throughout the note, A denotes a C^*-algebra. Let Prim A denote the structure space of A, that is the set of all primitive ideals of A, with the hull-kernel topology. Let $M(A)$ denote the double centralizer algebra of A and $Z(M(A))$ the center of $M(A)$. Busby [1] has noted that the algebra $C^b(\mathrm{Prim}\,A)$ of all bounded continuous complex-valued functions on Prim A can be canonically identified with $Z(M(A))$, which is equivalent with a result of Dixmier ([5], Theorem 5). Moreover, we can regard the algebra $Z(M(A))$ as the algebra of all bounded linear operators T on A such that $(Tx)y = x(Ty)$ for all $x, y \in A$. In its final form, this identification Φ between $Z(M(A))$ and $C^b(\mathrm{Prim}\,A)$ can be described as follows: If $T \in Z(M(A))$, then $Ta + P = \Phi(T)(P)(a + P)$ for all $a \in A$ and $P \in \mathrm{Prim}\,A$, where $a + P$ for $P \in \mathrm{Prim}\,A$ denotes the canonical image of a in A/P (Dauns and Hofmann theorem [3] shows that every functions in $C^b(\mathrm{Prim}\,A)$ can be realized uniquely in this way). We will characterize the set of all compact central double centralizers on A in view of this representation theorem of $Z(M(A))$. Our characterization is similar to ones established by Kellogg [6] and Ching and Wong [2] for H^*-algebras, and this is also a generalization of one proved by Rowlands [7] for dual B^*-algebras.

Let $Z_C(M(A))$ denote the compact central double centralizers on A. If $LC(A)$ is the algebra of all compact operators on A, then $Z_C(M(A)) = Z(M(A)) \cap LC(A)$, so that $Z_C(M(A))$ is a closed ideal of $Z(M(A))$. Let I_C be the set of all functions f in $C^b(\mathrm{Prim}\,A)$ such that for any closed compact subset K in $\mathrm{supp}\,(f)$, A/I_K is finite dimensional. Here $\mathrm{supp}\,(f)$ denotes the set of all $P \in \mathrm{Prim}\,A$ such that $f(P) \neq 0$, and I_K denotes a closed two-sided ideal of A with $\mathrm{Prim}\,(A/I_K) \simeq K$ (cf. [4], §3.2). Note that if K is the empty set, then A/I_K is zero-dimensional, so that I_C contains the zero function. Now I_C is a closed ideal in $C^b(\mathrm{Prim}\,A)$. For since $\mathrm{supp}\,(f) \supset \mathrm{supp}\,(fg)$ for each f, g in $C^b(\mathrm{Prim}\,A)$, I_C is an ideal in $C^b(\mathrm{Prim}\,A)$. Let $\{f_n\}$ be a sequence of functions in I_C which converges uniformly to a function f in $C^b(\text{Prim}\,A)$. Let K be any nonempty closed compact subset in $\text{supp}\,(f)$. Set

$$\delta = \inf\{|f(P)|: P \in K\}.$$

Then $\delta > 0$ and $\|f_N - f\| < \delta$ for sufficiently large number N. This implies $K \subset \text{supp}\,(f_N)$. Then A/I_K is finite dimensional since $f_N \in I_C$. Hence $f \in I_C$ and so I_C is uniformly closed. Let $C_0(\text{Prim}\,A)$ be the set of all bounded continuous complex-valued functions on Prim A which vanish at infinity. Let $I_{C0} = I_C \cap C_0(\text{Prim}\,A)$. Then I_{C0} is a closed ideal of $C^b(\text{Prim}\,A)$.

We now show that these ideals $Z_C(M(A))$ and I_{C0} can be canonically identified and thus obtain a characterization for $Z_C(M(A))$.

THEOREM 1. *$Z_C(M(A))$ is isometrically *-isomorphic to I_{C0}.*

To show the above theorem, we need the following Urysohn's lemma for arbitrary C^*-algebras.

LEMMA 2 ([8], Theorem). *Let $\hat{A}$ be the spectrum of A and let S_1, S_2 be two nonempty closed subsets in $\hat{A}$. Then the following two conditions are equivalent*

(i) $S_1 \cap S_2 = \varnothing$.

