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MONOTONICITY AND ALTERNATIVE METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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Let X be a Hilbert space, E a linear operator with finite dimensional null space, and N a nonlinear operator. In this paper we study the nonlinear equation

$$Ex = Nx \quad x \in X . \tag{1}$$

Equations of this form arise in the study of boundary value problems for elliptic differential equations.

We use the alternative scheme of Bancroft, Hale, and Sweet and results from monotone operator theory with suitable monotonicity assumptions on E and N to reduce equation (1) to an alternative problem. We then use results from monotone operator theory to solve the alternative problem, hence prove the existence of solutions to equation (1). This extends to nonselfadjoint operators the results of Cesari and Kannan.

1. Introduction. The reduction of equation (1) to a finite dimensional alternative problem has been done using the contraction mapping principle by Cesari [4] for selfadjoint operators and by Bancroft, Hale, and Sweet [1] for nonselfadjoint operators; and using the theory of monotone operators by Gustafson and Sather [8] for selfadjoint operators where E may have continuous spectrum, by Cesari and Kannan [7] for selfadjoint operators with a complete set of eigenfunctions and eigenvalues approaching $-\infty$, by Cesari [6] for nonselfadjoint operators, and by Osborn and Sather [11] for nonselfadjoint operators generated by a coercive bilinear form. Only the papers of Gustafson and Sather [8] and Cesari [6] avoid using a compactness argument such as assuming that E has a compact resolvent $(E - aI)^{-1}$. For a survey of recent results see Cesari [5] for selfadjoint problems and Cesari [6] for nonselfadjoint problems.

Since the alternative problem is now on a finite dimensional subspace of X, it has been the practice to use either degree theory or the implicit function theorem to solve the alternative problem. An exception to this is the paper by Cesari and Kannan [7] which uses monotone operator theory to solve the alternative problem hence obtain a solution to equation (1).

In §3 of this paper we use the alternative scheme of Bancroft, Hale, and Sweet [1] but with the theory of monotone operators to reduce equation (1) to a finite dimensional alternative problem (Theorem 4). This reduction is a modification of the method used by Cesari [6] but only requires N to be quasimonotone instead of monotone. The results we obtain apply to nonselfadjoint operators and do not require any compactness arguments. Our assumption that N be quasimonotone is weaker than the monotonicity assumptions of Osborn and Sather [11], but we must place a stronger restriction on the domain of N.

In § 4 we prove an existence theorem for equation (1) using monotone operator theory which extends to nonselfadjoint operators the results of Cesari and Kannan [7] for selfadjoint operators. Again no compactness arguement is necessary.

Finally in § 5 we apply our results to nonlinear boundary value problems of elliptic differential equations.

We will consider the case when the linear operator E has a continuous spectrum in a subsequent paper.

2. Basic concepts in monotone operator theory. Let H be a real Hilbert space and let 2^H be the set of all subsets of H. Let T be a map $T:D(T) \to 2^H$ such that for some constant c, $(u - v, x - y) \geqq c \|x - y\|^2$ for all $x, y \in D(T)$, $u \in Tx$, and $v \in Ty$. We say T is *monotone* if $c = 0$, *strongly monotone* with constant c if $c > 0$, and *quasimonotone* with constant $-c$ if $c < 0$. The map T is *coercive* if there exists $x_0 \in H$ such that $(T^0x, x - x_0) \|x\|^{-1} \to +\infty$ as $\|x\| \to +\infty$ for $x \in D(T)$ where T^0x is the element of Tx with minimal norm. If T is a single-valued strongly monotone map, then T is coercive.

In addition to the standard results for monotone operators (See Brézis [2] and Browder [3]), we will need the following result concerning quasimonotone operators.

THEOREM 1 (Nagle [10]). *Let $A: D(A) \to 2^H$ and $B: H \to 2^H$ be hemicontinuous. If A is strongly monotone with constant μ, A maximal monotone, and B quasimonotone with constant η, $\eta < \mu$, then $A + B$ is maximal monotone. Moreover, if A and B are single-valued, then the range of $A + B$ is all of H.*

The concept of monotonicity may be extended to maps $T: S \to 2^{S^*}$ where S is a reflexive Banach space and S^* is the dual of S (see Browder [3]).

