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# ON PRODUCT OF SHAPE AND A QUESTION OF SHER

Y. KODAMA

**In this paper the products in various shape categories are investigated. In the weak shape category defined by Borsuk, for arbitrary metrizable spaces $X$ and $Y$ there exists always the product $\mathrm{Sh}_W(X) \times \mathrm{Sh}_W(Y)$. In the shape category in the sense of Fox, if $X$ is a pointed FANR and $Y$ is an arbitrary metrizable space, there exists the product $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y)$ and the relation $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y) = \mathrm{Sh}_F(X \times Y)$ holds. In the proper shape category in the sense of Ball and Sher, the product does not exist generally. If $X$ is a compactum and $Y$ is a locally compact metrizable space, the proper shape of the product space $X \times Y$ is determined uniquely by the proper shapes of $X$ and $Y$.**

**1. Introduction.** The notion of shape was introduced by Borsuk [4] for compact metric spaces in 1968. The concept was extended to arbitrary metrizable spaces by Borsuk [5] and by Fox [10], to locally compact metrizable spaces by Ball and Sher [1]. These extensions form different shape categories to each other. We denote by $\mathscr{C}_S$ (resp. $\mathscr{C}_W$) the strong (resp. weak) shape category in the sense of Borsuk [6], by $\mathscr{C}_F$ the shape category in the sense of Fox [10] and by $\mathscr{C}_p$ the proper shape category in the sense of Ball and Sher [1]. The shapes of a metrizable space $X$ in $\mathscr{C}_S$, $\mathscr{C}_W$, $\mathscr{C}_F$ and $\mathscr{C}_p$ are denoted by $\mathrm{Sh}_S(X)$, $\mathrm{Sh}_W(X)$, $\mathrm{Sh}_F(X)$ and $\mathrm{Sh}_p(X)$ respectively.

The purpose of this paper is to investigate the existence of the products in the categories $\mathscr{C}_S$, $\mathscr{C}_W$, $\mathscr{C}_F$ and $\mathscr{C}_p$. The following was proved essentially by Borsuk [6].

(1) If $X$ and $Y$ are metrizable then the product $\mathrm{Sh}_W(X) \times \mathrm{Sh}_W(Y)$ in the category $C_W$ exists and it equals $\mathrm{Sh}_W(X \times Y)$.

We shall prove that

(2) If $X$ is a compact metric space which has the same shape as a compact ANR, then for every metrizable space $Y$ there exists the product $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y)$ in $\mathscr{C}_S$.

It is known that the equality $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y) = \mathrm{Sh}_S(X \times Y)$ does not generally hold.

(3) If $X$ is a pointed FANR in the sense of Borsuk [6], then for every metrizable space $Y$ there exists the product $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y)$ in $\mathscr{C}_F$.

(4) In the category $\mathscr{C}_p$ the product does not generally exist.

(5) If $X$ is a compactum and $Y$ is a locally compact metrizable space, then $\mathrm{Sh}_p(X \times Y)$ is determined uniquely by $\mathrm{Sh}_p(X)$ and $\mathrm{Sh}_p(Y)$.

The assertion (5) solves a question raised by R. B. Sher in the

Winter school "Shape theory and pro-homotopy" held in Dubrovnik, January, 1976.

Throughout this paper all spaces are metrizable and maps are continuous. AR and ANR mean those for metric spaces.

**2. Definition and notations.** A familiarity with the basic terminology and notations of Borsuk's shape theory for metrizable spaces [6], of Fox's shape theory for metrizable spaces [10] and of Ball-Sher's proper shape theory for locally compact metrizable spaces [1] is assumed. A number of other technical or specialized definitions and notations are given in this section.

We denote by $\mathscr{C}$ the shape category defined by Borsuk [6, Chap. VII] for compacta whose morphisms are the equivalence classes of fundamental sequences. Namely, we take for objects of $\mathscr{C}$ the class of all compacta, assign to each $X \in \mathscr{C}$ a subset $X'$ of the Hilbert cube $M$ homeomorphic to $X$, and define a morphism from $X$ to $Y$ in $\mathscr{C}$ to be an equivalence class of fundamental sequences from $X'$ to $Y'$ in $(M, M)$. We denote by $\mathscr{C}_S$ (resp. $\mathscr{C}_W$) the strong (resp. weak) shape category defined by Borsuk [6, Chap. III] for metrizable spaces whose morphisms are the equivalence classes of strong (resp. weak) fundamental sequences, by $\mathscr{C}_F$ the shape category defined by Fox [10] for metrizable spaces whose morphisms are the equivalence classes of mutations and by $\mathscr{C}_p$ the proper shape category of Ball and Sher [1] for locally compact metrizable spaces whose morphisms are the equivalence classes of proper fundamental nets. (cf. Ball [3, p. 17].) The category $\mathscr{C}$ is regarded as a full subcategory of each of the categories $\mathscr{C}_S, \mathscr{C}_W, \mathscr{C}_F$ and $\mathscr{C}_p$. The shapes of a space $X$ in $\mathscr{C}, \mathscr{C}_S, \mathscr{C}_W, \mathscr{C}_F$ and $\mathscr{C}_p$ are denoted by $\mathrm{Sh}(X)$, $\mathrm{Sh}_S(X)$, $\mathrm{Sh}_W(X)$, $\mathrm{Sh}_F(X)$ and $\mathrm{Sh}_p(X)$ respectively.

A space $X$ is said to be a *strong fundamental absolute retract* (SFAR) if, for every space $Y$ containing $X$ as a closed subset, $X$ is an $S$-retract of $Y$ (cf. Borsuk [6, Chap. VI, §2]). A spaces $X$ is said to be a *strong fundamental absolute neighborhood retract* (SFANR) if, for every space $Y$ containing $X$ as a closed set, $X$ is an $S$-retract of some neighborhood of $X$ in $Y$. For compacta, these concepts coincide with Borsuk's original one's, FAR and FANR (cf. [6, Chap. VIII]). Obviously every AR is an SFAR and every ANR is an SFANR.

