CHAIN CONDITIONS AND PURE-EXACTNESS

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Let R be a ring and M any (right) R-module. For any set I let $M^{(I)}$ and M^I denote the direct sum and respectively the direct product of copies of M indexed by the set I. For any cardinal number r, let $\mathscr{C}_r$ denote the class of R-modules admitting a generating set of cardinality $\leq r$. In this paper we study the relationship between the pure-exactness of the sequence $0 \to M^{(I)} \to M^I \to M^I/M^{(I)} \to 0$ with respect to $\mathscr{C}_r$ under the functor Hom_R and chain conditions on suitably defined families of R-modules. This study led us to the introduction of five properties $A_r, A_{(r)}, D_r, D_{(r)}$, and P_r for any R-module M. We also study the effect of base extension (both covariant and contravariant) of the ring R on modules having any (or some) of the above mentioned properties. Finally we obtain necessary and sufficient conditions for

$$0 \longrightarrow \oplus M_\alpha \longrightarrow \pi M_\alpha \longrightarrow \pi M_\alpha/\oplus M_\alpha \longrightarrow 0$$

to be pure-exact with respect to $\mathscr{C}_k$, where $\{M_\alpha\}$ is any family of R-modules and k any integer ≥ 1.

Introduction. Let $\mathscr{C}$ be any family of R-modules. An exact sequence $0 \to M' \to M \xrightarrow{\varepsilon} M'' \to 0$ of R-modules is said to be pure-exact with respect to the family $\mathscr{C}$ (for the functor $\mathrm{Hom} = \mathrm{Hom}_R$) if $\varepsilon_*: \mathrm{Hom}\,(E, M) \to \mathrm{Hom}\,(E, M'')$ is onto for all $E \in \mathscr{C}$. Given a module M and a set I let $M^{(I)}$ and M^I denote respectively the direct sum and the direct product of copies of the same module M indexed by I. In his paper [6] H. Lenzing, among other things, studies equivalent conditions under which the sequence

$$(1) \qquad 0 \longrightarrow M^{(I)} \longrightarrow M^I \longrightarrow M^I/M^{(I)} \longrightarrow 0$$

is pure-exact with respect to the family of cyclic modules. His result (Proposition 1 of [6]) is a generalization of Carl Faith's result [4] concerning necessary and sufficient conditions for $M^{(I)}$ to be injective. Let r be any cardinal number and $\mathscr{C}_r$ the class of R-modules admitting a generating set with cardinality $\leq r$. One of the objectives of the present paper is to study the extent to which Lenzing's results could be generalized to furnish conditions under which (1) will be pure-exact with respect to $\mathscr{C}_r$.

Throughout this paper k will denote an integer ≥ 1. R will denote a ring with 1, R-mod and mod-R will denote the categories of left resp. right unitary R-modules. Unless otherwise mentioned, by an R-module we mean an object of mod-R. All the concepts

used in this paper are right sided unless otherwise mentioned. When we talk of homomorphisms between rings we always assume that they preserve the identity elements. When R is a subring of S we assume $1_R = 1_S$ so that the inclusion map $j: R \to S$ is a ring homomorphism. For any set J, the cardinality of J will be denoted by $|J|$.

The results of Lenzing in [6] suggest the introduction and study of five properties A_r, $A_{(r)}$, D_r, $D_{(r)}$, and P_r for any $M \in \mathrm{mod}\text{-}R$ and any cardinal number $r \neq 0$. Let J be any set with $|J| = r$. For any $\underline{x} = (x_j)_{j \in J} \in M^J$ and $\underline{\lambda} = (\lambda_j)_{j \in J} \in R^{(J)}$ let $\langle \underline{x}, \underline{\lambda} \rangle = \sum_j x_j \lambda_j$. For any subset T of M^J let $L_T = \{ \underline{\lambda} \in R^{(J)} \mid \langle \underline{x}, \underline{\lambda} \rangle = 0 \text{ for all } \underline{x} \in T \}$. We write $\mathscr{F}_r(M)$ (resp. $\mathscr{F}_{(r)}(M)$) for the family $\{ L_T \mid T \text{ subset of } M^J \}$ (resp. $\{ L_T \mid T \text{ subset of } M^{(J)} \}$). $M \in \mathrm{mod}\text{-}R$ is said to have the property A_r (resp. $A_{(r)}$) if $\mathscr{F}_r(M)$ (resp. $\mathscr{F}_{(r)}(M)$) has the ascending chain condition. Similarly, M is said to have the property D_r (resp. $D_{(r)}$) if $\mathscr{F}_r(M)$ (resp. $\mathscr{F}_{(r)}(M)$) has the descending chain condition. When $r = k$, A_k and $A_{(k)}$ are equivalent by definition and we use the common notation A_k. Similarly D_k and $D_{(k)}$ are equivalent and we use the common notation D_k. It turns out A_r and $A_{(r)}$ are equivalent for any cardinal r. However, we use either notation depending on the context. M is said to have P_r if for any set I, $0 \to M^{(I)} \to M^I \to M^I/M^{(I)} \to 0$ is pure-exact with respect to $\mathscr{C}_r$. It also turns out that A_k and P_k are equivalent, whereas for an infinite cardinal r one has only the implication $A_r \Rightarrow P_r$. We show by means of an example that the implication $P_r \Rightarrow A_r$ is not true. It turns out that the properties A_r, D_r, and $D_{(r)}$ are hereditary, in the sense that if $M \in \mathrm{mod}\text{-}R$ has any one of these properties and N is a submodule of M, then N also has the same property. Since A_k and P_k are equivalent, it follows that P_k is hereditary. We give an example to show that P_r is not hereditary for an infinite cardinal r.