(ii) *For any element $a \geqq 0$ in A there exists an element x in A such that $0 \leqq x \leqq a$, $\pi(x) = 0$ for all $\pi \in S_1$, and $\pi(x) = \pi(a)$ for all $\pi \in S_2$.*

Proof of Theorem 1. Let Φ be the canonical *-isomorphism of $Z(M(A))$ onto $C^b(\text{Prim}\,A)$ as be stated above. We will show that $\Phi(Z_C(M(A))) = I_{C0}$ going through three steps.

(I) $\Phi(Z_C(M(A))) \supset I_{C0}$. Let $f \in I_{C0}$ and $\varepsilon > 0$ be chosen arbitrarily. Set

$$K_\varepsilon = \{P \in \text{Prim}\,A: |f(P)| \geqq \varepsilon\}$$

and

$$F_\varepsilon = \{P \in \text{Prim}\,A: |f(P)| \leqq \varepsilon/2\}.$$

Let $\{u_\lambda\}$ be a positive approximate identity for A (in the sense of Appendice B29 in [4]). By Lemma 2, for each λ there exists an element $x_{\lambda,\varepsilon}$ in A such that $0 \leqq x_{\lambda,\varepsilon} \leqq u_\lambda$, $x_{\lambda,\varepsilon} + P = u_\lambda + P$ for all $P \in K_\varepsilon$ and $x_{\lambda,\varepsilon} + P = 0$ for all $P \in F_\varepsilon$. Set $T = \Phi^{-1}(f)$, so that T is a central double centralizer on A. Moreover, set

$$T_{\lambda,\varepsilon}(a) = T(x_{\lambda,\varepsilon}a)$$

for each λ and $a \in A$. Then $T_{\lambda,\varepsilon}$ is a bounded linear operator on A.

We will show that $T_{\lambda,\varepsilon}$ is an element of $LC(A)$. Let $\operatorname{supp}(Tx_{\lambda,\varepsilon})$ be the set of all $P \in \operatorname{Prim} A$ such that $Tx_{\lambda,\varepsilon} \notin P$. Since $Tx_{\lambda,\varepsilon} \in T(P) \subset P$ for all $P \in F_\varepsilon$, we have F_ε is included $\operatorname{Prim}(A) \backslash \operatorname{supp}(Tx_{\lambda,\varepsilon})$. This implies that

$$\operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon})) \subset \operatorname{cl}(\operatorname{Prim}(A) \backslash F_\varepsilon) \subset K_{\varepsilon/2} ,$$

where cl denotes closure in the hull-kernel topology. Since $K_{\varepsilon/2}$ is compact, it follows that $\operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))$ is a closed compact subset in $\operatorname{supp}(f)$. Let $I_{\lambda,\varepsilon}$ is a closed two-sided ideal of A such that $\operatorname{Prim}(A/I_{\lambda,\varepsilon}) \simeq \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))$. Then $A/I_{\lambda,\varepsilon}$ is finite dimensional since $f \in I_C$. Let $\{a_n\}$ be a sequence of A with $\|a_n\| \leqq 1$ for all $n = 1, 2, \cdots$. Then $\{a_n + I_{\lambda,\varepsilon}\}$ is also a bounded sequence in $A/I_{\lambda,\varepsilon}$, so that there exists a convergent subsequence $\{a_{n_j} + I_{\lambda,\varepsilon}\}$. We now have

$$\begin{aligned} &\| T_{\lambda,\varepsilon}(a_{n_j}) - T_{\lambda,\varepsilon}(a_{n_k}) \| \\ &\quad = \sup\{\| (Tx_{\lambda,\varepsilon})(a_{n_j} - a_{n_k}) + P \| : P \in \operatorname{Prim} A\} \\ &\quad = \sup\{\| (Tx_{\lambda,\varepsilon} + P)(a_{n_j} - a_{n_k} + P) \| : P \in \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))\} \\ &\quad \leqq \sup\{\| T \| \, \| a_{n_j} - a_{n_k} + P \| : P \in \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))\} \\ &\quad = \| T \| \, \| (a_{n_j} + I_{\lambda,\varepsilon}) - (a_{n_k} + I_{\lambda,\varepsilon}) \| \end{aligned}$$

for all $j, k = 1, 2, \cdots$. Then $\{T_{\lambda,\varepsilon}(a_{n_j})\}$ is Cauchy and hence converges in A. Thus $T_{\lambda,\varepsilon}$ is compact for each λ. Now since $f \in I_C$ and K_ε is a closed compact subset in $\operatorname{supp}(f)$, it follows that A/I_{K_ε} is finite dimensional C^*-algebra and hence $\{u_\lambda + I_{K_\varepsilon}\}$ converges to the identity 1_ε of A/I_{K_ε}. Then there exists a λ_ε such that $\| 1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon}) \| < \varepsilon$. Set $T_\varepsilon = T_{\lambda_\varepsilon,\varepsilon}$ and $x_\varepsilon = x_{\lambda_\varepsilon,\varepsilon}$. For any $a \in A$ we further set

$$\alpha = \sup\{\| (Ta - x_\varepsilon Ta) + P \| : P \in K_\varepsilon\} ,$$

$$\beta = \sup\{\| T(a - x_\varepsilon a) + P \| : P \in \operatorname{Prim}(A) \backslash K_\varepsilon\} .$$