Let $X_i$, $i = 1, 2$, be compacta. A *product* of $\mathrm{Sh}(X_1)$ and $\mathrm{Sh}(X_2)$ is a triple $(\mathrm{Sh}(X), \boldsymbol{p}_1, \boldsymbol{p}_2)$ where $X$ is a compactum and $\boldsymbol{p}_1: X \to X_1$, $\boldsymbol{p}_2: X \to X_2$ are fundamental sequences (called *projections*) with the property if $Z$ is any compactum (that is, an object of $\mathscr{C}$) and $\boldsymbol{f}_1: Z \to X_1$, $\boldsymbol{f}_2: Z \to X_2$ are arbitrary fundamental sequences there exists a unique fundamental sequence $\boldsymbol{f}: Z \to X$ (up to the equivalence class) such that $\boldsymbol{p}_1\boldsymbol{f} \cong \boldsymbol{f}_1$ and $\boldsymbol{p}_2\boldsymbol{f} \cong \boldsymbol{f}_2$. The product of $\mathrm{Sh}(X_1)$ and $\mathrm{Sh}(X_2)$ is denoted simply by $\mathrm{Sh}(X_1) \times \mathrm{Sh}(X_2)$ (suppressing projections). Products in the categories $\mathscr{C}_S$, $\mathscr{C}_W$, $\mathscr{C}_F$ and $\mathscr{C}_p$ are defined similarly. We denote $\mathrm{Sh}_S(X_1) \times \mathrm{Sh}_S(X_2)$, $\mathrm{Sh}_W(X_1) \times \mathrm{Sh}_W(X_2)$, $\mathrm{Sh}_F(X_1) \times \mathrm{Sh}_F(X_2)$ and $\mathrm{Sh}_p(X_1) \times \mathrm{Sh}_p(X_2)$ the products in $\mathscr{C}_S$, $\mathscr{C}_W$, $\mathscr{C}_F$ and $\mathscr{C}_p$ respectively.

It is known that in the category $\mathscr{C}$ there exists the product $\mathrm{Sh}(X) \times \mathrm{Sh}(Y)$ for arbitrary compacta $X$ and $Y$. Keesling [13] has proved that in the shape category of Mardešić and Segal [16] whose objects are compact Hausdorff spaces there exists the product for any family of objects; namely, if $\{X_\alpha : \alpha \in \Lambda\}$ is a family of compact Hausdorff spaces there exists the product $\prod_{\alpha \in \Lambda} \mathrm{Sh}_{MS}(X_\alpha)$ and the relation $\prod_{\alpha \in \Lambda} \mathrm{Sh}_{MS}(X_\alpha) = \mathrm{Sh}_{MS}(\prod_{\alpha \in \Lambda} X_\alpha)$ holds, where $\mathrm{Sh}_{MS}(X)$ means the shape of $X$ in the sense of Mardešić and Segal. Since $\mathrm{Sh}(X) = \mathrm{Sh}_{MS}(X)$ for a compact metric space $X$ (we consider only metrizable spaces), the theorem of Keesling means the existence of the product in the category $\mathscr{C}$.

Let $M$ be a space and $X$ a subspace of $M$. Throughout the paper, by $\boldsymbol{U}(X, M)$ we mean the set of all neighborhoods $X$ in $M$.

## 3. Products in the shape categories.

THEOREM 1. *Let $X$ be a compactum with the same shape as a compact* ANR *$K$. Then for every space $Y$ there exists the product $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y)$ in the category $\mathscr{C}_S$ and $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y) = \mathrm{Sh}_S(K \times Y)$.*

From the example given by Godlewski and Nowak [11, p. 391] it is known that in Theorem 1 the equality

$$\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y) = \mathrm{Sh}_S(X \times Y)$$

does not hold generally and also the compactness of $K$ is essential.

*Proof of Theorem* 1. Let $L, M, N$ be AR's containing $X, K, Y$ as closed sets respectively. Since $\mathrm{Sh}(X) = \mathrm{Sh}(K)$, there exist fundamental sequences $\boldsymbol{\alpha} = \{\alpha_k : K \to X\}_{M,L}$ and $\boldsymbol{\beta} = \{\beta_k : X \to K\}_{L,M}$ such that

$$\boldsymbol{\beta\alpha} \simeq \boldsymbol{i}_{K,M} \quad \text{and} \quad \boldsymbol{\alpha\beta} \simeq \boldsymbol{i}_{X,L}\,. \tag{3.1}$$

(See [6, Chap. III and VII] for notations.) Let $\pi: M \times N \to M$ and $\mu: M \times N \to N$ be the projections. Consider the $S$-sequences $\boldsymbol{p} = \boldsymbol{\alpha\pi}: K \times Y \to X$ and $\boldsymbol{\mu}: K \times Y \to Y$, where $\boldsymbol{\pi}: K \times Y \to K$ and $\boldsymbol{\mu}$ are generated by $\pi$ and $\mu$ respectively. We shall prove that the triple $(\mathrm{Sh}_S(K \times Y), \boldsymbol{p}, \boldsymbol{\mu})$ equals $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y)$. Let $Z$ be a space and $P$ be an AR containing $Z$ as a closed get. Suppose that

$$\boldsymbol{f} = \{f_k : Z \longrightarrow K \times Y\}_{P, M \times N} \text{ and } \boldsymbol{g} = \{g_k : Z \longrightarrow K \times Y\}_{P, M \times N}$$

are $S$-sequences such that $\boldsymbol{pf} \underset{S}{\simeq} \boldsymbol{pg}$ and $\boldsymbol{\mu f} \underset{S}{\simeq} \boldsymbol{\mu g}$. Let us prove $\boldsymbol{f} \underset{S}{\simeq} \boldsymbol{g}$. Take a map $r: M \to M$ such that for some $U \in \boldsymbol{U}(K, M)$ the restriction $r|U$ is a retraction of $U$ to $K$. For each $k$, consider the map $\varphi_k = r\pi f_k \times \mu f_k: P \to M \times N$. Since $K$ is compact, it is easy to know that the sequence $\{\varphi_k\}$ forms an $S$-sequence which is $S$-homotopic to $\boldsymbol{f}$. Since $K$ is an ANR and $\{f_k\}$ is an $S$-sequence, there exist a $V \in \boldsymbol{U}(Z, P)$ and a number $k_0$ such that $r\pi f_k|V \simeq r\pi f_{k'}|V$ in $K$ for $k, k' \geqq k_0$. Hence

$$\boldsymbol{f} \underset{S}{\simeq} \{r\pi f_{k_0} \times \mu f_k\} . \tag{3.2}$$

Similarly we know that there exist a $V' \in \boldsymbol{U}(Z, P)$ and a $k_1$ such that $r\pi g_k V' \simeq r\pi g_{k'} V'$ in $K$ for $k, k' \geqq k_1$. Hence

$$\boldsymbol{g} \underset{S}{\simeq} \{r\pi_{k_1} \times \mu g_k\} . \tag{3.3}$$

Since $\boldsymbol{pf} \underset{S}{\simeq} \boldsymbol{pg}$, there exists a $W \in \boldsymbol{U}(Z, P)$, $W \subset V \cap V'$, such that $r\pi f_{k_0}|W \simeq r\pi g_{k_1}|W$ in $K$. Thus we have

$$\{r\pi f_{k_0} \times \mu f_k\} \underset{S}{\simeq} \{r\pi g_{k_1} \times \mu f_k\} . \tag{3.4}$$

Since $\boldsymbol{\mu f} \underset{S}{\simeq} \boldsymbol{\mu g}$, it is proved that

$$\{r\pi g_{k_1} \times \mu f_k\} \underset{S}{\simeq} \{r\pi g_{k_1} \times \mu g_k\} . \tag{3.5}$$

By (3.2), (3, 3), (3.4) and (3.5) we have $\boldsymbol{f} \underset{S}{\simeq} \boldsymbol{g}$.