Let R be a subring of S and $j: R \to S$ the inclusion. Using j we regard S (and also S/R) as an element of R-mod. Let $M \in \mathrm{mod}\text{-}R$. If $M' = M \otimes_R S \in \mathrm{mod}\text{-}S$ has property $A_{(r)}$ (resp. $D_{(r)}$) and if $\mathrm{Tor}_1^R(M, S/R) = 0$, then we show that $M \in \mathrm{mod}\text{-}R$ property $A_{(r)}$ (resp. $D_{(r)}$). We say that a ring R has a certain property if R considered as an element of $\mathrm{mod}\text{-}R$ in the usual way has that property. An immediate consequence of the above result is that if S has $A_{(r)}$ (resp. $D_{(r)}$), then R has $A_{(r)}$ (resp. $D_{(r)}$). Also the above results yield an alternative proof of the following well-known result which can be found in Chap. I, § 3, n°5 of [2]: If R is a subring of a right noetherian (artinian) ring S and if S/R is left flat over R, then R is right noetherian (artinian). Also we prove that a regular ring R has property A_1 if and only if it is semisimple.

Next we consider the situation where there exists a ring homo-

morphism $f: R \to S$. Let $M \in \text{mod-}R$ and $M' = \text{Hom}_R(S, M) \in \text{mod-}S$. We study conditions under which M' having property A_r (resp. D_r) implies M has the same property. We apply these results to derive conditions under which Σ-injectivity of M' implies Σ-injectivity of M.

In [7] B. Sarath and K. Varadarajan obtain necessary and sufficient conditions for the direct sum $\bigoplus M_\alpha$ of a given family $\{M_\alpha\}_{\alpha \in I}$ of modules to be injective. Finally we generalize results in [7] and furnish conditions under which the sequence

$$(2) \qquad 0 \longrightarrow \bigoplus M_\alpha \longrightarrow \Pi M_\alpha \longrightarrow \Pi M_\alpha / \bigoplus M_\alpha \longrightarrow 0$$

is pure-exact with respect to $\mathscr{C}_k$, where $\{M_\alpha\}_{\alpha \in I}$ is an arbitrary family in mod-R.

1. Preliminaries. Let R be a ring, $U \in \text{mod-}R$, $W \in R\text{-mod}$ and M an abelian group. Let $\theta: U \otimes_R W \to M$ be a given homomorphism of abelian groups. For any $(u, w) \in U \times W$ we write $\langle u, w \rangle$ for the element $\theta(u \otimes w)$. Given any subset X of U let $L_X = \{w \in W \mid \langle u, w \rangle = 0 \text{ for all } u \in X\}$. Clearly, L_X is an additive subgroup of W. Let $\mathscr{F}_W(\theta) = \{L_X \mid X \text{ any subset of } U\}$. We remark that L_X has been called an annihilator with respect to $\langle, \rangle$ in [1, § 24].

PROPOSITION 1.1. *With the above notation, the following conditions are equivalent.*

(a) $\mathscr{F}_W(\theta)$ *satisfies the ascending chain condition.*

(b) *There do not exist any sequences* $\{u^{(n)}\}_{n \geq 1}$, $\{w^{(n)}\}_{n \geq 1}$ *with* $u^{(n)} \in U$, $w^{(n)} \in W$ *and satisfying*

(i) $\langle u^{(n)}, w^{(n)} \rangle \neq 0$
(ii) $\langle u^{(m)}, w^{(n)} \rangle = 0$ *for all* $m > n \geq 1$.

Proof. (a) $\Rightarrow$ (b). Suppose there exist sequences $u^{(n)}$ in U, $w^{(n)}$ in W satisfying (i) and (ii) of (b). Let X_n be the subset $X_n = \{u^{(k)} \mid k \geq n\}$ of U. Then $L_{X_n} \subset L_{X_{n+1}}$. From $\langle u^{(n)}, w^{(n)} \rangle \neq 0$ and $\langle u^{(m)}, w^{(n)} \rangle = 0$ for $m > n$ we see that $w^{(n)} \in L_{X_{n+1}} - L_{X_n}$. Thus $L_{X_1} \subset L_{X_2} \cdots$ is a strict ascending chain in $\mathscr{F}_W(\theta)$ contradicting (a).

(b) $\Rightarrow$ (a). Suppose $L_{\Delta_1} \subset L_{\Delta_2} \subset \cdots$ is a strictly increasing chain of objects in $\mathscr{F}_W(\theta)$, where Δ_i are subsets of U. Let $w^{(n)} \in L_{\Delta_{n+1}} - L_{\Delta_n}$. Then there exists a $u^{(n)} \in \Delta_n$ with $\langle u^{(n)}, w^{(n)} \rangle \neq 0$. Since $u^{(m)} \in \Delta_m$ and $w^{(n)} \in L_{\Delta_m}$ for all $m > n$ we get $\langle u^{(m)}, w^{(n)} \rangle = 0$ for $m > n$. This contradicts (b).

PROPOSITION 1.2. *With the same notation as in Proposition 1.1, the following are equivalent.*

(a) $\mathscr{F}_W(\theta)$ *satisfies the descending chain condition.*

(b)   *There do not exist sequences* $\{u^{(n)}\}_{n \geq 1}$, $\{w^{(n)}\}_{n \geq 1}$ *with* $u^{(n)} \in U$, $w^{(n)} \in W$ *and satisfying*

   (i) $\langle u^{(n)}, w^{(n)} \rangle \neq 0$
   (ii) $\langle u^{(n)}, w^{(m)} \rangle = 0$ $\Big\}$ *for all* $m > n \geq 1$.

The proof is similar to that of Proposition 1.1 and hence omitted.

For any subset Y of W let $L_Y = \{u \in R \,|\, \langle u, y \rangle = 0 \text{ for all } y \in Y\}$. Then L_Y is a $\mathbf{Z}$-submodule of U.   Let $\mathscr{F}_U(\theta) = \{L_Y \,|\, Y \text{ any subset of } W\}$.   The following proposition is an immediate consequence of Propositions 1.1 and 1.2 (cf. Proposition 24.3 of [1]).