3. Reduction to an alternative problem. Let us consider the equation

$$Ex = Nx \tag{1}$$

where E is a linear operator whose domain $D(E)$ is a subspace of a real Hilbert space X and whose range $R(E)$ lies in X. Here N is an operator, not necessarily linear, with $D(N) \subset X$, and $R(N) \subset X$. Moreover, we assume $D(E) \subset D(N)$.

We will use the alternative scheme of Bancroft, Hale, and Sweet [1] to split equation (1) into a system of two equations. We assume there are bounded projection operators $P: X \rightarrow X$ and $Q: X \rightarrow X$ and a linear operator $H: D(H) \rightarrow X$, $D(H) \subset X$, such that for all $x \in D(E)$:

$$H(I - Q)Ex = (I - P)x \tag{A_1}$$

$$QEx = EPx \tag{A_2}$$

$$EH(I - Q)Nx = (I - Q)Nx \tag{A_3}$$

and where $R(P) \subset D(E)$, $R(H) \subset D(E)$, $R(E) = D(H)$, and $(I - Q)R(E) = D(H)$.

In view of (A_1) and (A_3), we may think of H as a partial inverse for E. It follows from (A_1) that $\ker(E) \subset PX$. Since $D(H) \subset R(E)$, this implies $(I - Q)R(E) \subset R(E) \cap \{(I - Q)X\}$ and since $I - Q$ is the identity on $(I - Q)X$, we must have $(I - Q)R(E) = R(E) \cap \{(I - Q)X\}$.

In the rest of the paper we will use the following notation: $X_0 = PX$ and $X_1 = (I - P)X$ hence $X = X_0 + X_1$; $Y_0 = QX$ and $Y_1 = (I - Q)X$ hence $X = Y_0 + Y_1$.

THEOREM 2. *If* $(A_{1,2,3})$ *are satisfied, then* $Ex = Nx$ *for some* $x \in D(E)$ *if and only if*

$$x = Px + H(I - Q)Nx \tag{2}$$

$$Q(Ex - Nx) = 0 . \tag{3}$$

Proof. For a proof see Cesari [6] or Bancroft, Hale, and Sweet [1].

Since by (A_1) $PH(I - Q)Nx = 0$, we may write equation (2) as

$$x = x_0 + H(I - Q)Nx \tag{4}$$

where $x_0 \in X_0$.

We will now show that with suitable monotonicity assumptions on E and N and a technical assumption on P and Q, that equation (4) is uniquely solvable for each $x_0 \in X_0$. We then reformulate equation (1) as an alternative problem in X_0.

(A_4) There is a constant $\mu > 0$ such that for $x \in D(E) \cap X_1$; $(-Ex, x) \geqq \mu \|x\|^2$ or for $y \in Y_1$; $(y, -Hy) \geqq \mu \|Hy\|^2$.

(A_5) Let P and Q be chosen so that $(y, x) = 0$ for every $y \in Y_0$ and $x \in X_1$.

It follows from the Fredholm Alternative that if P is the projection onto the ker E and Q is the projection onto the ker E^*, then assumption (A_5) is satisfied.

Assumption (A_4) is often satisfied for elliptic partial (or ordinary) differential operators. In particular, Osborn and Sather [11] prove a stronger result for a certain class of operators generated by a coercive bilinear form.

The next theorem is a generalization to quasimonotone operators of a theorem due to Cesari [6]. The method of proof is a slight modification of the proof given in Cesari [6].

THEOREM 3. *Let conditions* (A_{1-5}) *be satisfied. Let* $N: D(N) = X \to X$ *be hemicontinuous and quasimonotone with constant* $\eta > 0$, $\eta < \mu$. *Then equation* (4) *has a unique solution for each* $x_0 \in X_0$.

Proof. We write equation (4) in the form $x - x_0 - H(I-Q)Nx = 0$ and since $0 \in [-H(I-Q)]^{-1}0$, we have

$$[-H(I-Q)]^{-1}(x - x_0) + Nx \ni 0 \qquad (5)$$
$$x \in x_0 + (X_1 \cap D(E)) .$$

Conversely, by applying $-H(I-Q)$ to both sides we get $x - x_0 - H(I-Q)Nx = -H(I-Q)0 = 0$. So the two equations are equivalent.

Equation (5) is of the form $Ax + Bx \ni 0$ where $Ax = [-H(I-Q)]^{-1}(x - x_0)$ and $Bx = Nx$. By our assumptions on N we find that B satisfies the hypothesis of Theorem 1. We will now show that A is strongly monotone with monotonicity constant $\mu > \eta$ and A is maximal monotone.