Finally, suppose that $S$-sequences $\boldsymbol{f} = \{f_k: Z \to X\}_{P,L}$ and $\boldsymbol{g} = \{g_k: Z \to Y\}_{P,N}$ are given. If we let $\boldsymbol{\beta f} = \{\varphi_k: Z \to K\}_{P,M}$ (cf. (3.1)), then it is proved by the same argument as in above that $\boldsymbol{\psi} = \{r\varphi_k \times g_k: Z \to K \times Y\}_{P,M\times N}$ forms an $S$-fequence such that $\boldsymbol{p\psi} \underset{S}{\simeq} \boldsymbol{f}$ and $\boldsymbol{\mu\psi} \underset{S}{\simeq} \boldsymbol{g}$. This completes the proof.

COROLLARY 1. *If $X$ is an* FAR, *then* $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y) = \mathrm{Sh}_S(Y)$ *for every space $Y$. If, in addition, $Y$ is an* SFAR, *then* $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y)$ *is an* SFAR.

For, an FAR has the same shape as one point space (Borsuk [6, p. 257]).

COROLLARY 2. *If a compactum $X$ has the same shape as a compact* ANR *$K$ and $Y$ is an* SFANR, *then* $\mathrm{Sh}_S(X) \times \mathrm{Sh}_S(Y) = \mathrm{Sh}_S(K \times Y) \in \mathrm{SFANR}$.

This is proved by the same argument as the proof of Theorem 1 and we omit the proof.

EXAMPLE 1. Let $X$ be the Warsaw circle. Then $X$ is an FANR but not an ANR. Let $J$ be the set of positive integers. The space $X \times J$ is not an SFANR. This is proved by the same argument as Godlewski and Nowak [11]. Imbedd $X \times J$ in an AR as a closed set. Suppose that there exist a $V \in U(X \times J, M)$ and an $S$-retraction $r = \{r_k: V \to X \times J\}_{M,M}$. We can assume that $V = \bigcup_{k \in J} V_k$, $V_k \supset X \times \{k\}$, $k \in J$, and $\{V_k\}$ is discrete. Since $X$ is not an ANR, for each $k \in J$ there exists a point $x_k \in V_k \cap r_k(V_k) - X \times \{k\}$. The set $W = M - \bigcap \{x_k: k \in J\}$ is a neighborhood of $X \times J$ in $M$. However, $r_k(V) \not\subset W$ for each $k \in J$. Hence $r$ is not an $S$-retraction.

In the category $\mathscr{C}_W$ the product exists always as shown in the following.

THEOREM 2 (Borsuk). *Let $X$ and $Y$ be arbitrary spaces. Then in the category $\mathscr{C}_W$ the product* $\mathrm{Sh}_W(X) \times \mathrm{Sh}_W(Y)$ *exists and* $\mathrm{Sh}_W(X) \times \mathrm{Sh}_W(Y) = \mathrm{Sh}_W(X \times Y)$.

This is proved essentially by Borsuk. The theorem is obvious from the following lemma.

LEMMA 1. *Let $X$, $Y$ and $Z$ be spaces and let $M$, $N$ and $L$ be* AR's *containing $X$, $Y$ and $Z$ as closed sets respectively. If $\boldsymbol{f} = \{f_k: Z \to X\}_{L,M}$ and $\boldsymbol{g} = \{g_k: Z \to Y\}_{L,N}$ are $W$-sequences, then $\boldsymbol{f} \times \boldsymbol{g} = \{f_k \times g_k: Z \to X \times Y\}_{L,M \times N}$ is a $W$-sequence and the $W$-class* $[\boldsymbol{f} \times \boldsymbol{g}]$ is *determined uniquely by* $[\boldsymbol{f}]$ *and* $[\boldsymbol{g}]$.

The proof follows from the definitions of $W$-sequences and $W$-classes. (cf. Borsuk [6, Chap. IV(7.1)].)

EXAMPLE 2. Let $Y$ be the subset of the plane as follows:

$$Y = \{(x, y): x = 0, -1 < y < 1\} \cup \{(x, y): y = \sin(\pi/2x), 0 < x \leqq 1\}.$$

Let $X = \{p\}$ be one point space. Consider the maps $f, g: X \to Y$ such that $f(p) = (0, 0)$ and $g(p) = (1, 1)$. Then $f$ and $g$ induce non equivalent morphisms in the category $\mathscr{C}_S$ or $\mathscr{C}_W$, and induce equivalent morphisms in $\mathscr{C}_F$ or $\mathscr{C}_p$. Next, let $Z = \{(x, y): x = 0, 1, 2, \cdots, -1 \leqq y \leqq 1\} \cup \{(x, y): y = \sin \pi/2(x - n), n < x \leqq n + 1, n = 0, 1, \cdots\}$, $J = \{1, 2, 3, \cdots\}$ and $R =$ the real line. Then $\mathrm{Sh}_S(J) = \mathrm{Sh}_S(R \times J) \neq \mathrm{Sh}_S(Z \times J)$, $\mathrm{Sh}_W(J) = \mathrm{Sh}_W(R \times J) = \mathrm{Sh}_W(Z \times J)$, $\mathrm{Sh}_F(J) = \mathrm{Sh}_F(R \times J) = \mathrm{Sh}_F(Z \times J)$ and $\mathrm{Sh}_p(J) \neq \mathrm{Sh}_p(R \times J) = \mathrm{Sh}_p(Z \times J)$.

Next, we shall consider the product in the category $\mathscr{C}_F$. A compactum $X$ is called a *pointed* FANR (cf. [6, pp. 254-255]) if for each $x \in X$ $(X, x)$ is shape dominated by a pointed finite $CW$ complex. Our theorem is

THEOREM 3. *If $X$ is a pointed* FANR, *then for every space $Y$ the product* $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y)$ *in* $\mathscr{C}_F$ *exists and* $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y) = \mathrm{Sh}_F(X \times Y)$.

For the proof we need a couple of lemmas. The following lemma gives a useful characterization of a pointed FANR which is proved by Siebenmann, Gillou and Hähl [20, Théorème 5.8] and Edwards and Geoghegan [9, Theorem 1.1].