PROPOSITION 1.3.   *The following are equivalent.*
   (a) $\mathscr{F}_W(\theta)$ *satisfies the ascending* (resp. descending) *chain condition.*
   (b) $\mathscr{F}_U(\theta)$ *satisfies the descending* (resp. ascending) *chain condition.*

2.   The five properties.   Let $M \in \text{mod-}R$ and J a nonempty set. We take $U = M^J$ and $W = R^{(J)}$.   For any $\underline{x} = (x_j)_{j \in J}$ in U and $\underline{\lambda} = (\lambda_j)_{j \in J}$ in $R^{(J)}$ let $\phi : U x W \to M$ be given by $\phi(\underline{x}, \underline{\lambda}) = \Sigma x_j \lambda_j$.   Notice that the sum $\Sigma x_j \lambda_j$ is only a finite sum since $\underline{\lambda} \in R^{(J)}$.   The map ϕ clearly induces a homomorphism $\theta : M^J \otimes_R R^{(J)} \to M$.   Also $R^{(J)}$ is a left, right R-bimodule.   Hence $M^J \otimes_R R^{(J)}$ is in a natural way a right R-module.   It is clear that θ is an R-homomorphism.   For any subset T of U, $L_T = \{\underline{\lambda} \in W \,|\, \langle \underline{x}, \underline{\lambda} \rangle = 0 \text{ for all } \underline{x} \in T\}$ is an R-submodule of $W = R^{(J)}$.   In this case the family $\mathscr{F}_W(\theta)$ introduced in §1 will be denoted by $\mathscr{F}^J(M)$.   If J and J' are any two sets with $|J| = |J'| = r \neq 0$, using a bijection of J with J' we can get a bijection of $\mathscr{F}^J(M)$ with $\mathscr{F}^{J'}(M)$ preserving inclusions.   Hence we write $\mathscr{F}_r(M)$ for any $\mathscr{F}^J(M)$ with $|J| = r$.

We can respect the above considerations with $U = M^{(J)}$, $W = R^{(J)}$, and $\phi(\underline{x}, \underline{\lambda}) = \Sigma x_j \lambda_j$ for any $\underline{x} = (x_j)$ in $M^{(J)}$ and $\underline{\lambda} = (\lambda_j)$ in $R^{(J)}$.   In this case we denote the family $\mathscr{F}_W(\theta)$ by either $\mathscr{F}^{(J)}(M)$ or $\mathscr{F}_{(r)}(M)$. Since $R^{(\phi)} = 0$, we do not consider the case $r = 0$ at all.   Thus throughout this paper r denotes a cardinal $\neq 0$.

DEFINITION 2.1. M is said to have the property A_r (resp. $A_{(r)}$) if $\mathscr{F}_r(M)$ (resp. $\mathscr{F}_{(r)}(M)$) has the ascending chain condition.

DEFINITION 2.2. M is said to have the property D_r (resp. $D_{(r)}$) if $\mathscr{F}_r(M)$ (resp. $\mathscr{F}_{(r)}(M)$) has the descending chain condition.

DEFINITION 2.3. M is said to have the property P_r if for every

set I, $0 \to M^{(I)} \to M^I \to M^I/M^{(I)} \to 0$ is pure-exact with respect to $\mathscr{C}_r$.

The following two lemmas are implicit in [5].

LEMMA 2.4 (Fieldhouse). *Let $B \subset A$, $N \subset M$ be R-modules, $\eta: A \to A/B$ and $\varepsilon: M \to M/N$ the canonical quotient maps. Suppose ε_*: $\mathrm{Hom}\,(A/B, M) \to \mathrm{Hom}\,(A/B, M/N)$ is onto. Then for any $f: A \to M$ satisfying $f(B) \subset N$, there exists a $g: A \to N$ such that $f/B = g/B$.*

LEMMA 2.5 (Fieldhouse). *With the same notation as in Lemma 2.4, suppose ε_*: $\mathrm{Hom}\,(A, M) \to \mathrm{Hom}\,(A, M/N)$ is onto. Suppose for any $f: A \to M$ satisfying $f(B) \subset N$, there exists a $g: A \to N$ such that $g/B = f/B$. Then ε_*: $\mathrm{Hom}\,(A/B, M) \to \mathrm{Hom}\,(A/B, M/N)$ is onto.*

The set of natural numbers will be denoted by N. For each cardinal number r we denote the class of modules isomorphic to $R^{(J)}/L$ with $|J| = r$ and L countably generated submodule of $R^{(J)}$ by $\mathscr{C}'_r$. In the statement of Propositions 2.6 and 2.7, M is a given object in mod-R, r a given cardinal number and J a set with $|J| = r$.

PROPOSITION 2.6. *Consider the following statements:*
  (a) *M has property A_r.*
  (a') *M has property $A_{(r)}$.*
  (b) *There do not exist any sequences $\{\underline{x}^{(n)}\}_{n \geq 1}$, $\{\underline{\lambda}^{(n)}\}_{n \geq 1}$ with $\underline{x}^{(n)} \in M^J$, $\underline{\lambda}^{(n)} \in R^{(J)}$ and satisfying*
$$\left.\begin{array}{ll} \text{(i)} & \langle \underline{x}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0 \\ \text{(ii)} & \langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0 \end{array}\right\} \text{ for } m > n \geq 1.$$
  (b') *Same as (b) with $M^{(J)}$ replacing M^J.*
  (c) *For any set I, any submodule L of $R^{(J)}$ and any map f: $R^{(J)} \to M^I$ satisfying $f(L) \subset M^{(I)}$ there exists a $g: R^{(J)} \to M^{(I)}$ such that $g/L = f/L$.*
  (d) *For any set I, the sequence*
$$0 \longrightarrow M^{(I)} \longrightarrow M^I \longrightarrow M^I/M^{(I)} \longrightarrow 0$$
is pure-exact with respect to $\mathscr{C}_r$.
  (e) *The sequence $0 \to M^{(N)} \to M^N \to M^N/M^{(N)} \to 0$ is pure-exact with respect to $\mathscr{C}'_r$.*
  In general, we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b') $\Leftrightarrow$ (a') and (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). If r is a finite cardinal all the statements (a) to (e) are equivalent.