With assumptions ($A_{4,5}$) we have for all $y \in X$,

$$(y, -H(I-Q)y) = ((I-Q)y, -H(I-Q)y) \qquad (6)$$
$$\geqq \mu \| -H(I-Q)y \|^2$$

so $-H(I-Q)$ is monotone. Since $-H(I-Q)$ is bounded linear operator defined on all of X, $-H(I-Q)$ is continuous over X, hence $-H(I-Q)$ is maximal monotone. Since x_0 is fixed, $-H(I-Q)+x_0$ is maximal monotone. Now A is just the inverse of the map $Ky = -H(I-Q)y + x_0$ so it follows that A is maximal monotone. To show A is strongly monotone, let $x, y \in D(A) = x_0 + (X_1 \cap D(E))$. Then $x - x_0 \in X_1 \cap D(E)$ and $y - x_0 \in X_1 \cap D(E)$. For $x^* \in Ax$, $x - x_0 =$

$-H(I-Q)x^*$ and for $y^* \in Ay$, $y - x_0 = H(I-Q)y^*$. Using equation (6), we have

$$\begin{aligned}(x^* - y^*, x - y) &= (x^* - y^*, -H(I-Q)x^* + H(I-Q)y^*) \\ &= ((I-Q)x^* - (I-Q)y^*, \ -H(I-Q)x^* + H(I-Q)y^*) \\ &\geqq \mu \| -H(I-Q)x^* + H(I-Q)y^* \|^2 \\ &\geqq \mu \|x - y\|^2 .\end{aligned}$$

So A is strongly monotone with monotonicity constant μ.

To show that $A + B$ is coercive we use the fact that $-H(I-Q)$ is a one to one map from Y_1 onto $X_1 \cap D(E)$. Thus for $y_1 \in Y_1$ there is a unique $x_1 \in X_1 \cap D(E)$ such that $x_1 = -H(I-Q)y_1$. Thus, $[-H(I-Q)]^{-1}x_1 = y_1 + Y_0$. Let $x = x_0 + x_1$, $x_1 \in X_1 \cap D(E)$, then $Ax = [-H(I-Q)]^{-1}(x - x_0) = [-H(I-Q)]^{-1}x_1 = y_1 + Y_0$. Let $A^0x = y_1 + y_0^*$, $y_0^* \in Y_0$ (in fact since $Y_0 \perp X_1$ we have $A^0x = y_1$). Using equation (6) and condition (A_3) we have $(A^0x, x - x_0) = (A^0x, x_1) = (y_1 + y_0^*, x_1) = (y_1, x_1) + (y_0^*, x_1) = (y_1, x_1) = (y_1, -H(I-Q)y_1) \geqq \mu \| -H(I-Q)y_1\|^2 = \mu \|x_1\|^2 = \mu \|x - x_0\|^2$. Since x_0 is fixed, this implies A is coercive. Since B is single-valued and monotone, $A + B$ is coercive.

Now $A + B$ is maximal monotone and coercive thus $R(A+B) = X$. The equation $Ax + Bx \ni 0$ has a solution, hence equation (4) has a solution for each $x_0 \in X_0$.

To prove uniqueness, let x_1 and x_2 both be solutions to equation (4) for a fixed $x_0 \in X_0$, then $x_1 - x_0 = H(I-Q)Nx_1$; $x_2 - x_0 = H(I-Q)Nx_2$; and $x_1 - x_2 = H(I-Q)Nx_1 - H(I-Q)Nx_2$. By assumption ($A_5$), equation (6), and the hypothesis of N we have $-\eta \|x_1 - x_2\|^2 \leqq (Nx_1 - Nx_2, x_1 - x_2) = -(Nx_1 - Nx_2, -H(I-Q)Nx_1 + H(I-Q)Nx_2) \leqq -\mu \| -H(I-Q)Nx_1 + H(I-Q)Nx_2\|^2 = -\mu \|x_1 - x_2\|^2$. That is $0 \leqq (\eta - \mu)\|x_1 - x_2\|^2$. Since $\mu > \eta$, we must have $x_1 = x_2$. This completes the proof of the theorem.