Since $x_\varepsilon + P = u_{\lambda_\varepsilon} + P$ for all $P \in K_\varepsilon$, we have

$$\begin{aligned} \alpha &= \sup\{\| (Ta + P) - (u_{\lambda_\varepsilon} + P)(Ta + P) \| : P \in K_\varepsilon\} \\ &= \| (1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon}))(Ta + I_{K_\varepsilon}) \| \\ &\leqq \| Ta \| \, \varepsilon . \end{aligned}$$

We further have

$$\begin{aligned} \beta &= \sup\{| f(P) | \, \| (a - x_\varepsilon a) + P \| : P \in \operatorname{Prim}(A) \backslash K_\varepsilon\} \\ &\leqq (\| a \| + \| u_{\lambda_\varepsilon} \| \, \| a \|)\varepsilon \\ &\leqq 2 \| a \| \, \varepsilon . \end{aligned}$$

Therefore $\|Ta - T_\varepsilon a\| \leqq \alpha + \beta \leqq (\|Ta\| + 2\|a\|)\varepsilon$ for all $a \in A$, so that $\|T - T_\varepsilon\| \leqq (\|T\| + 2)\varepsilon$. Since T_ε is compact and ε is arbitrary, T is also compact and (I) is proved.

(II) $\Phi(Z_C(M(A))) \subset I_C$. Let $f \in \Phi(Z_C(M(A)))$ and $T \in Z_C(M(A))$ with $f = \Phi(T)$. Suppose that $f \notin I_C$, so that there exists a non-empty closed compact subset K in supp(f) such that A/I_K is infinite dimensional. Then there exist elements a_n in A such that $\|a_n + I_K\| = 1$ $(n = 1, 2, \cdots)$ and $\|(a_n + I_K) - (a_m + I_K)\| \geqq 1/2$ $(n \neq m)$. We can assume that $\|a_n\| \leqq 2$ $(n = 1, 2, \cdots)$. Set

$$\delta = \inf\{|f(P)|: P \in K\} .$$

Then $\delta > 0$ since K is compact and we have

$$\begin{aligned} \|Ta_n - Ta_m\| &\geqq \sup\{|f(P)|\,\|(a_n - a_m) + P\|: P \in K\} \\ &\geqq \sup\{\|(a_n - a_m) + P\|\delta: P \in K\} \\ &= \|(a_n + I_K) - (a_m + I_K)\|\delta \\ &\geqq \delta/2 \end{aligned}$$

for all distinct numbers n, m. Then $\{Ta_n\}$ contains no convergent subsequence. But this is imposible since T is compact and (II) is proved.

(III) $\Phi(Z_C(M(A))) \subset C_0(\mathrm{Prim}\, A)$. Let $T \in Z_C(M(A))$ and $\varepsilon > 0$. Set

$$f = \Phi(T) \text{ and } K_\varepsilon = \{P \in \mathrm{Prim}\, A: |f(P)| \geqq \varepsilon\} .$$

We only show that K_ε is compact. Let I_{K_ε} be a closed two-sided ideal of A with Prim$(A/I_{K_\varepsilon}) \simeq K_\varepsilon$, as be stated above. Suppose that A/I_{K_ε} is infinite dimensional. Then, as in the proof of (II), there exist elements a_n in A such that $\|a_n\| \leqq 2$, $\|a_n + I_{K_\varepsilon}\| = 1$ $(n = 1, 2, \cdots)$ and $\|(a_n + I_{K_\varepsilon}) - (a_m + I_{K_\varepsilon})\| \geqq 1/2$ $(n \neq m)$. By the same computation in the proof of (II), we have $\|Ta_n - Ta_m\| \geqq \varepsilon/2$, so that $\{Ta_n\}$ contains no convergent subsequence, which contradicts T is compact. Thus A/I_{K_ε} is a finite dimensinal C^*-algebra. Then A/I_{K_ε} can be canonically identified with its enveloping von Neumann algebra. Suppose that Prim(A/I_{K_ε}) contains an infinite countable subset $\{P_1, P_2, \cdots\}$. Let π_i be a nonzero irreducible representation of A/I_{K_ε} with $P_i = \mathrm{Ker}\, \pi_i$ and ξ_i a norm one element in the Hilbert space associated with π_i for each i. Set

