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THE SECOND DUAL OF $C(X)$

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In this paper, we undertake a study of the order dual, denoted M, of the radon measures of compact support on a locally compact space X. In the case that X is realcompact, M is the second (order) dual of the space of continuous functions on X, $C(X)$. We define the sublattice of semi-continuous elements, $S(X)$, and prove that each member of M is dominated by a member of $S(X)$. It follows that the ideal generated by $S(X)$ in M is all of M. On the other hand, the ideal generated by $C(X)$ in M is all of M if and only if X is a cb-space.

Finally, we show that $S(X)$ and $C(X)$ can be identified in M as certain spaces of multiplication operators which are continuous with respect to certain weak topologies. This extends the work of J. Mack, who first characterized M as the (continuous) multiplication operators on the Radon measures.

Introduction. In [3] Kaplan considered $C_k(X) = C_k$, the continuous functions of compact support on a locally compact space, and its order dual L_k (the space of Radon measures). In the process, he singled out $\cup L(K)$, the ideal of those measures having compact support. It is the order dual of this space, denoted M, in which we will be interested. In the case that X is realcompact, M is the second dual of the space of continuous functions and therefore of particular interest.

M has already been studied by Mack [5], who characterized it as the set of (order) continuous multiplication operators on L_k. It is our purpose to extend his work. In considering the case where X is compact, Kaplan studied various sublattices of M including what he called the semi-continuous elements $S(X)$. We will extend the study to our more general setting and show that $S(X)$ and $C(X)$ can be identified in M as spaces of multiplication operators on L_k, continuous with respect to certain weak topologies. Thus we will relate the work of the two authors.

1. Preliminaries. The information and results summarized here will be used frequently in the rest of the paper. We assume a knowledge of the basic results on Riesz spaces.

1.1. A subset B of a Riesz space (vector lattice) E is called bounded if it is contained in some interval $[a, b] = \{c \in E: a \leqq c \leqq b\}$. E is called (Dedekind) complete if the supremum, $\vee B$, and the infimum, $\wedge B$, exist for all bounded sets. If E and F are vector lattices, a map from E to F is called bounded if it transforms bounded sets into bounded sets. A linear mapping is called positive if it maps the positive cone E_+ into F_+. If F is a complete vector lattice, then a linear mapping from E into F is bounded if and only if it is the difference of two positive mappings, and the set of all such mappings is a complete vector lattice. The set of bounded linear functionals is denoted E^b. If $A \subset E^b$ is directed upward, that is for each f_1 and f_2 in A, there exists $f^* \in A$ such that $f^* \geqq f_1$ and $f^* \geqq f_2$, then $f = \vee A$ if and only if $\langle a, f\rangle = \vee \{\langle a, f_\alpha\rangle | f_\alpha \in A\}$ for all a in E_+. Given a subset B of E^b one may adjoin to it all suprema of finite subsets and the resulting set will be directed upward and have the same supremum, if it exists, as the subset B.

1.2. Given a directed net $\{a_\alpha\}$ in E, $a_\alpha \uparrow a$ means $a_\alpha \geqq a_\beta$ for $\alpha \geqq \beta$ and $a = \vee a_\alpha$; $a_\alpha \downarrow a$ is defined similarly. A net $\{a_\alpha\}$ converges to a if there exists a net $\{b_\alpha\}$ such that $b_\alpha \downarrow 0$ and $|a - a_\alpha| \leqq b_\alpha$ for all α. We will write in this case $a_\alpha \rightarrow a$. A linear functional ϕ is called continuous if $a_\alpha \rightarrow a$ implies $\langle a_\alpha, \phi\rangle \rightarrow \langle a, \phi\rangle$. E^c denotes the space of continuous linear functionals.

1.3. A subset A is closed if $\{a_\alpha\} \subset A$ and $a_\alpha \rightarrow a$ imply $a \in A$. An ideal is a linear subspace I of E such that $a \in I$ and $|b| \leqq |a|$ imply $b \in I$. If A is a subspace of E, the ideal generated by A, $I(A) = \{b \in E: |b| \leqq |a| \text{ for some } a \in A\}$. If $E = I \oplus J$, then a_I will denote the component of $a \in E$ in I. I will be called a band. Given a subset A in E, A' will be the set of elements disjoint from A: $A' = \{x: |x| \wedge |a| = 0 \text{ for all } a \in A\}$. A' is a closed ideal and if $E = I \oplus J$, $J = I'$, so bands are closed ideals. In a complete space, closed ideals are bands (Riesz). Finally, if $E = I \oplus J$, then $E^b = J^\perp \oplus I^\perp$ where $I^\perp$ has the usual definition. It is also true that $I^b = J^\perp = (I')^\perp = I^{\perp\prime}$. If $\phi \in E^b_+$, then the component of ϕ in $I^{\perp\prime}$ is given by $\langle \phi_{I^{\perp\prime}}, \mu\rangle = \vee \{\langle \phi, \nu\rangle | 0 \leqq \nu \leqq \mu, \nu \in I\}$ for all $\mu \in E_+$.

1.4. In this paper, X is locally compact, $C = C(X)$ is the space of continuous functions, $C_k = C_k(X)$ is the subset of those having compact support. 1_X will denote the function identically equal to one on X. If $f \in C(X)$, the symbols $S(f)$ and coz f represent respectively, $cl_X\ \{x: f(x) \neq 0\}$ and $\{x: f(x) \neq 0\}$. $L_k(X) = C_k^b$, the space of Radon measures. Unless there is danger of confusion, we will not indicate the underlying space in the above notation. If $\mu \in L_k$ and if $\sup \{|\langle h, \mu\rangle|: h \in C_k \text{ and } |h| \leqq 1\}$ exists and is finite then μ is called a bounded Radon measure and the supremum is defined to be $\|\mu\|$. All measures of compact support are bounded. For compact sets, we follow Kaplan's notation: if K is a compact set, $C(K)$ is the Banach lattice of continuous functions, $L(K)$ its dual and $M(K)$ the bidual.

1.5. Let K be a compact subset of X. In general, $C(K)$ cannot be identified with an ideal in C_k. Its dual, however, is a direct summand of L_k. Indeed, let $I = \{f \in C_k: f|_K = 0\}$. Then $C(K)$ can be identified with the quotient space C_k/I. It follows that $L(K) = I^\perp$ and since L_k is complete, $L_k = L(K) \oplus I^{\perp\prime}$. If M_k is the (continuous) second dual of C_k, we also have $M_k = M(K) \oplus L(K)^\perp$.

1.6. The set $\cup L(K)$ as K ranges over all compact subsets of X is an ideal in L_k [3, (4.2)]. Let $M = M(X) = (\cup L(K))^b$. If X is realcompact, M is the second dual of C. If $f \in M$ and $\sup\{|\langle f, \mu\rangle|: \mu \in \cup L(K), \|\mu\| \leqq 1\}$ exists and is finite, then f is called bounded and the supremum is denoted $\|f\|$. Since for each compact set $K, L(K)$ is a closed ideal in $\cup L(K)$, $M(X) = M(K) \oplus L(K)^\perp$. For each compact set K, $M(K)$ consists of bounded elements.