For each $x_0 \in X_0$, the unique solution to (4) can be expressed by $x = [I - H(I-Q)N]^{-1}x_0$. Substituting into equation (3) we find that solving our original equation (1) is equivalent to solving the alternative problem: $QN[I - H(I-Q)N]^{-1}x_0 - QEx = 0$. Since $QEx = EPx = Ex_0$, the alternative problem becomes $QN[I - H(I-Q)N]^{-1}x_0 - Ex_0 = 0$, where $x_0 \in X_0$.

4. An existence theorem. The main result of this paper is the following existence theorem which extends to nonselfadjoint problems the results of Cesari and Kannan [7]. We will need an additional assumption; however, the remark concerning assumption (A_5) also applies here.

(A_6) $$(y, x) = 0 \text{ for } y \in Y_1 \text{ and } x \in X_0 .$$

THEOREM 4. *Let* X_0 *and* Y_0 *be finite dimensional subspaces of* X, *and assume* $\dim X_0 = \dim Y_0$. *Under assumptions* (A_{1-6}):

(a) *If* $X_1 = X$ *and* $Y_1 = Y$, i.e., $(-Ex, x) \geqq \mu \|x\|^2$ *for all* $x \in D(E)$; *and* $N: D(N) = X \to X$ *is hemicontinuous and quasimonotone with constant* $\eta \geqq 0$ *and* $\eta < \mu$, *then* $Ex = Nx$ *has a unique solution.*

(b) *If* $(-Ex, x) \geqq -\alpha \|x\|^2$ *for all* $x \in D(E)$, $\alpha > 0$, *and if* $N: D(N) \to X$ *is hemicontinuous, bounded mapping, that is strongly monotone with constant* $\eta > 0$, *then the equation* $Ex = Nx$ *has at least one solution provided* $\eta > \alpha \|P\|^2$ *or* $\eta = \alpha = 0$ *and* N *is coercive.*

Proof. (a) Since $Y_1 = X$, Q is the zero operator on X, hence equation (3) is always satisfied. Part (a) of the theorem now follows from Theorem 3.

(b) Since the dimension of X_0 and Y_0 are equal and finite, we may identify X_0^* with Y_0. The identification may be made as follows: let $y \in Y_0 \subset X$, for $x_0 \in X_0$ let $y_0(x_0) = (y_0, x_0)$, where on the right hand side we view y_0 as an element of X and x_0 as an element of X. This identification defines a continuous, linear mapping of Y_0 into X_0^*. Our identification of Y_0 with X_0^* is complete if we can show that this map is one to one and onto. Since the dimension is finite we need only show the map is one to one or the kernel of the map is just the zero element of Y_0. For this let y_0 be an element of Y_0. If $(y_0, x_0) = 0$ for all $x_0 \in X_0$ then by (A_5), $(y_0, x_1) = 0$ for all $x_1 \in X_1$ hence $(y_0, x) = 0$ for all $x \in X$ hence y_0 must be the zero element of X and Y_0.

Since the assumptions of Theorem 3 hold, the equation $Ex = Nx$ is reduced to the alternative problem $QN[I - H(I - Q)]^{-1}x_0 - Ex_0 = 0$ for $x_0 \in X_0$. Since $QEx = EPx = Ex_0$, and $Y_0 = X_0^*$, $-E$ maps X_0 into X_0^*, and $T = QN[I - H(I - Q)N]^{-1}$ maps X_0 into X_0^*, so $T - E$ maps X_0 into X_0^*.

We will now show that $T - E$ is monotone, continuous, and coercive. Then, since $T - E$ is defined on all X_0, $R(T - E) = X_0^*$ and the alternative problem will be solved.

Let $x, y \in X_0$, and let $u = [I - H(I - Q)N]^{-1}x$, $v = [I - H(I - Q)N]^{-1}y$. Then $u - H(I - Q)Nu = x$ and $v - H(I - Q)Nv = y$. Since $P(u - v) = x - y$, $\|x - y\| \leqq \|P\| \|u - v\|$. So

$$\begin{aligned}(Tx - Ty, x - y) &= (QNu - QNv, x - y) = (Nu - Nv, x - y)\\ &= (Nu - Nv, u - v) + (Nu - Nv, -H(I - Q)Nu + H(I - Q)Nv)\\ &= (Nu - Nv, u - v) + ((I - Q)Nu - (I - Q)Nv, -H(I - Q)Nu\\ &\quad + H(I - Q)Nv)\\ &\geqq (Nu - Nv, u - v) \geqq \eta \|u - v\|^2\\ &\geqq \eta(\|P\|^{-1})^2 \|x - y\|^2 .\end{aligned}$$

Now by hypothesis, $x, y \in X_0$, $(-E(x-y), x-y) \geqq -\alpha \|x-y\|^2$. Thus $((T-E)x-(T-E)y, x-y) \geqq (\eta\|P\|^{-2}-\alpha)\|x-y\|^2$. This proves $T-E$ is monotone. If $\eta\|P\|^{-2} > \alpha$, then it follows that $T-E$ is also coercive since T and E are single-valued maps.