Proof. The equivalences (a) $\Leftrightarrow$ (b) and (a') $\Leftrightarrow$ (b') follow from Proposition 1.1. (b) $\Rightarrow$ (b') is trivial since $M^{(J)} \subset M^J$.

(b') $\Rightarrow$ (b): For each $n \geq 1$ let $J_n = \{j \in J \mid \lambda_j^{(i)} \neq 0$ for some i in $1 \leq i \leq n\}$. Since $\underline{\lambda}^{(i)} \in R^{(J)}$ we see that each J_n is a finite set. Let $\underline{y}^{(n)} \in M^{(J)}$ be defined by $y_j^{(n)} = x_j^{(n)}$ for $j \in J_n$ and $y_j^{(n)} = 0$ for $j \in J - J_n$. Then clearly $\langle \underline{y}^{(m)}, \underline{\lambda}^{(n)} \rangle = \langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle$ for all $m \geq n \geq 1$. In particular, $\langle \underline{y}^{(n)}, \underline{\lambda}^{(n)} \rangle = 0$, $\langle \underline{y}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0$ for $m > n \geq 1$.

(a) $\Rightarrow$ (c): Let L be a submodule of $R^{(J)}$ and $f: R^{(J)} \to M^I$ be a map satisfying $f(L) \subset M^{(I)}$. Let $\{\underline{e}^j\}$ denote the canonical basis of $R^{(J)}$, namely, $\underline{e}^j = (u_{j'})_{j' \in J}$ with $u_{j'} = 0$ for $j' \neq j$ and $u_j = 1$. Let $f(\underline{e}^j) = (x_i^j)_{i \in I}$ with $x_i^j \in M$. For any $i \in I$ define $\underline{y}^i \in M^J$ by $y_j^i = x_i^j$ $\forall j \in J$. Let $H(I)$ denote the set of finite subsets of I. For any $F \in H(I)$ let $K_F = \{\underline{\lambda} \in R^{(J)} \mid \langle \underline{y}^i, \underline{\lambda} \rangle = 0$ for $i \in I - F\}$. If F, F' in $H(I)$ satisfy $F \subset F'$ we have $K_F \subset K_{F'}$. Each K_F is an element of $\mathscr{F}_r(M)$. Since $\mathscr{F}_r(M)$ satisfies the ascending chain condition, there exists an $F_0 \in H(I)$ such that K_{F_0} is maximal in $\{K_F\}_{F \in H(I)}$. For any $F \in H(I)$ we have $K_F \subset K_{F \cup F_0} \supset K_{F_0}$. The maximality of K_{F_0} yields $K_{F \cup F_0} = K_{F_0}$. Hence $K_{F_0} \supset K_F$ for all $F \in H(I)$.

Let $\underline{\lambda} = (\lambda_j)_{j \in J}$ be any element of L. The i coordinate of $f(\underline{\lambda})$ is $\sum_j x_i^j \lambda_j$. Since $f(\underline{\lambda}) \in M^{(I)}$ we see that there exists an $F \in H(I)$ (depending on $\underline{\lambda}$) such that $\sum_j x_i^j \lambda_j = 0$ for $i \in I - F$. Hence $\langle \underline{y}^i, \underline{\lambda} \rangle = \sum_j y_j^i \lambda_j = \sum_j x_i^j \lambda_j = 0$ for $i \in I - F$. This proves that $L \subset \bigcup_{F \in H(I)} K_F$. From $K_F \subset K_{F_0}$ for all $F \in H(I)$ we get $L \subset K_{F_0}$. Let $\underline{v}^j \in M^{(I)}$ be defined by $v_i^j = \begin{cases} x_i^j & \text{for } i \in F_0 \\ 0 & \text{for } i \in I - F_0 \end{cases}$. The map $g: R^{(J)} \to M^{(I)}$ carrying $\underline{e}^j$ to $\underline{v}^j$ is easily seen to fulfill the requirements of (c).

(c) $\Rightarrow$ (d): Every $E \in \mathscr{C}_r$ can be identified with $R^{(J)}/L$ for some $L \subset R^{(J)}$. If $\varepsilon: M^I \to M^I/M^{(I)}$ denotes the canonical quotient map, then $\varepsilon_*: \operatorname{Hom}(R^{(J)}, M^I) \to \operatorname{Hom}(R^{(J)}, M^I/M^{(I)})$ is onto since $R^{(J)}$ is free. Lemma 2.5 now shows that $\varepsilon_*: \operatorname{Hom}(E, M^I) \to \operatorname{Hom}(E, M^I/M^{(I)})$ is onto.

(d) $\Rightarrow$ (e) is trivial.