1.7. Let $(L_k)_0$ be the closed ideal generated by X when considered as a subset of L_k and $(L_k)_1$ its complementary ideal. Clearly, $(L_k)_0$ consists of the purely atomic Radon measures on X. If $x \in X$, we will represent the atomic measure at x by x. For any subset $A \subset L_k$, we let A_i be the projection on $(L_k)_i$. Then since $\cup L(K) = (\cup L(K))_0 \oplus (\cup L(K))_1$, we have $M = M_0 \oplus M_1$ where $M_i = (\cup L(K))_i^{\perp\prime}$ in M. M_0 is lattice and ring isomorphic to the locally bounded functions on X. [5, (5.7)]. For convenience, if x is the atomic measure at x and $f \in M$, we will usually write $\langle f, x\rangle$ as $f_0(x)$.

2. The ideals $M(K)$. Since for each compact set K, $M = M(K) \oplus L(K)^\perp$ in M, the problem of identifying $C(K)$ with an ideal in C_k is partially alleviated. Indeed, since $C \subset M$, $C(K)$ can be identified with $C_{M(K)}$, the projection of C on $M(K)$.

PROPOSITION 2.1. *$C_{M(K)} = (C_k)_{M(K)}$ for every compact subset $K \subset X$.*

Proof. Clearly $(C_k)_{M(K)} \subset C_{M(K)}$. So let $g_{M(K)} \in C_{M(K)+}$. We show there exists $h \in C_{k+}$ such that $h_{M(K)} = q_{M(K)}$. Choose $h \in C_k$ such that $h = g$ on K. This can be done since K is compact. Then $g_{M(K)} = h_{M(K)}$. Indeed, let $\mu \in \cup L(K)_+$. Since $M(K) = L(K)^{\perp\prime}$, (1, 3) gives $\langle g_{M(K)}, \mu\rangle = \sup\{\langle g, \nu\rangle | 0 \leqq \nu \leqq \mu \text{ and } \nu \in L(K)\}$. The result follows since $\langle g, \nu\rangle = \langle h, \nu\rangle$ for every such measure ν and hence

$$\langle g_{M(K)}, \mu\rangle = \sup\{\langle h, \nu\rangle | 0 \leqq \nu \leqq \mu, \nu \in L(K) = \langle h_{M(K)}, \mu\rangle .$$

Since $C(K)$ can be identified with $C_{M(K)}$, the vague topology on $L(K)$, $\sigma(L(K), C(K))$, is the same as $\sigma(L(K), C_{M(K)})$ which equals $\sigma(L(K), (C_K)_{M(K)})$ by the above. The following is easily checked.

PROPOSITION 2.2. *The following topologies on $L(K)$ are equivalent*:

(a) $\sigma(L_k, C_k)|_{L(K)}$
(b) $\sigma(L(K), C_k)$
(c) $\sigma(L(K), (C_k)_{M(K)})$
(d) $\sigma(L(K), C(K))$.

3. **$M(X)$ as multiplication operators on L_k.** A bounded operator on a vector lattice E is called a multiplication operator if each closed ideal is invariant with respect to the operator. Mack has shown each $f \in M$ defines an order continuous multiplication operator on L_k by the following definition: for $\mu \in L_k$ and $h \in C_k$, $\langle h, f^t\mu \rangle = \langle f, h^t\mu \rangle$ where $h^t\mu$ is the element of $\cup L(K)$ defined by $\langle g, h^t\mu \rangle = \langle gh, \mu \rangle$ for all $g \in C(X)$. Indeed he was able to show every such operator arises in this way.

THEOREM 3.1 (Mack [5, (4.4)]). *M is lattice isomorphic with the vector lattice of all multiplication operators on L_k.*

If $\sigma(E, F)$ is a weak topology on a vector lattice E, we say a linear operator T from E to itself is $\sigma(E, F)$ continuous if $\{\mu_\alpha\} \to 0$ $\sigma(E, F)$ implies $\{T\mu_\alpha\} \to 0$ $\sigma(E, F)$. We now determine those elements of M for which f^t is a $\sigma(L_k, C_k)$ continuous operator.

THEOREM 3.2. *Let $f \in M$. Then f^t is a $\sigma(L_k, C_k)$ continuous operator on L_k if and only if $f \in C(X)$.*

Proof. Suppose $f \in C(X)$ and assume $\langle h, \mu_\alpha \rangle \to 0$ for all $h \in C_k$. We must show $\langle g, f^t\mu_\alpha \rangle \to 0$ for all $g \in C_k$. But this is clear because $\langle g, f^t\mu_\alpha \rangle = \langle f, g^t\mu_\alpha \rangle = \langle fg, \mu_\alpha \rangle \to 0$ since $fg \in C_k$.

For the converse we need two lemmas

LEMMA 3.3. *Suppose X is compact and $f \in M(X)$. If f^t is a $\sigma(L, C)$ continuous operator on L, then $f \in C(X)$.*

Proof. We show f *is* $\sigma(L, C)$ continuous on L. Suppose $\{\mu_\alpha\} \subset L(X)$ and $\langle h, \mu_\alpha \rangle \to 0$ for all $h \in C$. We show $\langle f, \mu_\alpha \rangle \to 0$. But $\langle f, \mu_\alpha \rangle = \langle f, 1^t\mu_\alpha \rangle = \langle 1, f^t\mu_\alpha \rangle \to 0$ since f^t is a $\sigma(L, C)$ continuous operator.

LEMMA 3.4. *Let $f \in M$. If for each compact set K, $f_{M(K)} \in C(K)$, then $f \in C(X)$.*

Proof. We first show that $f_0 = g_0$ for some $g \in C$. We then show $f = g$. Let $p \in X$. Since X is locally compact, p has a compact neighborhood K. Since $f_{M(K)} \in C(K)$, $(f_{M(K)})_0 \in (C_{M(K)})_0$ so f_0 is continuous on a neighborhood of p. Since p was arbitrary, f_0 is continuous as a function on X. Therefore, let g be the continuous function such that $g_0 = f_0$. We claim that $f = g$. Let $\mu \in \cup L(K)$ and $S = S(\mu)$. Since S is compact, $f_{M(S)} \in C(S)$. Furthermore, $g_{M(S)} \in C(S)$ and $(f_{M(S)})_0 = (g_{M(S)})_0$. Therefore, $g_{M(S)} = f_{M(S)}$ [1, (5.4)]. So we have $\langle f, \mu\rangle = \langle f_{M(S)}, \mu\rangle = \langle g_{M(S)}, \mu\rangle = \langle g, \mu\rangle$. To complete the proof of the proposition:

By Lemma 3.4 it suffices to show $f_{M(K)} \in C(K)$ for all compact sets K in X. By Lemma 3.3 it then suffices to show $f^t_{M(K)}$ is a $\sigma(L(K), C(K))$ continuous multiplication operator on $L(K)$. So let $\{\mu_\alpha\} \subset L(K)$ and $\langle h, \mu_\alpha\rangle \to 0$ for all $h \in C(K)$. We must show $\langle h, f^t_{M(K)}\mu_\alpha\rangle \to 0$ for all $h \in C(K)$.

By Proposition 2.2 $\langle h, f^t_{M(K)}\mu_\alpha\rangle \to 0$ for all $h \in C(K)$ if and only if $\langle g, f^t_{M(K)}\mu_\alpha\rangle \to 0$ for all $g \in C_k$. Since by hypothesis f^t is a $\sigma(L_k, C_k)$ continuous operator, $\langle g, f^t\mu_\alpha\rangle \to 0$ for all $g \in C_k$. Now $g^t\mu_\alpha \in L(K)$, so by (1.5) we have $\langle g, f^t\mu_\alpha\rangle = \langle f, g^t\mu_\alpha\rangle = \langle f_{M(K)}, g^t\mu_\alpha\rangle$. Thus $\langle g, f^t_{M(K)}\mu_\alpha\rangle = \langle f_{M(K)}, g^t\mu_\alpha\rangle \to 0$ for all $g \in C_k$.