If $\eta = \alpha = 0$, then we must show that N coercive implies that T is coercive. Let $x_0 \in X_0$ and $x = [I-H(I-Q)N]^{-1}x_0$; $Px = x_0$ thus $\|x_0\| \leqq \|P\|\,\|x\|$, and $\|x_0\| \to +\infty$ implies $\|x\| \to +\infty$. We now have $(Tx_0, x_0) = (QN[I-H(I-Q)N]^{-1}x_0, x_0) = (Nx, x_0) = (Nx, x - H(I-Q)Nx) = (Nx, x) + (Nx, -H(I-Q)Nx) \geqq (Nx, x)$. So $\|x_0\|^{-1}(Tx_0, x_0) \geqq \|x_0\|^{-1}(Nx, x) \geqq \|P\|^{-1}\|x\|^{-1}(Nx, x)$. Since N is coercive, T must be coercive. Since $-E$ is linear and monotone, $T-E$ is coercive.

It remains to show that $T-E$ is continuous. We begin by showing that $[I-H(I-Q)N]^{-1}$ is bounded. Let u, v be such that $u - H(I-Q)Nu = v$, then $\mu\|u-v\|^2 = \mu\|-H(I-Q)Nu\|^2 \leqq (-H(I-Q)Nu, (I-Q)Nu) = (-H(I-Q)Nu, Nu) = (v-u, Nu) = (v-u, Nu - Nv + Nv) = -(u-v, Nu-Nv) + (v-u, Nv) \leqq (v-u, Nv) \leqq \|v-u\|\,\|Nv\|$ hence $\|u-v\| \leqq \mu^{-1}\|Nv\|$. Thus if $\|v\| \leqq R$, then $\|Nv\| < R'$ for some R' depending only on R since N is bounded, and $\|u\| \leqq \|v\| + \mu^{-1}\|Nv\| \leqq R + \mu^{-1}R'$. Thus $[I-H(I-Q)N]^{-1}$ is bounded. For x, y in X we have $\|x+y\|^2 + \|y\|^2 \geqq 1/4\|x\|^2$. To prove this consider the two cases $\|x\| \geqq 1/2\|y\|$ and $\|x\| \geqq 1/2\|y\|$. To show $[I-H(I-Q)N]^{-1}$ is continuous let $u - H(I-Q)Nu = u^*$ and $v - H(I-Q)Nv = v^*$ then $u^* - v^* = u - v - H(I-Q)Nu + H(I-Q)Nv$. Now $\mu\|-H(I-Q)Nu + H(I-Q)Nv\|^2 \leqq (Nu - Nv, u-v) + (Nu-Nv, -H(I-Q)Nu + H(I-Q)Nv) = (Nu-Nv, u - H(I-Q)Nu - v + H(I-Q)Nv) = (Nu-Nv, u^*-v^*) \leqq \|Nu - Nv\|\,\|u^*-v^*\|$. Now let $x = v-u$ and $y = u^*-v^*$ in the equation $\|x+y\|^2 + \|y\|^2 \geqq 1/4\|x\|^2$, then $1/4\|u-v\|^2 - \|u^*-v^*\|^2 \leqq \|u^*-v^*+v-u\|^2 = \|-H(I-Q)Nu + H(I-Q)Nv\|^2 \leqq \mu^{-1}\|Nu - Nv\|\,\|u^*-v^*\|$. If $\|u^*\|, \|v^*\| < R$, then since $[I-H(I-Q)N]^{-1}$ is bounded, we have $\|u\|, \|v\| \leqq R'$ and $\|Nu\|, \|Nv\| \leqq R''$ since N is bounded. Thus $\|u-v\|^2 \leqq 4\|u^*-v^*\|^2 + 4\mu^{-1}\|Nu-Nv\|\,\|u^*-v^*\| \leqq (8R + 8\mu^{-1}R'')\|u^*-v^*\|$. Hence $[I-H(I-Q)N]^{-1}$ is continuous.