LEMMA 2. *Let $X$ be a pointed* FANR *lying in the Hilbert space $M$. Then there exist a decreasing sequence $\{M_k: k = 0, 1, 2, \cdots\}$ of neighborhood of $X$ in $M$ and a map $\varphi: M \times [0, \infty) \to M$ satisfying the following conditions.*

(3.6) *For each $r \in [0, \infty)$ the map $\tau: M \to M$ defined by $\tau(x) = \varphi(x, r)$, $x \in M$, is a homeomorphism onto.*

(3.7) *$\{M_k\}$ forms a neighborhood basis of $X$ in $M$.*

(3.8) $\varphi(x, r) = x$, $(x, r) \in M \times \{0\} \cup (M - M_0) \times [0, \infty) \cup \bigcup_{k=1}^{\infty} M_{k+2} \times [0, k]$.

(3.9) $\varphi(M_k \times [k + n, \infty)) \subset M_{k+n-1}$, $k = 1, 2, \cdots$, $n = -1, 0, 1, 2, \cdots$, *where* $M_{-1} = M_0$.

(3.10) *For every point $(x, r) \in M_0 \times [0, \infty)$, if $\varphi(x, r) \in M_k$ for some $k$, $\varphi(x, r') \in M_{k-2}$ for each $r' \in [r, \infty)$, where $M_{-2} = M_{-1} = M_0$.*

The proof is easily obtained by making use of [20, Théorème 5.8] and by a simple induction.

REMARK 1. Dydak, Nowak and Strok [8, Lemma 4] have proved that if we replace the Hilbert space by an arbitrary *AR* in Lemma 2 then there exist a decreasing sequence $\{M_k\}$ of neighborhoods and a map $\varphi$ satisfying (3.7), (3.8) and (3.9). If $M$ is an AR containing $X$ and $X$ is unstable in $M$ in the sense of Sher [18, p. 346], then it is proved that there exist $\{M_k\}$ satisfying (3.7), (3.8), (3.9) and (3.10). As seen in the proof of Theorem 3, the condition (3.6) is not necessary to prove it. Hence it is enough for us to assume that $X$ is embedded unstably in an AR $M$.

From now on we assume that $X$ is a pointed FANR lying in the Hilbert space $M$ and $Y$ is a closed set of an AR $N$. We denote by $\{M_k, k = 0, 1, 2, \cdots\}$ a decreasing sequence of neighborhoods of $X$ in $M$ and by $\varphi$ a map from $M \times [0, \infty)$ to $M$ satisfying the conditions (3.6)-(3.10).

A neighborhood $W \in \boldsymbol{U}(X \times Y, M \times N)$ is said to be *basic* if there exist a closed neighborhood $V \in \boldsymbol{U}(Y, N)$, an open covering $\{V_\alpha: \alpha \in \Lambda\}$ of $V$ and a collection $\{U_\alpha: U_\alpha \in \boldsymbol{U}(X, M), \alpha \in \Lambda\}$ such that $W = \bigcup_{\alpha \in \Lambda} U_\alpha \times V_\alpha$. The closed neighborhood $V$ is said to be a *base* of $W$ and denoted by $B(W)$.

From the compactness of $X$ the following lemma is obvious.

LEMMA 3. *The set of basic neighborhoods forms a cofinal subsystem of* $\boldsymbol{U}(X \times Y, M \times N)$.

LEMMA 4. *For every basic neighborhood* $W \in \boldsymbol{U}(X \times Y, M \times N)$ *there exists a map* $\beta: B(W) \to [0, \infty)$ *such that*

(3.11) *if* $\tilde{\varphi}: M_1 \times B(W) \to M \times B(W)$ *is defined by* $\tilde{\varphi}(x, y) = (\varphi(x, \beta(y)), y)$ *for* $(x, y) \in M_1 \times B(W)$, *then* $\tilde{\varphi}(M_1 \times B(W)) \subset W$,

(3.12) *for every* $(x, y) \in M_1 \times B(W)$ *and every* $r \in [\beta(y), \infty)$, $(\varphi(x, r), y) \in W$.

*Here* $M_1$ *is the neighborhood of* $X$ *in* $M$ *described in above and* $B(W)$ *is the base of* $W$.

*Proof*. There exists an open cover $\{V_\alpha: \alpha \in \Lambda\}$ of $B(W)$ and a collection $\{U_\alpha: U_\alpha \in \boldsymbol{U}(X, M), \alpha \in \Lambda\}$ such that $W = \bigcup_{\alpha \in \Lambda} U_\alpha \times V_\alpha$. By the paracompactness of $B(W)$ and (3.7), we can find a locally finite open cover $\{V'_\lambda: \lambda \in \Omega\}$ of $B(W)$ and a map $i: \Omega \to \{1, 2, 3, \cdots\}$ such that $\{V'_\lambda\}$ refines $\{V_\alpha\}$ and $\bigcup_{\lambda \in \Omega} M_{i(\lambda)} \times V'_\lambda \subset W$. Construct a map $\beta: B(W) \to [0, \infty)$ such that $\beta(y) \geqq i(\lambda) + 1$ for each $y \in V'_\lambda$. To prove that $\beta$ satisfies the lemma, let $(x, y) \in M_1 \times B(W)$ and $r \geqq \beta(y)$. If $y \in V'_\lambda$, then $r \geqq \beta(y) \geqq i(\lambda) + 1$. By the property (3.9) of $\varphi$,

$$\varphi(M_1 \times [i(\lambda) + 1, \infty)) \subset M_{i(\lambda)} .$$

Since $(x, r) \in M_1 \times [i(\lambda) + 1, \infty)$, we have $(\varphi(x, r), y) \in M_{i(\lambda)} \times V'_\lambda \subset W$. This completes the proof.

*Proof of Theorem* 3. Let $\pi: M \times N \to M$ and $\mu: M \times N \to N$ be the projections. We shall prove that the triple $(\mathrm{Sh}_F(X \times Y), \boldsymbol{\pi}, \boldsymbol{\mu})$ is the product to $\mathrm{Sh}_F(X)$ and $\mathrm{Sh}_F(X)$ where $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ mean the mutations generated by $\pi$ and $\mu$ respectively.

Let $Z$ be a space and let $L$ be an AR containing $Z$ as a closed set. We must prove: Suppose that mutations $\boldsymbol{f}: \boldsymbol{U}(Z, L) \to \boldsymbol{U}(X, M)$ and $\boldsymbol{g}: \boldsymbol{U}(Z, L) \to \boldsymbol{U}(Y, N)$ are given. Then there exists a mutation $\boldsymbol{h}: \boldsymbol{U}(Z, L) \to \boldsymbol{U}(X \times Y, M \times N)$ satisfying the following conditions.

(3.13) $$\pi \boldsymbol{h} \simeq \boldsymbol{f} \quad \text{and} \quad \mu \boldsymbol{h} \simeq \boldsymbol{g} .$$

(3.14) If $\boldsymbol{h}': \boldsymbol{U}(Z, L) \to \boldsymbol{U}(X \times Y, M \times N)$ is a mutation such that $\pi \boldsymbol{h}' \simeq \boldsymbol{f}$ and $\mu \boldsymbol{h}' \simeq \boldsymbol{g}$, then $\boldsymbol{h} \simeq \boldsymbol{h}'$ .