4. The semi-continuous elements. We now proceed in a mannar analogous to Kaplan's for the compact case and employ some methods from integration theory. Unless otherwise indicated, all infima and suprema will be taken in M.

DEFINITION 4.1. An element $f \in M$ is usc if for each real number r there exists a subset A_r of C such that $f \wedge r1_X = \wedge A_r$.

REMARK 4.2. Clearly if $f \in C$, then f is *usc*. Furthermore if $f = \wedge f_\alpha$ for some collection $\{f_\alpha\} \subset C$, then f is *usc*. If X is compact, then this definition is equivalent to that of Kaplan.

If K is a compact space, we follow Kaplan's notation and let $S(K)$ be the sublattice generated by the *usc* elements in $M(K)$. In our more general case we still have the result that $S(K)$ is a subset of $M(K)$ for every compact set $K \subset X$. We show now that $f \in M$ is usc exactly in the case that $f_{M(K)}$ is a usc element in $M(K)$ for every compact set $K \subset X$. For this we need several lemmas.

LEMMA 4.3. *If $B \subset C(X)$ and $f = \wedge B$, then $f_{M(K)} \in S(K)$. Indeed,*

$f_{M(K)}$ is a usc element in $M(K)$.

Proof. If $f = \wedge B$, then $f_{M(K)} = \wedge\{g_{M(K)}: g \in B\}$. Since $C(K)$ is identified with the projection of $C(X)$ on $M(K)$, the result follows immediately.

We say a real valued function on X is upper semi-continuous if $\{x: f(x) < r\}$ is open for every real number r. The following is easily proved.

LEMMA 4.4. *If $f \in M$ is usc, then f_0 is an upper semi-continuous function on X.*

LEMMA 4.5. *Let $f: X \to R$ be a function such that for each compact set K there exist positive upper semi-continuous function f_i^K so that $f|_K = f_1^K - f_2^K$. Then $f = f_1 - f_2$ where each f_i is a positive upper semi-continuous function.*

Proof. Let $f_1(x) = \wedge\{u_1(x): f(x) = u_1(x) - u_2(x)$ on a neighborhood of x for some positive upper semi-continuous functions $u_i\}$. Then f_1 is well defined, positive and upper semi-continuous. Letting $f_2(x) = f_1(x) - f(x)$ we have $f_2(x) = \wedge\{u_1(x): f(x) = u_1(x) - u_2(x)$ on a neighborhood of x for some positive upper semi-continuous functions $u_i\} - f(x) = \wedge\{u_2(x): f(x) = u_1(x) - u_2(x)$ on a neghborhood of x for some positive upper semi-continuous functions $u_i\}$. So f_2 is also a positive upper semi-continuous function.

LEMMA 4.6. *Let $f: X \to R$ be a locally bounded upper semi-continuous function. Then there exists a usc element $g \in M$ so that $g_0 = f$.*

Proof. For each $n \in N$, let $g_n = \wedge H_{f,n} = \wedge\{h \in C(X): h \geqq f \wedge n1_X\}$. Then $H_{f,n}$ is filtering downward and to show g_n is well defined, it suffices to show $\langle g_n, \mu\rangle$ is finite for each $\mu \in \cup L(K)_+$. [See 1.1 and make appropriate changes.] This follows because f is locally bounded and hence bounded below on compact sets. If $\mu \in \cup L(K)_+$ and $K' = S(\mu)$, choose a natural number r so that $f|_{K'} \geqq -r1_X|_{K'}$. Then $\langle h, \mu\rangle \geqq \langle -r1_X, \mu\rangle > -\infty$ for each $h \in H_{f,n}$ and the infimum exists. Similarly, to show $g = \vee g_n$ exists, we choose an arbitrary $\mu \in \cup L(K)_+$ and a compact set K' so that $S(\mu) \subset \operatorname{int} K'$. Then if $r \in N$ is such that $f|_{K'} \leqq r$, we claim that for $n > r$, $\langle g_n, \mu\rangle = \langle g_r, \mu\rangle$. Indeed it is clear that $\langle g_n, \mu\rangle \geqq \langle g_r, \mu\rangle$. For the opposite inequality, let $\varepsilon > 0$ be given and choose $h_1 \in H_{f,r}$ so that $\langle g_r, \mu\rangle \geqq \langle h_1, \mu\rangle - \varepsilon$. Let $h_2 \in C(X)$ be chosen so that $h_2 = h_1$ on $S(\mu)$ and $h_2 = n1_X$ on $X\backslash K'$. If $h = h_1 \vee h_2$ then $h \geqq f \wedge n1_X$ and $\langle h, \mu\rangle = \langle h_1, \mu\rangle$. Thus $\langle g_r, \mu\rangle \geqq$

$\langle h_1, \mu\rangle - \varepsilon = \langle h, \mu\rangle - \varepsilon \geqq \langle g_n, \mu\rangle - \varepsilon$. Thus the supremum equals $\langle g_r, \mu\rangle$.

Furthermore, g is usc, for if $n \in N$, we claim $g \wedge n1_X = \bigwedge\{h \in C(X); h \geqq g \wedge n1_X\}$. It suffices to show $g \wedge n1_X \geqq \bigwedge\{h \in C(X): h \geqq g \wedge n1_X\}$ and consequently that for each $\mu \in \bigcup L(K)_+$, $\langle g \wedge n1_X, \mu\rangle \geqq \bigwedge\{\langle h, \mu\rangle: h \in C(X)$ and $h \geqq g \wedge n1_X\}$. Let $\mu \in \bigcup L(K)_+$ and $\varepsilon > 0$. By definition of the infimum in M [1, (2.1)] there exist μ_1 and $\mu_2 \in \bigcup L(K)_+$ so $\mu = \mu_1 + \mu_2$ and $\langle g \wedge n1_X, \mu\rangle \geqq \langle g, \mu_1\rangle + \langle n1_X, \mu_2\rangle - 1/2\varepsilon$. Let K' and H be compact sets such that $S(\mu) \subset \operatorname{Int} H \subset H \subset \operatorname{Int} K' \subset K'$. As above, if $f|_{K'} \leqq r$, then $\langle g, \mu_1\rangle = \langle g_r, \mu_1\rangle$. Choose $h_1 \in C(X)$ such that $h_1 \geqq f \wedge r1_X$ and such that $\langle g_r, \mu_1\rangle \geqq \langle h_1, \mu_1\rangle - \varepsilon/2$ so that $\langle g \wedge n1_X, \mu\rangle \geqq \langle h_1, \mu_1\rangle + \langle n1_X, \mu_2\rangle - \varepsilon$. Let $h_2 \in C(X)$ be chosen so that $h_2 = h_1$ on $S(\mu_1)$ and $h_2 = n1_X$ on $X\backslash\operatorname{Int} H$ and let $h_3 = h_1 \vee h_2$. We claim $h_3 \geqq g \wedge n1_X$. Indeed if $\nu \in \bigcup L(K)_+$ and $\Phi \in C_k$ such that $\Phi = 1$ on H and $\Phi = 0$ on $X\backslash\operatorname{Int}\ K'$ then $\nu = (\Phi\nu) + (1_X - \Phi)\nu$. Thus $\langle h_3, (\Phi\nu)\rangle + \langle h_3, (1_X - \Phi)\nu\rangle = \langle h_3, \nu\rangle$. Now $h_3 \geqq h_1$ implies $\langle h_3, (\Phi\nu)\rangle \geqq \langle g_r, (\Phi\nu)\rangle$ and $S((1_X - \Phi)\nu) \subset X\backslash\operatorname{Int} H$ implies $\langle h_3, (1_X - \Phi)\nu\rangle \geqq \langle n1_X, (1_X - \Phi)\nu\rangle$. Thus $\langle h_3, \nu\rangle \geqq \langle g_r, (\Phi\nu)\rangle + \langle n1_X, (1_X - \Phi)\nu\rangle$. Since $S(\Phi\nu) \subset K'$, $\langle g_r, \Phi\nu\rangle = \langle g, \Phi\nu\rangle$ so $\langle h_3, \nu\rangle \geqq \langle g, (\Phi\nu)\rangle + \langle n1_X, (1_X - \Phi)\nu\rangle \geqq (g \wedge n1_X, \nu\rangle$. Finally since ν was arbitrary we have $h_3 \geqq g \wedge n1_X$ and $h_3 \wedge n1_X \geqq g \wedge n1_X$. As a result,