Before proving the continuity of T, recall a map S is demicontinuous if $x_n \to x$ strongly in X implies $Sx_n \to Sx$ weakly in X^*. Kato [9] has shown that for monotone operators defined on a Banach space X with range in X^*, then hemicontinuity and demicontinuity agree. Hence, since $[I-H(I-Q)N]^{-1}$ is continuous, $N[I-H(I-Q)N]^{-1}$ is demicontinuous and $QN[I-H(I-Q)N]^{-1}$ is demicontinuous. Since the dimension of X_0^* is finite, weak convergence in X_0^* is the same as strong convergence. Hence $T = QN[I-H(I-Q)N]^{-1}$ is continuous. This completes the proof of Theorem 4.

5. Applications. In this section we demonstrate how Theorem 4 applies to boundary value problems for nonlinear elliptic differential equations.

EXAMPLE 1. Let us consider the existence of solutions to the partial differential equation

$$\Delta u + au_x + bu_y - g(x, y)u = f(x, y) + \operatorname{Sin}(u)$$

for x, y in D with Dirichlet boundary conditions $u = 0$ on ∂D, where $D = [0, \pi] \times [0, \pi]$, a and b are constants, $g \in C(D)$, $g(x, y) \geqq 0$, and $f \in L_2(D)$.

Let $Eu = \Delta u + au_x + bu_y - g(x, y)u$, and let $D(E) = \{u \in L_2(D)$: $Eu \in L_2(D)$ and $u = 0$ on $\partial D\}$. Now E is a one to one, invertible, nonselfadjoint operator so let H be the bounded inverse for E. Let $P = Q$ be the zero, operator. It is easy to show that assumptions $(A_{1-3,5,6})$ are all satisfied. To prove (A_4) is satisfied, let $u \in D(E)$. Then, $(-au_x, u) = -a\int_0^\pi\int_0^\pi \partial/\partial x(u^2/2)dxdy = 0$ since $u = 0$ on ∂D. Similarly $(-bu_y, u) = 0$. It is easy to show that $(gu, u) \geqq 0$. Now $-\Delta u$ has a series of eigenvalues increasing from 2 toward $+\infty$. So by the spectral theorem, $(-\Delta u, u) \geqq 2\|u\|^2$. Hence $(-Eu, u) \geqq 2\|u\|^2$ so (A_4) is satisfied with $\mu = 2$.

Since $f \in L_2(D)$ and $\operatorname{Sin}(u)$ is a bounded function, then $Nu = f + \operatorname{Sin}(u)$ is a continuous map from $X = L_2(D)$ into X. Using the Mean Value Theorem it is easy to show that $(Nu - Nv, u - v) \geqq -\|u - v\|^2$ so N is quasimonotone with constant $\eta = 1$.

Since all the assumptions of Theorem 4 part (a) are satisfied, it follows that our problem has a unique solution for each $f \in L_2(D)$.

EXAMPLE 2. Let us now consider the partial differential equation

$$\Delta u + au_x + bu_y = f(x, y) + u^3/(1 + u^2) \tag{7}$$

for (x, y) in $D = [0, 2\pi] \times [0, 2\pi]$ where a and b are constants and $f \in L_2(D)$. We want solutions to equation (7) which are doubly periodic; i.e., u 2π-periodic in both x and y. We assume also that f is doubly periodic.

Let X be the Hilbert space of functions in $L_2(D)$ which are doubly periodic. Let $Eu = \Delta u + au_x + bu_y$ and let $D(E) = \{u \in X$: $Eu \in X\}$. Now E is a nonselfadjoint elliptic operator whose kernel consists of only the constant functions. Let P be the projection onto the constants and let $Q = P$. On $(\ker E)^\perp$, E has a bounded right inverse H. The adjoint of E is $E^*u = \Delta u - au_x - bu_y$ with $D(E^*) = D(E)$. It now follows from the Fredholm Alternative

Theorem that $D(H) = R(E) = (I - P)X = (I - Q)X$. It is easy to show that $(A_{1-3,5,6})$ are satisfied.