Without loss of generality (by taking a cofinal subsystem of $\{M_i\}$ if necessary), we can assume that the mutation $\boldsymbol{f}$ has the following property.

(3.15) Let $f: U \to V$ be any map in $\boldsymbol{f}$. Then there exists an $M_i$, $i \geqq 3$, such that $f(U) \subset M_{i+2} \subset M_i \subset V$ .

First, let us construct a mutation $\boldsymbol{h}$. Define $\boldsymbol{h}$ as the set of all maps $h$ satisfying the conditions:

(3.16) $h: U \longrightarrow W'$, $U \in \boldsymbol{U}(Z, L)$, $W' \in \boldsymbol{U}(X \times Y, M \times N)$ ;

(3.17) there exists a basic neighborhood $W \in \boldsymbol{U}(X \times Y, M \times N)$ such that $h(U) \subset W \subset W'$ ;

(3.18) there exist $f \in \boldsymbol{f}$, $g \in \boldsymbol{g}$, domain of $f$ = domain of $g$, and $\alpha: U \to [0, \infty)$ such that range of $f \subset M_3$ and range of $g \subset B(W)$ ;

(3.19) $$h(x) = (\varphi(f(x), \alpha(x)), g(x)), \ x \in U ;$$

(3.20) for every $x \in U$ and for every $r \in [\alpha(x), \infty)$ $(\varphi(f(x), r), g(x)) \in W$ ;

(3.21) if $\pi h(x) \in M_{i+2}$ for $x \in U$ then $\varphi(f(x), r) \in M_i$ for every $r \in [\alpha(x), \infty)$ .

Let us prove that $\boldsymbol{h}$ forms a mutation. We have to prove that $\boldsymbol{h}$ satisfies the conditions (2.1), (2.2) and (2.3) of [10, p. 49]. That $\boldsymbol{h}$ satisfies (2.1) is obvious. Let $W' \in \boldsymbol{U}(X \times Y, M \times N)$. We must find a member $h$ of $\boldsymbol{h}$ whose range is $W'$. Take a basic neighborhood $W$ such that $W \subset W'$ (cf. Lemma 3). By Lemma 4 there exists a map $\beta: B(W) \to [0, \infty)$ satisfying (3.11) and (3.12). Choose maps $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ such that $f: U \to M_3$ and $g: U \to B(W)$, where $U \in \boldsymbol{U}(Z, L)$. Set $\alpha = \beta g: U \to [0, \infty)$ and consider the map $h$ defined on the set $U$ as follows:

$$h(x) = (\varphi(f(x), \alpha(x)), g(x)) , \quad x \in U .$$

By (3.11) and (3.12) $h(U) \subset W$, and for each $x \in U$ and each $r \geqq \alpha(x)$ $(\varphi(f(x), r), g(x)) \in W$. Moreover (3.21) is satisfied by the property (3.10) of $\varphi$. Hence $h: U \to W$ belongs to $\boldsymbol{h}$. Therefore $\boldsymbol{h}$ satisfies (2.2) of [10].

Finally, let $h_0, h_1: U \to W'$ be any maps in $\boldsymbol{h}$. By the definition of $\boldsymbol{h}$ there exist $f_i \in \boldsymbol{f}$, $g_i \in \boldsymbol{g}$, basic neighborhoods $W_i$ and $\alpha_i: U \to [0, \infty)$ for $i = 0, 1$ satisfying (3.16)-(3.21). Since $\boldsymbol{f}$ and $\boldsymbol{g}$ are mutations, there exist a $U' \in \boldsymbol{U}(Z, L)$, homotopies $\xi: U' \times I \to M_3$ and $\eta: U' \times I \to B(W_0) \cup B(W_1)$ such that $\xi(x, i) = f_i(x)$ and $\eta(x, i) = g_i(x)$ for $x \in U'$ and $i = 0, 1$. By Lemma 4 we can find a map $\beta: B(W_0) \cup B(W_1) \to [0, \infty)$ such that if $\tilde{\varphi}: M_3 \times (B(W_0) \cup B(W_1)) \to M \times (B(W_0) \cup B(W_1))$ is defined by $\tilde{\varphi}(x, y) = (\varphi(x, \beta(y)), y)$ for $(x, y) \in M_3 \times (B(W_0) \cup B(W_1))$, then

$$\tilde{\varphi}(M_3) \times (B(W_0) \cup B(W_1)) \subset W', \text{ for each } (x, y) \in M_3 \times (B(W_0) \cup B(W_1)) \text{ and for each } r \in [\beta(y), \infty), (\varphi(x, r), y) \in W' . \tag{3.22}$$

Define a homotopy $H_2: U' \times I \to M \times N$ by

$$H_2(x, t) = (\varphi(\xi(x, t), \beta(\eta(x, t))), \eta(x, t)), (x, t) \in U' \times I .$$

By (3.22) $H_2(U' \times I) \subset W'$. We have

$$H_2(x, i) = (\varphi(f_i(x), \beta g_i(x)), g_i(x)), x \in U', \text{and for each } x \in U' \text{ and for each } r \in [\beta g_i(x), \infty), (\varphi(f_i(x), r), g_i(x)) \in W', i = 0, 1 . \tag{3.23}$$

Let $H_i$, $i = 0, 1$, be the homotopies defined on the set $U' \times I$ as follows.

$$H_i(x, t) = (\varphi(f_i(x), t\alpha_i(x) + (1-t)\beta g_i(x)), g_i(x))(x, t) \in U' \times I \text{ and } i = 0, 1 . \tag{3.24}$$

Then, by (3.24) and (3.19)

$$\begin{aligned} H_i(x, 0) &= (\varphi(f_i(x), \beta g_i(x)), g_i(x)) , \\ H_i(x, 1) &= h_i(x) , \end{aligned} \qquad x \in U' \text{ and } i = 0, 1 . \tag{3.25}$$

Since $t\alpha_i(x) + (1-t)\beta g_i(x) \geqq \min\{\alpha_i(x), \beta g_i(x)\}$ for each $x \in U'$, by (3.20) and (3.22)

$$H_i(U' \times I) \subset W' .$$

By (3.23) and (3.25) we have $h_0|U' \simeq h_1|U'$ in $W'$. Thus $\boldsymbol{h}$ forms a mutation.