$$\begin{aligned}\langle g \wedge n1_X, \mu\rangle &\geqq \langle h_1, \mu_1\rangle + \langle n1_X, \mu_2\rangle - \varepsilon \geqq \langle h_3, \mu_1\rangle + \langle n1_X, \mu_2\rangle - \varepsilon \\ &\geqq \langle h_3 \wedge n1_X, \mu\rangle - \varepsilon \geqq \bigwedge\{\langle h, \mu\rangle: h \\ &\quad \in C(X), h \geqq g \wedge n1_X\} - \varepsilon\,.\end{aligned}$$

Since ε was arbitrary we have the desired result.

Finally, we check that $g_0(x) = f(x)$ for each $x \in X$. Let $x \in X$, K' a compact neighborhood of x and $\varepsilon > 0$. Suppose $f|_{K'} \leqq r$ for some natural number r. Then $g(x) = g_0(x) = (g_r)_0(x)$. Since $f \wedge r1_X$ is upper semi-continuous there exists $h \in C(X)$ such that $h \geqq f \wedge r1_X$ and $h(x) < f \wedge r1_X(x) + \varepsilon$. Thus $(g_r)_0(x) \leqq h(x) \leqq f \wedge r1_X(x) + \varepsilon = f(x) + \varepsilon$. Since the other inequality is clear, the proof is complete.

LEMMA 4.7. *If g is usc in M, then $g_{M(K)}$ is usc in $M(K)$ for every compact set $K \subset X$.*

Proof. Let K be a compact set and r be an integer so that $g_{M(K)} \leqq r1_K$. Then $g_{M(K)} = (g \wedge r1_X)_{M(K)}$. The result now follows from the definition of *usc* elements Lemma (4.3).

THEOREM 4.8. *Let $f \in M$. Then f is usc if and only if $f_{M(K)}$ is a usc element in $M(K)$ for every compact set K.*

Proof. Suppose $f_{M(K)}$ is a *usc* element in $M(K)$ for every compact set K. Consider f_0. Since $f_{M(K)}$ is bounded and *usc* in $M(K)$, f_0 is locally bounded and upper semi-continuous as a function on X. By Lemma 4.6, there exists a *usc* element g in M so that $g_0 = f_0$. We show $g = f$. This follows immediately from the fact that for each compact set K, $f_{M(K)}$ and $g_{M(K)}$ are *usc* members of $M(K)$ (Lemma 4.7). Since $(g_{M(K)})_0 = (f_{M(K)})_0$ it follows from the compact case that $g_{M(K)} = f_{M(K)}$. Since this is true for each compact set K, it follows that $f = g$.

PROPOSITION 4.9. *If f and g are usc elements in M, then so is $f + g$.*

Proof. Let K be a compact set. Then $(f + g)_{M(K)} = f_{M(K)} + g_{M(K)}$ is the sum of two usc elements in $M(K)$ and hence is *usc*. The result now follows from Theorem 4.8.

Similarly, the following propositions are easily verified using corresponding results for the compact case.

PROPOSITION 4.10. *If f and g are usc elements in M, then $f \wedge g$ and $f \vee g$ are also. If $a > 0$, af is usc.*

PROPOSITION 4.11. *If A is a subset of usc elements in M and $f = \bigwedge A$, then f is usc.*

Let $S = S(X)$ denote the linear subspace of M generated by the positive *usc* elements. It follows from (4.9) and (4.10) that each element in S can be written as $f - g$ where f and g are positive *usc*. The fact that S is a sublattice follows from the fact that $(f_1 - g_1) \wedge (f_2 - g_2) = \{(f_1 + g_2) \wedge (f_2 + g_1)\} - (g_1 + g_2)$ and (4.10).

Of course for each compact set K, $S(K)$, the semi-continuous elements studied by Kaplan [1] is a subset of $M(K)$ and hence of M. In the following we assume a knowledge of the compact case.

PROPOSITION 4.12. *If $f \in M$, then $f \in S$ if and only if $f_{M(K)} \in S(K)$ for each compact set K.*

Proof. If $f \in S$, then $f = f_1 - f_2$ where each f_i is positive *usc*. But then $f_{M(K)} = (f_1)_{M(K)} - (f_2)_{M(K)} \in S(K)$ by Lemma 4.7. Conversely, if $f_{M(K)} \in S(K)$ for each compact set K, we need only observe that in $S(K)$ element can be written as the difference of positive *usc* elements and the proof follows as in Theorem 4.8 using Lemma 4.5.

PROPOSITION 4.13. *If f and g are members of S, then $f \leqq g$ if* *and only if* $f_0 \leqq g_0$.

Proof. It suffices to show $f_0 \leqq g_0$ implies $f \leqq g$. Let $\mu \in \bigcup L(K)_+$ and $K' = S(\mu)$. Then $f_0 \leqq g_0$ implies $(f_{M(K')})_0 \leqq (g_{M(K')})_0$. By Proposition 4.12 $f_{M(K')}$ and $g_{M(K')}$ are members of $S(K')$ and consequently using results from the compact case, $f_{M(K')} \leqq g_{M(K')}$. Thus $\langle f, \mu\rangle = \langle f_{M(K')}, \mu\rangle \leqq \langle g_{M(K')}, \mu\rangle = \langle g, \mu\rangle$. Since this is true for every $\mu \in \bigcup L(K)_+$ $f \leqq g$.

COROLLARY 4.14. *If f and g are members of S, then $f = g$ is equivalent to $f_0 = g_0$. In particular, the projection of S onto S_0 is one-to-one.*

PROPOSITION 4.15. $S(K) = S(X)_{M(K)}$ *for each compact set* K.

Proof. By (4.12), $S(X)_{M(K)} \subset S(K)$. It now suffices to show that if f is a positive *usc* element in $S(K)$, then $f = g_{M(K)}$ for some $g \in S(X)$. Suppose $\{f_\alpha\} \subset C(K)$ and $f = \bigwedge f_\alpha$ in $M(K)$. Let $g_\alpha \in C_{k+}$ be such that $g_\alpha|_K = f_\alpha$. Then $g = \bigwedge g_\alpha$ exists in M and is *usc*. By Lemma 4.7, $g_{M(K)} \in S(K)$. Thus, since $(g_{M(K)})_0 = f_0$ it follows from Corollary 4.14 applied to the compact case that $g_{M(K)} = f$.