To complete the proof of Proposition 2.6 we have only to show that (e) $\Rightarrow$ (b) when r is a finite cardinal. If possible let $\underline{x}^{(n)} \in M^J$, $\underline{\lambda}^{(n)} \in R^{(J)}$ satisfy (i) and (ii) of (b). Let $\underline{x}^{(n)} = (x_j^{(n)})$, $\underline{\lambda}^{(n)} = (\lambda_j^{(n)})$. Let $\underline{y}^j = (y_n^j)_{n \geq 1} \in M^N$ be given by $y_n^j = x_j^{(n)}$. Let $f: R^{(J)} \to M^N$ be the map carrying $\underline{e}^j$ onto $\underline{y}^j$. Let L be the submodule of $R^{(J)}$ generated by $\underline{\lambda}^{(n)}$, $n \geq 1$. Then $R^{(J)}/L \in \mathscr{C}_r'$. The mth coordinate of $f(\underline{\lambda}^{(n)})$ is $\sum_j y_m^j \lambda_j^{(n)}$. But $\sum_j y_m^j \lambda_j^{(n)} = \sum_j x_j^{(m)} \lambda_j^{(n)} = \langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle$. Since $\langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0$ for $m > n$, we get $f(\underline{\lambda}^{(n)}) \in M^{(N)}$. Hence $f(L) \subset M^{(N)}$. Let $\varepsilon: M^N \to M^N/M^{(N)}$ denote the quotient map. By (e) the map $\varepsilon_*: \operatorname{Hom}(R^{(J)}/L, M^N) \to \operatorname{Hom}(R^{(J)}/L, M^N/M^{(N)})$ is onto. Hence by Lemma 2.4 there exists a map $g: R^{(J)} \to M^{(N)}$ satisfying $g/L = f/L$. Now let $g(\underline{e}^j) = (u_n^j)_{n \geq 1}$. From $f(\underline{\lambda}^{(n)}) = g(\underline{\lambda}^{(n)})$ we immediately get $\sum_j y_m^j \lambda_j^{(n)} = \sum_j u_m^j \lambda_j^{(n)}$ for all m. In particular, $\sum_j u_n^j \lambda_j^{(n)} = \sum_j y_n^j \lambda_j^{(n)} = \sum_j x_j^{(n)} \lambda_j^{(n)} \neq 0$ by (i)

of (b). Hence for any $n \geq 1$, there exists at least one $j \in J$ such that $u_n^j \neq 0$. Since J is a finite set and n varies over the infinite set N, it follows that there exists at least one $j \in J$ with $u_n^j \neq 0$ for infinitely many n. This contradicts the fact that (u_n^j) is in $M^{(N)}$.

This completes the proof of Proposition 2.6. Actually the proof is patterned after the proof of Lenzing [Proposition 1 of [6]].

It follows from this proposition that $A_r \Leftrightarrow A_{(r)}$ for any cardinal r. However, we will continue to use both the notations. Another consequence of the proposition is the equivalence of A_k and P_k. In general, we only have $A_r \Rightarrow P_r$.

PROPOSITION 2.7. *Consider the following conditions:*

(a) *M has property D_r.*

(b) *There do not exist sequences $\{\underline{x}^{(n)}\}$ in M^J and $\{\underline{\lambda}^{(n)}\}$ is $R^{(J)}$ satisfying*

(i) $\langle \underline{x}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0$

(ii) $\langle \underline{x}^{(n)}, \underline{\lambda}^{(m)} \rangle = 0$ $\Bigg\}$ *for $m > n \geq 1$.*

(a') *M has property $D_{(r)}$.*

(b') *Same as (b) with $M^{(J)}$ replacing M^J.*

Then we have (a) $\Leftrightarrow$ (b), (a') $\Leftrightarrow$ (b') and (b) $\Rightarrow$ (b') (thus (a) $\Rightarrow$ (b')).

Proof. The equivalences (a) $\Leftrightarrow$ (b), (a') $\Leftrightarrow$ (b') follow from Proposition 1.2. The implication (b) $\Rightarrow$ (b') is trivial since $M^{(J)} \subset M^J$.

COROLLARY 2.8. *Let G_r denote any one of the properties A_r, D_r, $D_{(r)}$.*

(i) *Let N be a submodule of M. If M has G_r so does N.*

(ii) *If M_1, M_2 are any two modules having G_r so does $M_1 \oplus M_2$.*

EXAMPLE 2.9. In mod-Z the sequence $0 \to Q^{(I)} \to Q^I \to Q^I/Q^{(I)} \to 0$ is split exact and hence pure-exact with respect to any family of modules. In particular, Q has property P_r for every cardinal r. However, we will show that Q does not have $A_{\mathcal{N}_0}$. Let $\underline{x}^{(k)} \in Q^N$, $\underline{\lambda}^{(k)} \in Z^{(N)}$ be defined by $\underline{\lambda}^{(k)} = (\lambda_n^{(k)})_{n \geq 1}$ where

$$\lambda_n^{(k)} = \begin{cases} (-1)^{n+1}(n+1) & \text{for } n \leq 2k \\ 0 & \text{for } n > 2k \end{cases}$$

for all $k \geq 1$; $\underline{x}^{(1)} = (1/n+3)_{n \geq 1}$ and $\underline{x}^{(k)} = (x_n^{(k)})$ where

$$x_n^{(k)} = \begin{cases} \dfrac{1}{n+1} & \text{for } n \leq 2k-2 \\ \dfrac{1}{n+3} & \text{for } n \geq 2k-1 \end{cases}$$

for $k \geqq 2$. Then $\langle \underline{x}^{(l)}, \underline{\lambda}^{(k)} \rangle = 0$ for all $l > k \geqq 1$ and $\langle \underline{x}^{(k)}, \underline{\lambda}^{(k)} \rangle \neq 0$ for all $k \geqq 1$. Hence by Proposition 2.6, Q does not have $A_{\mathscr{N}_0}$.

EXAMPLE 2.10. Since $A_k \Leftrightarrow P_k$ it follows from Corollary 2.8 that P_k is a hereditary property. However, for an infinite cardinal r, P_r is not hereditary. In mod-Z, Q has P_r for all r. But Z does not have $P_{\mathscr{N}_0}$. Let p be any prime and $\underline{u}^{(k)} \in Z^N$ be defined by $u_n^{(k)} = \begin{cases} 0 & \text{if } n \leqq k - 1 \\ p^{n-k} & \text{if } n \geqq k \end{cases}$. Let $\varepsilon: Z^N \to Z^N / Z^{(N)}$ denote the quotient map. If $v^{(k)} = \varepsilon(\underline{u}^{(k)})$ then $pv^{(k+1)} = v^{(k)}$ for all k. Let Γ denote the subgroup of Q consisting of elements of the form m/p^r with $m \in Z$, $r \geqq 1$. There exists a unique homomorphism $f: \Gamma \to Z^N / Z^{(N)}$ satisfying $f(1) = v^{(1)}$, $f(1/p^k) = v^{(k+1)}$. However, $\text{Hom}(\Gamma, Z^N) = 0$.