It remains to show that $\boldsymbol{h}$ satisfies (3.13) and (3.14). By (3.19), since $\mu h = g|U$ for each $h \in \boldsymbol{h}$, obviously $\boldsymbol{\mu h} \simeq \boldsymbol{g}$. Let us show that $\boldsymbol{\pi h} \simeq \boldsymbol{f}$. Suppose $h \in \boldsymbol{h}$, $f \in \boldsymbol{f}$ and $\pi h, f: U \to V$, where $U \in \boldsymbol{U}(Z, L)$ and $V \in \boldsymbol{U}(X, M)$. By the definition of $\boldsymbol{h}$ there exist $f' \in \boldsymbol{f}$ and $\alpha: U \to [0, \infty)$ such that the range of $f' \subset M_3$ and $\pi h(x) = \varphi(f'(x), \alpha(x))$ for $x \in U$. Since $\boldsymbol{f}$ is a mutation, there is a $U' \in \boldsymbol{U}(Z, L)$ and a homotopy

$\xi: U' \times I \to M_3$ such that $\xi(x, 0) = f(x)$ and $\xi(x, 1) = f'(x)$ for $x \in U'$. By (3.15) we can find $M_i$ such that $f(U') \subset M_{i+2} \subset M_i \subset V$. Take a map $\beta: U' \to [0, \infty)$ such that $\beta(x) \geqq \alpha(x)$ and $\varphi(\xi(x, t), \beta(x)) \in V$ for each $x \in U'$. Define homotopies $H, H'$ and $H''$ on the set $U'$ by

$$\begin{aligned} H(x, t) &= \varphi(\xi(x, t), \beta(x)) \\ H'(x, t) &= \varphi(f(x), t\beta(x)) , \qquad\qquad x \in U' . \\ H''(x, t) &= \varphi(f'(x), t\alpha(x) + (1-t)\beta(x)) \end{aligned}$$

Obviously $H(U' \times I) \subset V$. Also, by (3.10) and (3.21) we have $H'(U' \times I) \cup H''(U' \times I) \subset M_i \subset V$. Since $\pi h(x) = H(x, 1)$ and $f(x) = H'(x, 0)$ for $x \in U'$, $\pi h|U' \simeq f|U'$ in $V$. Thus $\boldsymbol{h}$ satisfies (3.13).

Finally, to prove (3.14), let $h \in \boldsymbol{h}$, $h' \in \boldsymbol{h}'$ and $h, h': U \to W$, where $U \in \boldsymbol{U}(Z, L)$ and $W \in \boldsymbol{U}(X \times Y, M \times N)$. By the definition of $\boldsymbol{h}$, there exist $f \in \boldsymbol{f}$, $g \in \boldsymbol{g}$ and $\alpha: U \to [0, \infty)$ such that $h(x) = (\varphi(f(x), \alpha(x)), g(x)) \in W$ for $x \in U$. Without loss of generality we can assume that

(3.26) (range of $\pi h$) $\cup$ (range of $\pi h'$) $\subset M_3$ and there exist a collection $\{V_\alpha: \alpha \in \Lambda\}$ of open subsets of $N$ and a map $i: \Lambda \to \{3, 4, 5, \cdots\}$ such that (range of $\mu h$) $\cup$ (range of $\mu h'$) $\subset V = \bigcup_{\alpha \in \Lambda} V_\alpha$ and $h(U) \cup h'(U) \subset \bigcup_{\alpha \in \Lambda} M_{i(\alpha)+2} \times V_\alpha \subset \bigcup_{\alpha \in \Lambda} M_{i(\alpha)} \times V_\alpha \subset W$.

Since $\pi\boldsymbol{h} \simeq \boldsymbol{f} \simeq \pi\boldsymbol{h}'$ and $\mu\boldsymbol{h} \simeq \boldsymbol{g} \simeq \mu\boldsymbol{h}'$, there exists a $U' \in \boldsymbol{U}(Z, L)$ such that $\pi h|U' \simeq \pi h'|U'$ in $M_3$ and $\mu h|U' \simeq \mu h'|U'$ in $V$. Let $\xi: U' \times I \to M_3 \times V$ be a homotopy such that $\xi(x, 0) = (\pi h(x), \mu h(x))$ and $\xi(x, 1) = (\pi h'(x), \mu h'(x)) = h'(x)$ for $x \in U'$. Choose a map $\beta: U' \to [0, \infty)$ such that if a homotopy $H$ is defined on the set $U' \times I$ by

$$H(x, t) = (\varphi(\pi\xi(x, t), \beta(x)), \mu\xi(x, t)) , \quad (x, t) \in U' \times I ,$$

then $H(U' \times I) \subset \bigcup_{\alpha \in \Lambda} M_{i(\alpha)+2} \times V_\alpha$. We have

$$\begin{aligned} H(x, 0) &= (\varphi(\pi h(x), \beta(x)), \mu h(x)) \\ H(x, 1) &= (\varphi(\pi h'(x), \beta(x)), \mu h'(x)) , \end{aligned} \qquad x \in U' . \tag{3.27}$$

Define homotopies $H'$ and $H''$ on $U' \times I$ by

$$H'(x, t) = (\varphi(\pi h(x), t\beta(x)), \mu h(x)) , \quad (x, t) \in U' \times I ,$$

and

$$H''(x, t) = (\varphi(\pi h'(x), t\beta(x)), \mu h'(x)) , \quad (x, t) \in U' \times I .$$

By (3.26) and (3.10) $H'(U' \times I) \cup H''(U' \times I) \subset \bigcup_{\alpha \in \Lambda} M_{i(\alpha)} \times V_\alpha \subset W$. Moreover

$$\begin{aligned}&H'(x, 0) = (\varphi(\pi h(x), 0), \mu h(x)) = h(x), H'(x, 1) = (\varphi(\pi h(x), \\ (3.28)\quad &\beta(x)), \mu h(x)), H''(x, 0) = (\varphi(\pi h'(x), 0), \mu h'(x)) = h'(x) \text{ and} \\ &H''(x, 1) = (\varphi(\pi h'(x), \beta(x)), \mu h'(x)) \text{ for } x \in U' .\end{aligned}$$

By (3.27) and (3.28) we have $h|U \simeq h'|U$. This completes the proof of Theorem 3.

COROLLARY 3. *Let $X$ be a pointed* FANR (resp. FAR) *and let $Y$ be an* MANR (resp. MAR) *in the sense of Godlewski* [12]. *Then the product* $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y)$ *exists and it is an* MANR (resp. MAR).

This follows from Theorem 3 and [16, Theorem 2].

*Problem.* If $X$ is a movable compactum in the sense of Borsuk [6, Chap. V], for every metrizable space $Y$ does there exist the product $\mathrm{Sh}_F(X) \times \mathrm{Sh}_F(Y)$?

REMARK 2. Theorem 3 is strengthened as follows. By $\mathscr{C}_M$ denote the shape category in the sense of Mardešić [17] whose objects are topological spaces and morphisms are shapings. Then it is proved:

THEOREM 3′. *If $X$ is a pointed* FANR, *then for every paracompact space $Y$ the product* $\mathrm{Sh}_M(X) \times \mathrm{Sh}_M(Y)$ *in $\mathscr{C}_M$ exists and* $\mathrm{Sh}_M(X) \times \mathrm{Sh}_M(Y) = \mathrm{Sh}_M(X \times Y)$, *where* $\mathrm{Sh}_M(Z)$ *is the shape of $Z$ in the sense of Mardešić* [17].