Since $S(X)$ is not an ideal, it is not obvious that projections onto the ideals $M(K)$ are still members of $S(X)$. We consider this next.

LEMMA 4.16. *Let K be a compact subset of X and $\{f_\alpha\}$ a net in C_k such that $\alpha \leqq \beta$ implies $f_\alpha(x) \geqq f_\beta(x)$ for all x and $f_\alpha(x) \downarrow 0$ for all x in $X\backslash K$. Then $\wedge f_\alpha \in M(K)$.*

Proof. Since $M(K) = L(K)'^{\perp}$ [see (1.3) and (1.5)], it suffices to show $\langle \wedge f_\alpha, \nu\rangle = \bigwedge \langle f_\alpha, \nu\rangle = 0$ for all $\nu \in L(K)'$. Assume the contrary and let $\nu \in L(K)'$, $\nu \geqq 0$ be such that $\langle f_\alpha, \nu\rangle \geqq r > 0$ for all α. Let $\bigwedge f_\alpha^t \nu = \mu$. That is, for

$$h \in C(X)_+ \ \langle h, \mu\rangle = \langle h, \wedge f_\alpha^t \nu\rangle = \bigwedge \langle h, f_\alpha^t \nu\rangle = \bigwedge \langle h f_\alpha, \nu\rangle \,.$$

Now, μ is not identically zero. Indeed, $\|\mu\| = \langle 1_X, \mu\rangle = \bigwedge \langle f_\alpha, \nu\rangle \geqq r$. Furthermore, $S(\mu) \subset K$, for if $h \in C(X)$, $h|_K = 0$, then $f_\alpha h \downarrow 0$ for all $x \in X$ and since ν has compact support, $\langle h, \mu\rangle = \bigwedge \langle h f_\alpha, \nu\rangle = 0$ by the usual argument. Thus $\mu \in L(K)$ and $\mu \wedge \nu = 0$. However, let $b = \|f_{\alpha_0}\| \vee 1$ for some arbitrary α_0. Then for

$$h \in C_{k+} \ \langle h, \mu\rangle = \bigwedge \langle h f_\alpha, \nu\rangle \leqq \langle h f_{\alpha_0}, \nu\rangle \leqq \langle b{<}h, \nu\rangle = \langle h, b\nu\rangle$$

so $\mu \leqq b\nu$ and $\mu \wedge \nu \geqq \mu \wedge b^{-1}\mu = b^{-1}\mu \neq 0$ which is a contradiction.

COROLLARY 4.17. *With the above hypothesis* $\wedge f_\alpha \in S(K)$ *and is positive usc.*

PROPOSITION 4.18. *Let* $f \in C(X)_+$ *and* K *be a compact subset of* X. *Then* $f_{M(K)} \in S(X)$ *and indeed* $f_{M(K)}$ *is positive usc.*

Proof. Let $\{f_\alpha\} = \{f_\alpha \in C_{k+}: f_\alpha|_K = f|_K\}$. We direct the index set as follow $\alpha \leqq \beta$ if $f_\alpha(x) \geqq f_\beta(x)$ for all $x \in X$. It is easy to check that this definition satisfies the conditions for a directed set. Furthermore, since for each $x \in X \backslash K$, there exists f_{α_0} such that $f_{\alpha_0}|_K = f|_K$ and $f_{\alpha_0}(x) = 0$ it is clear that $f_\alpha(x) \downarrow 0$ for all x in $X \backslash K$. Let $g = \bigwedge f_\alpha$. By Corollary 4.17 $g \in S(K)$ and is positive *usc*. We show $f_{M(K)} = g$. But this is clear, for if $\mu \in L(K)_+$ then $\langle f_{M(K)}, \mu \rangle = \langle f, \mu \rangle = \langle f_\alpha, \mu \rangle$ for all α. Thus $\langle f_{M(K)}, \mu \rangle = \bigwedge \langle f_\alpha, \mu \rangle = \langle g, \mu \rangle$ and consequently $f_{M(K)} = g$ and is positive *usc*.

PROPOSITION 4.19. *Let* $f \in S(X)$, *then* $f_{M(K)} \in S(X)$ *for each compact set* K. *In particular, if* f *is a positive usc element, then for each real number* r, *there exists a collection* $B_r \subset C_{k+}$ *so that* $f_{M(K)} \wedge r1_X = \bigwedge B_r$.

Proof. Assume f is positive *usc*. Then for every real number r, there is a collection $A_r \subset C(X)_+$ so that $f \wedge r1_X = \bigwedge A_r$. Then

$$\begin{aligned} f_{M(K)} \wedge r1_X &= f_{M(K)} \wedge (r1_X)_{M(K)} = (f \wedge r1_X)_{M(K)} \\ &= (\wedge A_r)_{M(K)} = \bigwedge \{g_{M(K)}: g \in A_r\} . \end{aligned}$$

If $g \in A_r$, then by the argument in Proposition 4.18 $g_{M(K)}$ is the infimum of a collection $A_g \subset C_{k+}$. Thus we have: $f_{M(K)} \wedge r1_X = \bigwedge \{g_{M(K)}: g \in A_r\} = \bigwedge \{\wedge A_g: g \in A_r\} = \bigwedge \{h: h \in \bigcup \{A_g: g \in A_r\}\}$. The result now follows by choosing $B_r = \bigcup \{A_g: g \in A_r\}$.

If f is an arbitrary element of S, then $f = g - h$ where g and h are positive *usc*. Then $f_{M(K)} = g_{M(K)} - h_{M(K)} \in S(X)$.

COROLLARY 4.20. *If* f *is positive usc and* K *is a compact set, then there exists a collection* $A_K \subset C_{k+}$ *so that* $f_{M(K)} = \bigwedge A_K$.

Proof. Since $f_{M(K)}$ is bounded, there exists a real number r so that $f_{M(K)} = f_{M(K)} \wedge r1_X$. The result now follows immediately from (4.19).

5. **$S(X)$ as multiplication operators.** Let $S_k(X) = \{f \in S(X): f = f_{M(K)}$ for some compact subset $K\}$. $S_k(X)$ can also be regarded as the union of all $S(K)$ as K ranges over the compact subset of X.

It is easy to see that $f \in S_k(X)$ if and only if f_0 (as a function on X) is the difference of two positive upper semi-continuous functions with compact support. It is also clear that $S_k(X) \subset \bigcup M(K) \subset M_k$ and that it is a sub vector lattice containing C_k. As such, it is separating on L_k and determines a Hausdorff weak topology on L_k, namely $\sigma(L_k, S_k)$.

We have already considered M as multiplication operators on L_k. Indeed if $f \in M$ and $\mu \in L_k(X)_+ \langle h, f^t\mu\rangle = \langle f, h^t\mu\rangle$ for all $h \in C_{k+}$. Now consider the special case that $f \in S(X)$. Suppose f is *usc*. Choose an integer r so that $\langle f, h^t\mu\rangle = \langle f \wedge r1_X, h^t\mu\rangle$. Since f is *usc* there exists a collection $\{f_\alpha\} \subset C(X)$ such that $f \wedge r1_X = \bigwedge \{f_\alpha\}$. Thus

$$\langle f, h^t\mu\rangle = \langle \wedge f_\alpha, h^t\mu\rangle = \bigwedge \langle f_\alpha, h^t\mu\rangle = \bigwedge \langle hf_\alpha, \mu\rangle = \langle \wedge hf_\alpha, \mu\rangle .$$

Observe that $\wedge\, hf_\alpha \in S_k(X)$.