$$f_i(x + I_{K_\varepsilon}) = (\pi_i(x + I_{K_\varepsilon})\xi_i | \xi_i) \quad (i = 1, 2, \cdots)$$

for each $x + I_{K_\varepsilon} \in A/I_{K_\varepsilon}$. Since $\pi_i \not\simeq \pi_j$ $(i \neq j)$, it follows that $\|f_i - f_j\| = 2$ $(i \neq j)$ (cf. [4], 2.12.1). Let p_i denote the support of f_i for each i. Then $\{p_i\}$ are mutually orthogonal (cf. [4], 12.3.1). But this is imposible since each p_i is an element in A/I_{K_ε} and so

Prim (A/I_{K_ε}) is finite set. Then K_ε is also a finite set, so that it is compact and (III) is proved.

We will next show that a result of Rowlands ([7], Theorem 2) is a special case of Theorem 1. Let $\Omega(A)$ be the space of minimal closed two-sided ideals of A with its discrete topology, in case A is dual. Let $\{I_\lambda: \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A and $\Lambda_0 = \{\lambda \in \Lambda: I_\lambda$ is infinite dimensional$\}$. Let I_0 be the set of all functions f in the algebra $C^b(\Omega(A))$ of all bounded complex-valued functions on $\Omega(A)$ such that $f(I_\lambda) = 0$ for all $\lambda \in \Lambda_0$; if $\Lambda_0 = \emptyset$, let $I_0 = C^b(\Omega(A))$. Let $C_0(\Omega(A))$ be the subalgebra of $C^b(\Omega(A))$ which consists of functions vanishing at infinity.

COROLLARY 3 ([7], Theorem 2). *If A is a dual C^*-algebra, then $Z_C(M(A))$ is isometrically *-isomorphic to $I_0 \cap C_0(\Omega(A))$.*

Proof. By ([4], 10.10.6), Prim A is discrete. For each $P \in$ Prim A, we define a function δ_P on Prim A by the equation: $\delta_P(P) = 1$ and $\delta_P(Q) = 0$ if $Q \neq P$, and set $\mu(P) = \Phi^{-1}(\delta_P)(A)$. Then we can easily see that $P \longrightarrow \mu(P)$ is a bijection of Prim A onto $\Omega(A)$. Let μ^* be the dual map of μ. Then μ^* is a isometric *-isomorphism of $C^b(\Omega(A))$ onto $C^b(\text{Prim } A)$. By the definitions of I_C and I_0, we see that $\mu^*(I_0 \cap C_0(\Omega(A))) = I_{C0}$. Set $\Psi(T) = (\mu^*)^{-1}(\Phi(T))$ for each $T \in Z_C(M(A))$. Then $\Psi(Z_C(M(A))) = I_0 \cap C_0(\Omega(A))$ by Theorem 1 and the corollary is proved.

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REFERENCES

1. R. Busby, *On structure spaces and extensions of C^*-algebras*, J. Functional Analysis, **1** (1967), 370-377.
2. W. Ching and J. Wong, *Multipliers and H^*-algebras*, Pacific J. Math., **22** (1967), 387-396.
3. J. Dauns and K. H. Hofmann, *Representations of rings by continuous sections*, Mem. Amer. Math. Soc., **83** (1968).
4. J. Dixmier, *Les C^* algèbres et leurs représentations*, (Cahiers Scientifique 24), Gauthier-Villars, Paris, 1964.
5. ———, *Ideal center of a C^*-algebra*, Duke Math. J., **35** (1968), 375-382.
6. C. N. Kellogg, *Centralizers and H^*-algebras*, Pacific J. Math., **17** (1966), 121-129.
7. K. Rowlands, *A note on multiplier operators and dual B^*-algebras*, Canad. Math. Bull., **17** (1974), 563-565.
8. S. Takahasi, *A remark on Urysohn's lemma for spectrums of C^*-algebras*, Bull. Fac. Sci. Ibaraki Univ. Ser. A, **4** (1972), 33-35.

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