Let us sketch the proof. We claim that the triple $(\mathrm{Sh}_M(X\times Y), \pi, \mu)$ is the produt of $\mathrm{Sh}_M(X)$ and $\mathrm{Sh}_M(Y)$, where $\pi$ and $\mu$ are the shapings of $X \times Y$ to $X$ and $Y$ generated by the projections $\pi\colon X \times Y \to X$ and $\mu\colon X \times Y \to Y$ respectively. (See for the notations Mardešić [17].) Let $Z$ be a topological space. Suppose that any shapings $\boldsymbol{f}\colon Z \to X$ and $\boldsymbol{g}\colon Z \to Y$ are given. We must find a unique shaping $\boldsymbol{h}\colon Z \to X \times Y$ such that $\boldsymbol{\pi h} = \boldsymbol{f}$ and $\boldsymbol{\mu h} = \boldsymbol{g}$. Let $P$ be a simplicial complex with metric topology and let $s\colon X \times Y \to P$ be a map. Since $X$ is compact and $Y$ is paracompact, there exist a locally finite open cover $V$ of $Y$ and a map $\xi\colon X \times K_V \to P$ such that $\xi \circ (1_X \times \psi) \simeq s$, where $K_V$ is the nerve of $V$ and $\psi\colon Y \to K_V$ is a canonical map. Consider $X$ as a subset of the Hilbert space $M$. We use the same notations as in the proof of Theorem 3. Since $P$ is an ANR, there exist a neighborhood $W$ of $X \times K_V$ in $M \times K_V$ and an extension $\tilde{\xi}\colon W \to P$ of $\xi$. Let $i\colon X \to M_3$ be the inclusion. Consider the maps $f(i)\colon Z \to M_3$ and $g(\psi)\colon Z \to K_V$. (We assume that $M_3$ is an ANR.) There exists a map $\alpha\colon K_V \to [0, \infty)$ such that if $\eta\colon Z \to M_3 \times K_V$ is defined by $\eta(z) = (\varphi(f(i)(z), \alpha(g(\psi)(z))), g(\psi)(z))$, $z \in Z$, then $\eta(Z) \subset W$.

Let $h(s) = \tilde{\xi}\eta\colon Z \to P$. For a given map $s\colon X \times Y \to P$, we have defined a map $h(s)\colon Z \to P$. The correspondence $\boldsymbol{h}$ is the required shaping: $Z \to X \times Y$. That $\boldsymbol{h}$ satisfies $\boldsymbol{\pi h} = \boldsymbol{f}$ and $\boldsymbol{\mu h} = \boldsymbol{g}$ and the uniqueness of $\boldsymbol{h}$ are proved by the same argument as the proof of Theorem 3.

In the remainder of this section we discuss the product in the category $\mathscr{C}_p$. We assume that all spaces are separable. As shown by Ball and Sher [1, 5], this does not loose the generality.

THEOREM 4. *Let $X$ be a 0-dimensional locally compact space and let $Y$ be a locally compact space with the same proper shape as a locally compact* AR. *If the product* $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ *exists, then* $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y) = \mathrm{Sh}_p(X)$.

For the proof we need a couple of lemmas. We let $H = M - \{w\}$, where $M$ is the Hilbert cube and $w$ is a point of $M$.

LEMMA 5. *Let $X$ and $Z$ be locally compact spaces contained in $H$ as closed sets. If $X$ is 0-dimensional, then every proper fundamental net $(\boldsymbol{f}, Z, X)$, where $\boldsymbol{f} = \{f_\lambda\colon \lambda \in \Lambda\}$, $f_\lambda\colon H \to H$, is generated by a proper map $f\colon Z \to X$. Moreover $f$ is uniquely determined by the proper fundamental class $[\boldsymbol{f}]$.*

(See Ball and Sher [1, 3] for notations and definitions.)

*Proof.* Since $\dim X = 0$, there exist collections $\{F_\alpha\colon \alpha \in \Omega_i\}$, $i = 1, 2, \cdots$, such that for each $i = 1, 2, \cdots$,

(3.29) the set $F_i = \bigcup \{F_\alpha\colon \alpha \in \Omega_i\}$ is a neighborhood of $X$ in $H$,

(3.30) if $\alpha \in \Omega_i$, then $F_\alpha$ is compact, $F_\alpha \cap X \neq \emptyset$ and the diameter of $F_\alpha < 1/i$,

(3.31) $$\{F_\alpha\colon \alpha \in \Omega_i\} \text{ is discrete in } H,$$

(3.32) $$\{F_\beta\colon \beta \in \Omega_{i+1}\} \text{ refines } \{F_\alpha\colon \alpha \in \Omega_i\}.$$

Since $\boldsymbol{f}$ is a proper fundamental net, there exists a sequence $\{\lambda_i\colon i = 1, 2, \cdots\}$ of elements of $\Lambda$ such that

(3.33) $$\lambda_i < \lambda_{i+1},\ i = 1, 2, \cdots,$$

(3.34) $$\text{if } \lambda, \mu \geqq \lambda_i,\ \lambda, \mu \in \Lambda, \text{ then } f_\lambda | Z \underset{p}{\simeq} f_\mu | Z \text{ in } F_i,\quad i = 1, 2, \cdots.$$

Take a point $x \in Z$. By (3.31), for each $i$ there exists a unique $\alpha_i \in \Omega_i$, such that $f_{\lambda_i}(x) \in F_{\alpha_i}$. By (3.33) and (3.34) $F_{\alpha_i} \supset F_{\alpha_{i+1}}$ for each $i$. If we put $f(x) = \bigcap_{i=1}^{\infty} F_{\alpha}$, then by (3.29) and (3.30) $f(x)$ consists of exactly one point of $X$. Consider the map $f: Z \to X$. Obviously $f$ is continuous. Let $x \in X$. Take $F_{\alpha_i}$, $\alpha_i \in \Omega_i$, $i = 1, 2, \cdots$, such that $x \in F_{\alpha_i}$. Since $f^{-1}(F_{\alpha_i}) \subset f_{\lambda_i}^{-1}(F_{\alpha_i})$, we have

$$f^{-1}(x) = \bigcap_{i=1}^{\infty} f_{\lambda_i}^{-1}(F_{\alpha_i}) \cap Z .$$

This means $f^{-1}(x)$ is compact. Finally, let $F$ be a closed set of $Z$. Set $H_i = \bigcup \{F_{\alpha_i}: F_{\alpha_i} \cap f_{\lambda_i}(F) \neq \emptyset, \alpha_i \in \Omega_i\}$, $i = 1, 2, \cdots$. Since $f_{\lambda_i}$ is closed, $H_i$ is closed by (3.30) and (3.31). From the equality $f(F) = \bigcap_{i=1}^{\infty} H_i$ the closedness of $f$ follows. Thus $f$ is a proper map. It is obvious that $f$ generates $\boldsymbol{f} = \{f_\lambda\}$ and is uniquely determined by the proper fundamental class $[\boldsymbol{f}]$. This completes the proof.