We have already shown that $S(K) = S(X)_{M(K)}$. We now show that these are the same as $(S_k)_{M(K)}$.

PROPOSITION 5.1. $S(X)_{M(K)} = (S_k)_{M(K)}$.

Proof. Clearly $(S_k)_{M(K)} \subset S_{M(K)}$. Thus let $g_{M(K)} \in S_{M(K)}$ and g be positive *usc*. We show there exists $h \in S_k$ so that $h_{M(K)} = g_{M(K)}$. Let r be a real number so that $g_{M(K)} = (g \wedge r1_X)_{M(K)}$. By hypothesis, there exists $\{g_\alpha\} \subset C(X)_+$ so that $g \wedge r1_X = \bigwedge g_\alpha$. Let H be a compact neighborhood of K and $\{h_\alpha\} \subset C(X)$ so that $h_{\alpha|K} = g_\alpha$ and $S(h_\alpha) \subset H$. Let $h = \bigwedge h_\alpha$. Clearly $h_{M(K)} = (g \wedge r1_X)_{M(K)} = g_{M(K)}$ and since $h = h_{M(H)}$ $h \in S_k$.

This proposition and the previous remark make it easy to verify the following:

PROPOSITION 5.2. *On $L(K)$ the following topologies are equivalent*:

(a) $\sigma(L_k, S_k)|_{L(K)}$

(b) $\sigma(L(K), (S_k)_{M(K)})$

(c) $\sigma(L(K), S(X)_{M(K)})$

(d) $\sigma(L(K), S(K))$

LEMMA 5.3. *Let X be compact and $f \in M$. If f^t is $\sigma(L, S)$ continuous on L then $f \in S$.*

Proof. The proof of (3.3) carries over by replacing C with S.

THEOREM 5.4. *Let $f \in M$, then f^t is a $\sigma(L_k, S_k)$ continuous multiplication operator on L_k if and only if $f \in S$.*

Proof. Let $f \in S$ and $\{\mu_\alpha\} \subset L_k$ such that $\langle h, \mu_\alpha\rangle \to 0$ for all $h \in S_k$. We show $\langle h, f^t\mu_\alpha\rangle \to 0$ for all $h \in S_k$. Without loss of generality, we may assume f is positive and *usc*. Since $h \in S_k$, $h = h_1 - h_2$ where the h_i are positive *usc* elements in S_k so we may also assume h is positive *usc*.

By Corollary 4.20, there exists a collection $\{h_\gamma\} \subset C_{k+}$ such that $h = \bigwedge h_\gamma$. Thus $\langle h, f^t\mu_\alpha\rangle = \bigwedge_\gamma \langle h_\gamma, f^t\mu_\alpha\rangle = \bigwedge_\gamma \langle f, h_\gamma^t\mu_\alpha\rangle$. We may as well assume there is a compact set K such that $S(h_\gamma) \subset K$ for all γ. Choose an integer r so that

$$\langle f, h_\gamma^t\mu_\alpha\rangle = \langle f \wedge r1_X, h_\gamma^t\mu_\alpha\rangle$$

for all α and γ. This can be done since $S(h_\gamma^t\mu_\alpha) \subset K$. By assumption, there exists a collection $\{f_\beta\} \subset C(X)$ so that $f \wedge r1_X = \bigwedge f_\beta$. Thus for each α, $\langle h, f^t\mu_\alpha\rangle = \bigwedge_\gamma \langle f \wedge r1_X, h_\gamma^t\mu_\alpha\rangle = \bigwedge_\gamma \langle \bigwedge_\beta f_\beta, h_\gamma^t\mu_\alpha\rangle = \bigwedge_\gamma \bigwedge_\beta \langle f_\beta, h_\gamma^t\mu_\alpha\rangle = \bigwedge_\gamma \bigwedge_\beta \langle f_\beta h_\gamma, \mu_\alpha\rangle = \langle \bigwedge_{\gamma\beta} f_\beta h_\gamma, \mu_\alpha\rangle$. We observe that $\bigwedge_{\gamma\beta} f_\beta h_\gamma \in S_k$ and hence $\langle h, f^t\mu_\alpha\rangle = \langle \bigwedge_{\gamma\beta} f_\beta h_\gamma, \mu_\alpha\rangle \to 0$.

For the converse we merely adapt the proof of Theorem 3.2 replacing $C(K)$ with $S(K)$ and the references to 2.2, 3.3 and 3.4 with 5.2, 5.3 and 4.12.

6. The ideals generated by $S(X)$ and $C(X)$. We have now singled out two sublattices in M, namely S and C. In the case that X is compact, the constant function 1_X is a strong order unit in the normed vector space M. That is, it is a positive element of unit norm such that $f \in M(X)$ and $\|f\| \leqq 1$ imply $|f| \leqq 1_X$. Since 1_X is a member of both S and C, it can be shown that the ideals these sublattices generate, denoted $I(C)$ and $I(S)$, must be all of M. In the more general case we are considering, however, there may not be a strong order unit. Unless X is pseudocompact, 1_X is only a weak order unit: that is, a positive element such that for each $f \in M$, $|f| \wedge 1_X = 0$ implies $f = 0$. It can be shown that under this condition, the closed ideals generated by S and C are all of M, although not necessarily the ideals themselves. We give necessary and sufficient conditions under which $I(S)$ (respectively $I(C)$) is all of M.

THEOREM 6.1. *$I(S) = M$. Indeed, each element of M is dominated by positive usc element.*

Proof. Let $f \in M$. Since $f = f^+ - f^- \leqq f^+ + f^- = |f|$, it suffices to assume $f \geqq 0$. For each $n \in N$, let $g_n = \bigwedge \{h \in C(X): h \geqq f \wedge n1_X\}$. To show $g = \bigvee g_n$ exists it suffices to show $\vee \langle g_n, \mu\rangle < \infty$ for each $\mu \in \bigcup L(K)_+$ (1.1). Let $\mu \in \bigcup L(K)$ and K' be a compact set so that $S(\mu)$ is contained in the interior of K'. Assume $|f_{M(K')}| \leqq$

$r1_{K'}$ for some $r \in N$. We can prove as in Lemma 4.6 that for all $n > r$ $\langle g_n, \mu\rangle = \langle g_r, \mu\rangle$, and hence that the supremum in question is finite. Also $\langle g, \mu\rangle = \langle g_r, \mu\rangle \geqq \langle f \wedge r1_X, \mu\rangle = \langle f \wedge r1_{M(K')}, \mu\rangle = \langle f_{M(K')}, \mu\rangle = \langle f, \mu\rangle$. It remains to show g is usc. The proof in Lemma 4.6 applies if the condition $f|_K \leqq r$ is replaced by $|f_{M(K')}| \leqq r1_{K'}$.

DEFINITION 6.2. A topological space X is called a *cb-space* if and only if for each locally bounded function h, there exists $f \in C(X)$ such that $|h| \leqq f$.

THEOREM 6.3 (Mack [4, Theorem 1]). *X is a cb space if and only if each countable increasing cover of X has a countable refinement by cozero sets; that is, sets of the form,* coz f *for some* $f \in C(X)$.