REMARK 2.11. Suppose R is a commutative ring. Using Proposition 1.3, it is easily seen that any artinian (resp. noetherian) module M has property A_k (resp. D_k) for all $k \geqq 1$. In particular, in mod-Z, the modules $M_n = Z/p^n Z$ have both A_k and D_k. If $M = \bigoplus_{n \geqq 1} M_n$, then $pZ \supset p^2 Z \supset \cdots$ is a strict descending chain in $\mathscr{F}_1(M)$. Thus M does not have D_1.

REMARK 2.12. Let $R = \prod_{j \in J} K_j$ be the direct product of infinitely many fields. Then each K_j as an R-module has property A_k for all k. However, $M = \bigoplus_{j \in J} K_j$ does not have property A_1.

Remarks 2.11 and 2.12 show that 2.8(ii) cannot be generalized to infinite direct sums (hence by 2.8(i) to infinite direct products of modules as well). The following result is essentially due to C. Faith [4].

PROPOSITION 2.13. *The following conditions on a ring R are equivalent.*
(a) *Every injective module has property A_1.*
(b) *Every module has property A_1.*
(c) *R is noetherian.*
(d) *Every module has property A_k for all $k \geqq 1$.*

Proof. (a) $\Rightarrow$ (b) is immediate from 2.8(i), since every module is a submodule of an injective module.

(b) $\Rightarrow$ (c): Let $\mathscr{H}$ denote the family of all (right) ideals of R and $M = \bigoplus_{L \in \mathscr{H}} R/L$. Then $\mathscr{F}_1(M) = \mathscr{H}$.

(c) $\Rightarrow$ (d): When R is noetherian, R^k is noetherian. Hence $\mathscr{F}_k(M)$ has the a.c.c. for any $M \in \text{mod-}R$.

(d) $\Rightarrow$ (a) is trivial.

PROPOSITION 2.14. *The following conditions on a ring R are*

equivalent.

   (a) *Every injective module has property D_1.*
   (b) *Every module has property D_1.*
   (c) *R is artinian.*
   (d) *Every module has property D_k for all $k \geq 1$.*

The proof is similar to that of Proposition 2.13 and hence omitted.

REMARK 2.15. Let M be a module having the property A_1. Then by Faith's theorem [4], if M is injective, then it is Σ-injective (namely, $M^{(I)}$ is injective for every set I). In particular, $M^{(I)}$ is a direct summand of M^I for any set I. However, this property does not characterize injective modules among modules having A_1. Let R be a left artinian ring which is not self-injective. Then by Proposition 3 of Lenzing [6] it follows that $R^{(I)}$ is a direct summand of R^I for all I. But R is not injective.

3. Behavior under base extension. Let R be a subring of S, $M \in \text{mod-}R$ and $M' = M \otimes_R S \in \text{mod-}S$. J denotes a set with $|J| = r$.

THEOREM 3.1. *Suppose* $\text{Tor}_1^R(M, S/R) = 0$. *If M' has property $A_{(r)}$ (resp. $D_{(r)}$), then so does M.*

Proof. We will prove this for $A_{(r)}$ (the proof for $D_{(r)}$ is similar and hence omitted). If M does not have $A_{(r)}$, there exist sequences $\underline{x}^{(n)} \in M^{(J)}$ and $\underline{\lambda}^{(n)} \in R^{(J)}$ such that $\langle \underline{x}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0$ and $\langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0$ for $m > n \geq 1$. Let $\underline{y}^{(n)} = x^{(n)} \otimes 1 \in M^{(J)} \otimes S = M'^{(J)}$. Regarding $\underline{\lambda}^{(n)}$ as elements of $S^{(J)}$ we have

$$\langle \underline{y}^{(m)}, \underline{\lambda}^{(n)} \rangle = \langle \underline{x}^{(m)} \otimes 1, \underline{\lambda}^{(n)} \rangle = \langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle \otimes 1 \in M' .$$

When $\text{Tor}_1^R(M, S/R) = 0$, the map $u \mapsto u \otimes 1$ of M in M' is injective. Hence $\langle \underline{y}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0$ and $\langle \underline{y}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0$ for $m > n \geq 1$. This means M' does not have $A_{(r)}$.

REMARK 3.2. In general $M^J \otimes S \neq (M \otimes S)^J$. If S is finitely presented as a (left) R-module we can identify $M^J \otimes S$ with $(M \otimes S)^J$. In this case the analogue of Theorem 3.1 is valid for the property D_r as well.

PROPOSITION 3.3. *Suppose R is a subring of a noetherian (resp. artinian) ring S. Then any $M \in \text{mod-}R$ with $\text{Tor}_1^R(M, S/R) = 0$ has property A_k (resp. D_k).*

Proof. Immediate consequence of Propositions 2.13,, 2.14 and

Theorem 3.1.

COROLLARY 3.4. *Let R be a subring of a ring S and let S/R be flat as a left R-module (equivalently, let S be a faithfully flat left R-module). If S is right noetherian (artinian), then R is right noetherian (artinian).*

This corollary is actually proved in § 3, No. 5, Chap. I, of [2] by different methods.

PROPOSITION 3.5. *Let R be a von Neumann regular ring. Then R is semi-simple $\Leftrightarrow R$ has A_1.*

Proof. Immediate consequence of Proposition 2.6 and the well-known fact that a regular ring R is semi-simple $\Leftrightarrow$ there does not exist an infinite family of orthogonal idempotents.

4. Behavior of $\mathrm{Hom}_R(S, M)$. Unless otherwise mentioned we consider the following situation. R is a subring of S and $i: R \to S$ denotes the inclusion. We assume that there exists an augmentation $\varepsilon: S \to R$ (namely, a ring homomorphism satisfying $\varepsilon \circ i = Id_R$). Let $K = \ker \varepsilon$ be the augmentation ideal in S. Then as an R-module we have $S = R \oplus K$. Moreover, $SK = K = KS$. In particular, $RK \subset K$. In what follows J denotes a set with $|J| = r$.