Let $u \in D(E)$, since u is doubly periodic, then

$$(-au_x, u) = (-a/4\pi^2)\int_0^{2\pi}\int_0^{2\pi} u_x(x, y)u(x, y)dxdy$$
$$= (-\alpha/8\pi^2)\int_0^{2\pi}\{u^2(x, y)\}_0^{2\pi}dy = 0 .$$

Similarly, $(-bu_y, u) = 0$. Now $(-\Delta u, u) = ||u_y||^2 + ||u_x||^2 \geqq 0$. Hence $(-Eu, u) \geqq 0$. Let $\alpha = 0$.

Now the operator $Lu = \Delta u D(L) = D(E)$ has eigenvalues $0, 1, \cdots$. Hence by the Spectral Theorem, $(Lu, u) \geqq ||u||^2$ for $u \in (I - P)X \cap D(L)$. Now since $(-au_x, u) = (-bu_y, u) = 0$ for all $u \in D(E)$, (A_4) is satisfied with $\mu = 1$.

Now $Nu = u^3/(1 + u^2) + f(x, y)$ is certainly a continuous map from X into X, and N maps bounded sets into bounded sets. Since the derivative of $t^3/(1 + t^2)$ is positive, it follows that N is a monotone map. In fact, N is coercive since for $u \in X$

$$(Nu, u) = (4\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi}\frac{u^4}{1 + u^2}dxdy + (4\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} fudxdy$$
$$\geqq (4\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} u^2dxdy - (4\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi}\frac{u^2}{1 + u^2}dxdy$$
$$- ||f|| \, ||u||$$
$$\geqq ||u||^2 - 1 - ||f|| \, ||u|| .$$

Thus all the conditions of Theorem 4 part (b) are satisfied with $\mu = 1$ and $\alpha = \eta = 0$ and hence equation (7) has at least one solution for each $f \in X$.

EXAMPLE 3. Finally we consider the partial differential equation

$$\Delta u + au_x + bu_y + \frac{1}{2}u = g(u) + F(x, y) \tag{8}$$

for (x, y) in $D = [0, 2\pi] \times [0, 2\pi]$. Again we are interested in doubly periodic solutions. We assume a and b are constants, $F \in L_2(D)$ is doubly periodic and

$$g(u) = \int_0^u \exp(1/(1 + t^2))dt .$$

Let X be the Hilbert space of functions in $L_2(D)$ which are doubly periodic. Let P be the projection onto the constants and let $Q = P$. With this choice of P and Q assumptions $(A_{5,6})$ are satisfied.

Let $Eu = \Delta u + au_x + bu_y + (1/2)u$ with $D(E) = \{u \in X: Eu \in X\}$. Now $E^*u = \Delta u - au_x - bu_y + (1/2)u$ and $D(E^*) = D(E)$. Consider

$Lu = (E - (1/2)I)u, D(L) = D(E)$. The adjoint of L is $L^*u = (E^* - (1/2)I)u$ with $D(L^*) = D(E^*) = D(E)$. The operator L is the operator considered in Example 2 and it follows from our calculations in Example 2 that if we choose $\alpha = 1/2$ and $\mu = 1/2$ then assumption (A_4) is satisfied. Again from the calculations in Example 2, $\ker (E - (1/2)I) = \ker (E - (1/2)I) =$ constants and $R(E - (1/2)I) = R(E - (1/2)I) = (I - P)X$. We now have $(E - (1/2)I)P = Q(E - (1/2)I)$, and since $P = Q$, $EP - (1/2)P = QE - (1/2)Q$ or $EP = QE$. Hence assumption (A_2) is satisfied. Now on $(I - P)X$, E is bounded below and if E: $(I - P)X \rightarrow (I - P)X$ then E has a bounded linear inverse H and assumptions $(A_{1,3})$ will be satisfied. Again from our calculations in Example 2 we know $E - (1/2)I: (I - P)X \rightarrow (I - P)X$, which implies $E: (I - P)X \rightarrow (I - P)X$.

We now consider the nonlinear operator $Nu = g(u) + F(x, y)$. N is a continuous map from X into X and maps bounded sets into bounded sets. Since $1 \leqq g'(u) \leqq e$, N is strongly monotone with constant $\eta = 1$.

Thus the conditions of Theorem 4 part (b) are satisfied with $\mu = 1/2$, $\alpha = 1/2$, $\eta = 1$, and $||P|| = 1$, hence equation (8) has at least one doubly periodic solution for each $F(x, y)$ in X.

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