LEMMA 6.4. *Let $f \in M_+$ and $\mathscr{K}$ be the collection of all open relatively compact subsets of X. Let $U_n = \bigcup \{K \in \mathscr{K}: ||f_{M(\bar{K})}|| \leqq n\}$. Let $\mu \in \bigcup L(K)_+$ and $K_0 = S(\mu)$. If $K_0 \subset U_n$ then $\langle f, \mu\rangle \leqq \langle n1_X, \mu\rangle = n||\mu||$.*

Proof. Since X is locally compact, $\mathscr{K}$ is a cover of X and the compactness of K_0 implies there exist subset $K_1, \cdots, K_m$ in U_n such that $K_0 \subset K_1 \cup K_2 \cdots \cup K_m$. Let h_i, $i = 1, \cdots, m$ be a partition of unity on K_0 subordinate to the cover $\{K_i\}_{i=1}^m$. That is, $h_i \in C_k$, $0 \leqq h_i \leqq 1$, $S(h_i) \subset K_i$ and $\sum_1^m h_i(x) = 1$ for all $x \in K_0$. Then

$$\langle f, \mu\rangle = \langle f, (\sum_1^m h_i)^t\mu\rangle = \sum_1^m \langle f, h_i{}^t\mu\rangle$$

and since $S(h_i{}^t\mu) \subset K_i$ this last expression is dominated by

$$\sum_1^m ||f_{M(\overline{K_i})}||\,||h_i{}^t\mu|| \leqq \sum_1^m n||h_i{}^t\mu|| = \langle n, (\sum_1^m h_i)^t\mu\rangle = n||\mu|| .$$

THEOREM 6.5. *$I(C) = M$ if and only if X is a cb-space.*

Proof. Assume first that $I(C) = M$. Let h be locally bounded and real valued on X. We must show there exists $f \in C(X)$ such that $|h| \leqq f$. Assume $h \geqq 0$. Since h is locally bounded, it determines a function $g \in M_+$ such that $g_0(x) = h(x)$ for all $x \in X$. By hypothesis and (1.3) there exists $p \in C(X)$ such that $p \geqq g$. Thus $p_0 \geqq g_0$ and hence $f(x) = p_0(x)$ is the required function.

For the converse, let $f \in M_+$. We show there exists $g \in C(X)$ such that $g \geqq f$. Let $\mathscr{K}$ and $\{U_n\}$ be defined as in Lemma 6.4. Then the collection $\{U_n\}$ is an increasing cover of X by open sets. Since X is a *cb*-space and the countable union of cozero sets is again a cozero set, we have by Theorem (6.3) a family $\{g_n\} \subset C(X)$ such that $\operatorname{coz} g_n \subset U_n$ and $\{\operatorname{coz} g_n\}$ is a cover of X. Assume $g_i \geqq 0$. Define $f_n = 1_X - \bigvee \{n1_X g_i \wedge 1_X : i \leqq n\}$. Given $x \in X$, there exists an i such that $g_i(x) > 0$. Therefore, there exists a j such that $g_i(y) > j^{-1}$ for each y in a neighborhood of x. Therefore f_n vanishes on that neighborhood for $n \geqq i \vee j$. This implies Σf_n is locally finite and thus $g = 2 + \Sigma f_n \in C(X)$. We claim $f \leqq g$. Indeed, let $\mu \in \bigcup L(K)_+$ and $K_0 = S(\mu)$. Let $W_n = U_n \cap \{x: g(x) > n\}$. Since g is continuous W_n is open. Furthermore, $\{W_n\}$ is a cover. Indeed, let $x \in X$ and $n_0 = \bigwedge \{n: x \in U_n\}$. Then $g_m(x) = 0$ for $m = 1, \cdots, n_0 - 1$ and $x \in W_{n_0}$. Since K_0 is compact, there exists a natural number s such that $K_0 \subset W_1 \cup \cdots \cup W_s$. For $i = 1, \cdots, s$, choose $h_i \in C_k$ such that $0 \leqq h_i \leqq 1$, $S(h_i) \subset W_i$ and $\sum_1^s h_i(x) = 1$ for all $x \in K_0$. Then $\langle f, \mu\rangle = \langle f, (\sum_1^s h_i)^t \mu\rangle \leqq \sum_1^s \langle i, h_i{}^t\mu\rangle$ by Lemma (6.4). For $i = 1, \cdots, s$, choose $w_i \in C_k$ such that $0 \leqq w_i \leqq 1$, $w_i(x) = 1$ for all $x \in S(h_i{}^t\mu)$ and $S(w_i) \subset W_i$. So $\langle f, \mu\rangle \leqq \sum_1^s \langle w_i i, h_i{}^t\mu\rangle$ and $0 \leqq iw_i \leqq i$ with $S(iw_i) \subset W_i$. Since $g > i$ on W_i, this implies $iw_i \leqq g$. Finally this gives $\langle f, \mu\rangle \leqq \sum_1^s \langle g, h_i{}^t\mu\rangle = \langle g, (\sum_1^s h_i)^t\mu\rangle = \langle g, \mu\rangle$.

APPENDIX. If one uses the definition of usc element as given in this paper, it would be natural to define an *lsc* element in M as one for which $-f$ is *usc*. However, bearing in mind the properties possessed by lower semi-continuous functions, it is also natural to define an *lsc* element as one for which $f = \bigvee f_\alpha$ for some collection $\{f_\alpha\} \subset C(X)$. These are not compatible definitions. In a manner similar to that used above, we can show (using the second definition) that the linear sublattice T formed by the positive *lsc* elements consists of all those members of M which can be written as the difference of positive *lsc* elements. If X is compact, then this sublattice is exactly the same as $S(X)$. In general, however, S and T are not the same.

We say a real valued function on X is lower semi-continuous if $\{x: f(x) > r\}$ is open for each real r.

The following is easily checked:

LEMMA 7.1. *If $f \in M$ is lsc, the f_0 is a lower semi-continuous function on X.*

PROPOSITION 7.2. *Let $f: X \to R$ be a positive lower semi-continuous function and locally bounded. Then $h = \bigvee \{f_\alpha \in C(X): 0 \leqq f_\alpha \leqq f\}$ exists in M, h is lsc and $h_0(x) = f(x)$ for all $x \in X$.*

Proof. Let $\mathscr{F} = \{f_\alpha \in C(X): 0 \leqq f_\alpha \leqq f\}$. Then $\mathscr{F}$ determines an ascending net and we may write $\vee \mathscr{F} = \bigvee f_\alpha$. It suffices to prove that for each $\mu \in \bigcup L(K)_+$, $\bigvee \langle f_\alpha, \mu\rangle$ exists and is finite. Since f is locally bounded and $S(\mu)$ is compact, $\sup\{f(x): x \in S(\mu)\}$ is some finite number r. Then for each α, $\langle f_\alpha, \mu\rangle = \langle f_\alpha \wedge r1_X, \mu\rangle + \langle (f_\alpha - r1_X)^+, \mu\rangle = \langle f_\alpha \wedge r1_X, \mu\rangle$ since $(f_\alpha - r1_X)^+ = 0$ on $S(\mu)$. So $\langle f_\alpha, \mu\rangle \leqq r\|\mu\|$ for each α, giving an increasing set of real numbers which is bounded above. Hence the supremum in question exists and is finite. By definition h is *lsc* and the last assertion follows from the fact that f is lower semi-continuous.

COROLLARY 7.3. *Let $f \in M_0$ and $f \geqq 0$. If as a function on X, f is lower semi-continuous, then there exists a unique lsc element $g \in M$ such that $g_0 = f$.*

Proof. We merely observe that the elements of M_0 are locally bounded as functions on X. (1.7). The uniqueness follows from the fact that for *lsc* elements $f = g$ if and only if $f_0 = g_0$. (Argument follows as in [1]).