LEMMA 4.1. *Let $\{\underline{u}^{(\mu)}\}$ be any family of elements of $R^{(J)}$. Let N be the R-submodule of $R^{(J)}$ generated by $\{\underline{u}^{(\mu)}\}$ and V the S-submodule of $S^{(J)}$ generated by $\{\underline{u}^{(\mu)}\}$.*
  Then $V \cap R^{(J)} = N$.

Proof. Clearly $V \cap R^{(J)} \supset N$. Let $l = \Sigma \underline{u}^{(\mu)} s_\mu \in V \cap R^{(J)}$ with $s_\mu \in S$ (and $s_\mu = 0$ for almost all μ). Let $s_\mu = r_\mu + t_\mu$ with $r_\mu \in R$, $t_\mu \in K$. If $\underline{u}^{(\mu)} = (u_j^{(\mu)})_{j \in J}$ we have $\underline{u}^{(\mu)} \cdot t_\mu = (u_j^{(\mu)} t_\mu)_{j \in J}$. Since $RK \subset K$, it follows that $u_j^{(\mu)} t_\mu \in K$ for each $j \in J$. Hence $\underline{u}^{(\mu)} t_\mu \in K^{(J)}$. As an R-module we have $S^{(J)} = R^{(J)} \oplus K^{(J)}$ and the element l of $R^{(J)}$ has the representation $(\sum_\mu \underline{u}^{(\mu)} r_\mu) + (\sum_\mu \underline{u}^{(\mu)} t_\mu)$ with $\sum_\mu \underline{u}^{(\mu)} r_\mu \in R^{(J)}$ and $\sum_\mu \underline{u}^{(\mu)} t_\mu \in K^{(J)}$. It follows that $l = \sum_\mu \underline{u}^{(\mu)} r_\mu$. In other words, $l \in N$. Thus $V \cap R^{(J)} \subset N$.

Let Γ_r (resp. Γ'_r) dedote the class of R-submodules of $S^{(J)}$ of the form $R^{(J)} + V$ where V is an S-submodule of $S^{(J)}$ generated by finitely many (resp. countably many) element of $R^{(J)}$.

THEOREM 4.2. *Suppose $M \in mod\text{-}R$ satisfies $\mathrm{Ext}_R^1(S^{(J)}/L, M) = 0$ for any $L \in \Gamma_r$ (resp. Γ'_r). Let $M' = \mathrm{Hom}_R(S, M)$. If M' has A_n*

(resp. D_r), *then M has A_r* (resp. D_r).

Proof. We prove the theorem only for A_r. The proof for D_r is similar and hence omitted. Suppose M does not have A_r. Then there exist $\underline{x}^{(n)} \in M^J$, $\underline{\lambda}^{(n)} \in R^{(J)}$ with $\langle \underline{x}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0$ and $\langle \underline{x}^{(m)}, \underline{\lambda}^{(n)} \rangle = 0$ for $m > n \geqq 1$. We identify M^J with $\mathrm{Hom}\,(R^{(J)}, M)$. An element $\underline{x} \in M^J$ corresponds to the homomorphism $\alpha_{\underline{x}} \colon R^{(J)} \to M$ given by $\alpha_{\underline{x}}(\underline{\lambda}) = \langle \underline{x}, \underline{\lambda} \rangle$. For $n \geqq 2$ let V_n be the S-submodule of $S^{(J)}$ generated by $\underline{\lambda}^{(1)}, \cdots, \underline{\lambda}^{(n-1)}$ and L_n the R-submodule of $R^{(J)}$ generated by $\underline{\lambda}^{(1)}, \cdots, \underline{\lambda}^{(n-1)}$. From Lemma 4.1 we get $L_n = V_n \cap R^{(J)}$. Denote the map $\alpha_{\underline{x}}(n) \colon R^{(J)} \to M$ by $f^{(n)}$. Then $f^{(n)}(L_n) = 0$. If $g^{(n)} \colon R^{(J)} + V_n \to M$ is defined by $g^{(n)}(\underline{u} + \underline{v}) = f^{(n)}(\underline{u})$ for any $\underline{u} \in R^{(J)}$, $\underline{v} \in V_n$, then using the facts that $V_n \cap R^{(J)} = L_n$ and that $f^{(n)}(L_n) = 0$, it can easily be shown that $g^{(n)}$ is a well-defined R-homomorphism. Clearly $R^{(J)} + V_n \in \Gamma_r$. By assumption, $\mathrm{Ext}_R^1 (S^{(J)}/R^{(J)} + V_n, M) = 0$. It follows that $g^{(n)}$ can be extended to an R-homomorphism $h^{(n)} \colon S^{(J)} \to M$. Under the usual identification of $\mathrm{Hom}_R(S^{(J)}, M)$ with $\mathrm{Hom}_R(S, M)^J = M'^J$ let $h^{(n)}$ correspond to the element $\underline{h}^{(n)} = (h_j^{(n)}) \in M'^J$ with $h_j^{(n)} \in M'$. Then the element $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} \rangle \in M' = \mathrm{Hom}_R(S, M)$ satisfies $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} \rangle(s) = \sum_j h_j^{(n)}(\lambda_j^{(i)} s) \in M$ for every $s \in S$. But $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} s \rangle(1) = \sum_j h_j^{(n)}(\lambda_j^{(i)} s)$. Since $\underline{h}^{(n)}/V_n = 0$, it follows that $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} s \rangle = 0$ for $1 \leqq i \leqq n - 1$. In other words, $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} \rangle(s) = 0$ $\forall s \in S$ whenever $1 \leqq i \leqq n - 1$. Hence $\langle \underline{h}^{(n)}, \underline{\lambda}^{(i)} \rangle = 0$ in M' for $1 \leqq i \leqq n - 1$. Also $\langle \underline{h}^{(n)}, \underline{\lambda}^{(n)} \rangle(1) = f^{(n)}(\underline{\lambda}^{(n)}) = \langle \underline{x}^{(n)}, \underline{\lambda}^{(n)} \rangle \neq 0$. Thus M' does not have A_r.