DEFINITION 7.4. *A space X is countably paracompact if and only if each countable open cover has a locally finite refinement.*

THEOREM 7.5 (Mack [4, Theorem 10]). *X is countably paracompact if and only if for each locally bounded function h defined on X there exists a locally bounded lower semi-continuous function g such that $|h| \leqq g$.*

THEOREM 7.6. *$I(T) = M$ if and only if X is countably paracompact.*

Proof. Assume first that the ideal generated by T, $I(T) = M$. By Theorem 7.5 it suffices to show that for each locally bounded function h on X there exists a locally bounded lower semi-continuous function g such that $|h| \leqq g$. Now M_0 is isomorphic to the locally bounded functions on X. Therefore, $|h|$ determines a member of M_0. By hypothesis and (1.3), there exists $g \in T_0$ such that $0 \leqq |h| \leqq g$. By definition $g = g_1 - g_2$ where each g_i is a positive, locally bounded *lsc*. The result follows from the observation that $0 \leqq |h| \leqq g_1 - g_2 \leqq g_1$.

For the converse, let $f \in M_+$. By (1.3) it suffices to show that there exists an *lsc* element g such that $g \geqq f$. Let $\mathscr{K}$ and U_n be as in the Lemma 6.4 for the given f. Since X is countably paracompact, there exists a refinement $\{V_n\}$ such that $\bar{V}_n \subset U_n$ [4, Theorem 10]. Let $p(x) = \inf\{n: x \in \bar{V}_n\}$. Then $p(x)$ is positive, locally bounded and lower semi-continuous. By Proposition 7.2 p determines a member g of T by $\langle g, \mu\rangle = \bigvee \{\langle g_\alpha, \mu\rangle: g_\alpha \in C(X)$ and $0 \leqq g_\alpha \leqq p\}$. We claim

$f \leqq g$. Indeed, let $\mu \in \bigcup L(K)_+$ and $K_0 = S(\mu)$. Let $W_n = U_n \cap \{x: p(x) > n - 1\}$ for $n = 1, 2, \cdots$. Then W_n is open since p is lower semi-continuous. If $x \in X$ and $p(x) = n_0$, then $x \in W_{n_0}$. Thus $\{W_n\}$ is a cover of X. Since K_0 is compact, there exists an integer m such that $K_0 \subset W_1 \cup \cdots \cup W_m$. There also exist functions $h_i \in C_k$ such that $0 \leqq h_i \leqq 1$, $S(h_i) \subset W_i$ and $\sum_1^m h_i(x) = 1$ for all $x \in K_0$. Then we have $\langle f, \mu \rangle = \langle f, (\sum_1^m h_i)^t \mu \rangle = \sum_1^m \langle f, h_i{}^t \mu \rangle \leqq \sum_1^m \langle i, h_i{}^t u \rangle$ since the fact that $S(h_i{}^t \mu) \subset U_i$ allows us to apply Lemma 6.4. Now, for each i, choose $g_i \in C_k$ such that $0 \leqq g_i \leqq 1$, $g_i = 1$ on $S(h_i)$ and $S(g_i) \subset W_i$. Then $\langle i, h_i{}^t \mu \rangle = \langle g_i i, h_i{}^t \mu \rangle$. Now $0 \leqq g_i i \leqq i$ and $S(g_i i) \subset W_i$. But $p \geqq i$ on W_i so $i g_i \leqq p$. Thus we finally have $\langle f, \mu \rangle \leqq \sum_1^m \langle i, h_i{}^t \mu \rangle = \sum_1^m \langle g_i i, h_i{}^t \mu \rangle \leqq \sum_1^m \sup \{\langle h_\alpha, h_i{}^t \mu \rangle : h_\alpha \in C(X), 0 \leqq h_\alpha \leqq p\} = \sum_1^m \langle g, h_i{}^t \mu \rangle = \langle g, (\sum_1^m h_i)^t \mu \rangle = \langle g, \mu \rangle$.

This last result gives a good way to tell when the two sublattices $S(X)$ and $T(X)$ are identical.

THEOREM 7.7. *$T = S$ if and only if X is countably paracompact.*

Proof. If $T = S$, then $I(T) = I(S) = M$ by Theorem 6.1 and thus by Theorem 7.6, X is countably paracompact.

Assume next that X is countably paracompact. It will suffice to show that each positive *lsc* element can be written as the difference of positive *usc* elements and that each positive usc element can be written as the difference of positive *lsc* elements. Suppose f is *lsc*. Then by Theorem 6.1, there is a *usc* element g so that $f \leqq g$. It follows that $f = g - (g - f)$ and hence it suffices to show $g - f$ is *usc*. Let $\{f_\alpha\} \subset C(X)$ be such that $f = \bigvee f_\alpha$. Then $g - f = g - \bigvee f_\alpha = g + \bigwedge (-f_\alpha) = \bigwedge g - f_\alpha$. Now $g - f_\alpha$ is *usc* for each α so by (4.11) $g - f$ is *usc*.

Next suppose f is *usc*. Since f_0 is locally bounded on X and X is countably paracompact, Theorem 7.5 implies there is a lower semicontinuous function g so that $f_0 \leqq g$. By Corollary 7.3, there exists an *lsc* element G so that $G_0 = g$. We have already shown that the *lsc* elements are members of S. By Propsition 4.13 it follows that $f \leqq G$. Thus $f = G - (G - f)$ and it will be sufficient to show that $G - f = \bigvee \{h \in C(X): h \leqq G - f\}$. Again, since the other inequality is clear it remains to show that $\langle G - f, \mu \rangle \leqq \bigvee \{\langle h, \mu \rangle : h \in C(X), h \leqq G - f\}$ for each $\mu \in \bigcup L(K)_+$. Let $\varepsilon > 0$ be given and $\mu \in \bigcup L(K)_+$. Let $K_0 = S(\mu)$ and H be a compact set so that K_0 is contained in the interior of H. Choose n so that $f_{M(H)} = f \wedge n1_{M(H)}$. Since G is *lsc* and f is *usc*, there exist subsets A and B of $C(X)$ such that $G = \bigvee A$ and $f \wedge n1_X = \bigwedge B$. Choose $f_1 \in A$ so that $f_1 \leqq G$ and $\langle f_1, \mu \rangle \geqq$

$\langle G, \mu\rangle - \varepsilon$ and $f_2 \in B$ so that $f_2 \geqq f \wedge n1_X$ and

$$\langle f_2, \mu\rangle \leqq \langle f \wedge n1_X, \mu\rangle + \varepsilon .$$

Then $f_1 - f_2 \leqq G - f \wedge n1_X$. Let $\Phi \in C_k$ be such that $\Phi = 1$ on K_0 and $S(\Phi) \subset H$. Then if $h = \Phi(f_1 - f_2)$, we have $h|_H \leqq f_1 - f_2|_H \leqq G - f \wedge n1_X|_H = G - f|_H$. Since $S(h) \subset H$, we have $h \leqq G - f$. Then $h|_{K_0} = (f_1 - f_2)|_{K_0}$ implies $\langle h, \mu\rangle = \langle f_1 - f_2, u\rangle = \langle f_1, \mu\rangle - \langle f_2, \mu\rangle \geqq \langle G, \mu\rangle - \varepsilon - \langle f \wedge n1_X, \mu\rangle - \varepsilon = \langle G - f \wedge n1_X, \mu\rangle - 2\varepsilon = \langle G - f, \mu\rangle - 2\varepsilon$. The result now follows.

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