Let M be an injective module in mod-R. Then $M' = \mathrm{Hom}_R(S, M)$ is injective in mod-S [Proposition 1.4, §1, Chap. VI, [3]].

COROLLARY 4.3. *With the same conventions as at the beginning of this section, let $M \in$ mod-R be injective. If M' is Σ-injective in mod-S, then so is M in mod-R.*

Proof. This is because an injective module is Σ-injective if and only if it has A_1.

REMARK 4.4. Theorem 4.2 and Corollary 4.3 are not valid for an arbitrary ring S containing R as a subring. For instance, let R be a non-noetherian integral domain and S its quotient field. Since R is non-noetherian, there exists an injective module $M \in$ mod-R which is not Σ-injective. However, since S is a field, $M' = \mathrm{Hom}_R(S, M)$ is Σ-injective in mod-S.

REMARK 4.5. Let R be any ring and $\mathfrak{A}$ a two-sided ideal of R. Let $f \colon R \to R/\mathfrak{A}$ be the canonical quotient map. Let $E \in$ mod-R and $E' = \{x \in E \mid x\mathfrak{A} = 0\} \cong \mathrm{Hom}_R(R/\mathfrak{A}, E) \in$ mod-$R/\mathfrak{A}$. Suppose E' does not

have A_1 in mod-$R/\mathfrak{A}$. Then there exist $x^{(n)} \in E'$ and $\lambda^{(n)} \in R$ such that $x^{(n)}f(\lambda^{(n)}) \neq 0$ and $x^{(m)}f(\lambda^{(n)}) = 0$ for $m > n \geqq 1$. But $x^{(n)}\lambda^{(i)} = x^{(n)}f(\lambda^{(i)})$. Thus $x^{(n)}\lambda^{(n)} \neq 0$ and $x^{(m)}\lambda^{(n)} = 0$ for $m > n \geq 1$. Since $x^{(n)} \in E$ it follows that E does not have A_1. Now if E is R-injective, then E' is $R/\mathfrak{A}$-injective. By the argument above if E is Σ-injective, then so is E' in mod-$R/\mathfrak{A}$. Since $E'^{(I)} = \{x \in E^{(I)} \mid x\mathfrak{A} = 0\}$ this last fact can also be seen directly.

5. **Arbitrary family M_α.** Let $\{M_\alpha\}_{\alpha \in I}$ be any family of modules. For any k-tuple $(\underline{x}^{(1)}, \cdots, \underline{x}^{(k)})$ with $\underline{x}^{(i)} \in \Pi M_\alpha$ let $T_{(\underline{x}^{(1)}, \dots, \underline{x}^{(k)})} = \{(\lambda^{(1)}, \cdots, \lambda^{(k)}) \in R^k \mid \sum_{i=1}^k \underline{x}^{(i)}\lambda^{(i)} \in \bigoplus M_\alpha\}$.

DEFINITION 5.1. A k-tuple $(\underline{x}^{(1)}, \cdots, \underline{x}^{(k)})$ with $\underline{x}^{(i)} \in \Pi M_\alpha$ is called "special" if there exists a finite subset F of I such that $\sum_{i=1}^k x_\alpha^{(i)}\lambda^{(i)} = 0$ for all $\alpha \in I - F$ and $(\lambda^{(1)}, \cdots, \lambda^{(k)}) \in T_{(\underline{x}^{(1)}, \dots, \underline{x}^{(k)})}$. Here $\underline{x}^{(i)} = (x_\alpha^{(i)})_{\alpha \in I}$. For any integer $k \geq 1$ the following result is the analogue of Proposition 2.6. Its proof is practically similar to that of Proposition 2.6 and hence is omitted.

PROPOSITION 5.2. *Let $\{M_\alpha\}_{\alpha \in I}$ be any family of R-modules and k an integer ≥ 1. Then the following statements are equivalent.*

(a) *Each k-tuple $(\underline{x}^{(1)}, \cdots, \underline{x}^{(k)})$ of elements from ΠM_α is special.*

(b) *For every countable infinite subset X of I and any enumeration of X, there do not exist families of elements $(x_\mu^{(1)})_{\mu \in X}, \cdots, (x_\mu^{(k)})_{\mu \in X}$ in $\prod_{\mu \in X} M_\mu$ and $(\lambda_\mu^{(1)})_{\mu \in X}, \cdots, (\lambda_\mu^{(k)})_{\mu \in X}$ is $R^{(X)}$ satisfying*

(i) *$\sum_{i=1}^k x_\mu^{(i)}\lambda_\mu^{(i)} \neq 0$ for all $\mu \in X$ and*

(ii) *$\sum_{i=1}^k x_\nu^{(i)}\lambda_\mu^{(i)} = 0$ for $\nu > \mu$ in X.*

(c) *The sequence $0 \to \bigoplus M_\alpha \to \Pi M_\alpha \to \Pi M_\alpha / \bigoplus M_\alpha \to 0$ is pure-exact with respect to $\mathscr{C}_k$.*

(d) *For every countable subset X of I the sequence $0 \to \bigoplus_{\alpha \in X} M_\alpha \to \prod_{\alpha \in X} M_\alpha \to \prod_{\alpha \in X} M_\alpha / \bigoplus_{\alpha \in X} M_\alpha \to 0$ is pure-exact with respect to $\mathscr{C}_k'$.*

(e) *For any $L \subset R^{(k)}$ and $f: R^{(k)} \to \Pi M_\alpha$ with $f(L) \subset \bigoplus M_\alpha$ there exists a $g: R^{(k)} \to \bigoplus M_\alpha$ satisfying $g/L = f/L$.*

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