The following lemma is proved by the same argument as the proof of [14, Theorem 2] and we omit the proof.

LEMMA 6. *Let $X$ and $Y$ be locally compact spaces. If there exists a proper onto map $f: X \to Y$ such that for each $y \in Y$ $f^{-1}(y)$ has a trivial shape and* $\dim Y < \infty$, *then* $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(Y)$.

*Proof of Theorem* 4. We consider $X$ and $Y$ as closed subsets in $H$. Suppose that $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ exists. There exist a locally compact space $Z$ contained in $H$ as a closed set, proper fundamental nets $\boldsymbol{f}: Z \to X$ and $\boldsymbol{g}: Z \to Y$ such that the triple $(\mathrm{Sh}_p(Z), \boldsymbol{f}, \boldsymbol{g})$ is the product of $\mathrm{Sh}_p(X)$ and $\mathrm{Sh}_p(Y)$. By Lemma 5, there is a proper map $f: Z \to X$ generating $\boldsymbol{f}$. Note $f$ is onto, because $\mathrm{Sh}_p(Z) = \mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ and $\dim X = 0$. Put $Z_x = f^{-1}(x)$, $x \in X$. Since $f$ is a closed map and $\dim X = 0$, $Z_x$ has a neighborhood basis in $Z$ consisting of open and closed sets in $Z$. Consider two proper fundamental nets $\boldsymbol{h}_1$ and $\boldsymbol{h}_2: Z_x \to Z$ such that $\boldsymbol{h}_1$ is generated by the inclusion map: $Z_x \subset Z$ and $\boldsymbol{h}_2$ is generated by a constant map of $Z_x$ to a point of $Z_x \subset Z$. Since $Y$ has the same proper shape as a locally compact AR, by the definition of the product, we have $\boldsymbol{h}_1 \underset{p}{\simeq} \boldsymbol{h}_2$. This means that $Z_x$ has a trivial shape. By Lemma 6, $\mathrm{Sh}_p(Z) = \mathrm{Sh}_p(X)$. This completes the proof.

EXAMPLE 3. Let $X$ be a 0-dimensional locally compact and non compact space and let $Y \in \mathrm{SUV}^\infty$. (See Sher [18, p. 349].)

(3.35) If $Y$ has the end $E(Y)$ consisting of only one point, then the product $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ exists and $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y) = \mathrm{Sh}_p(X) \neq \mathrm{Sh}_p(X \times Y)$, where $E(Y) = F(Y) - Y$ and $F(Y)$ means the Freudenthal compactification of $Y$.

(3.36) If $E(Y)$ consists of more than one point, then the product $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ does not exist.

Let us show (3.35). Since $E(Y)$ is one point, we can assume by Sher [18, Theorem (3.1)] that $Y = R^+(=[0, \infty))$. Choose a proper map $\mu: X \rightarrow R^+$. We claim that the triple $(\mathrm{Sh}_p(X), \boldsymbol{i}, \boldsymbol{\mu})$ is the product of $\mathrm{Sh}_p(X)$ and $\mathrm{Sh}_p(Y)$, where $i$ is the identity map: $X \rightarrow X$, $\boldsymbol{i}$ and $\boldsymbol{\mu}$ are the proper fundamental nets generated by $i$ and $\mu$. Let $Z$ be any locally compact space. Give proper fundamental nets $\boldsymbol{f}: Z \rightarrow X$ and $\boldsymbol{g}: Z \rightarrow R^+$. There exist proper maps $f: Z \rightarrow X$ and $g: Z \rightarrow R^+$ generating $\boldsymbol{f}$ and $\boldsymbol{g}$. Since any proper maps of $Z$ into $R^+$ are properly homotopic to each other, we have $g \underset{p}{\simeq} \mu f$. This implies $(\mathrm{Sh}_p(X), \boldsymbol{i}, \boldsymbol{\mu}) = \mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$.

Next, let $Y$ be an $\mathrm{SUV}^\infty$ such that $E(Y)$ consists of more than one point. By [18, Theorem (3.1)] we can assume that $Y$ is a tree. Since $E(Y)$ has at least two points, $Y$ contains a real line $R$ as a closed set. There exists a proper retraction $r: Y \rightarrow R$. Suppose that $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ exists. By the proof of Theorem 4, we have $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y) = (\mathrm{Sh}_p(X), \boldsymbol{i}, \boldsymbol{\mu})$, where $i$ is the identity: $X \rightarrow X$ and $\mu$ is a proper map of $X$ into $Y$. Consider two proper maps $f_1, f_2: X \rightarrow Y$ defined as follows.

$$\begin{aligned} f_1(x) &= r\mu(x) \\ f_2(x) &= -r\mu(x) \end{aligned}, \qquad x \in X .$$

Since $(\mathrm{Sh}_p(X), \boldsymbol{i}, \boldsymbol{\mu}) = \mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ and $R$ is a proper retract of $Y$, we have $f_1 \underset{p}{\simeq} f_2$ in $R$. On the other hand, since $Ff_i(E(X)) \subset E(R) = \{-\infty\} \cup \{\infty\}$, where $Ff_i$ is the extension of $f_i$ over $F(X)$, it is easy to prove that $f_1 \underset{p}{\not\simeq} f_2$ in $R$. This contradiction means that the product $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ does not exist.

EXAMPLE 4. Let $X$ be a 0-dimensional locally compact and non-compact space. Then the product $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(R^n)$ does not exist. Here $R^n$ is the $n$-dimensional euclidean space. The proof is similar to (3.36). Namely, for $n \geqq 2$, one can give a similar argument using a proper map $g: S^{n-1} \times J \rightarrow R^n - \{0\}$ such that $g | S^{n-1} \times \{i\}$ fails to be null homotopic in $R^n - \{0\}$ for $i = 1, 2, \cdots$. (Note that $g$ is not properly homotopic in $R^n$ to any map which factors through a 0-dimensional space.)

## 4. Proper shapes of the products.

THEOREM 5. *Suppose that $X$ is compact and $X'$, $Y$ and $Y'$ are* *locally compact. If* $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(X')$ *and* $\mathrm{Sh}_p(Y) = \mathrm{Sh}_p(Y')$, *then* $\mathrm{Sh}_p(X \times Y) = \mathrm{Sh}_p(X \times Y')$.

In the Winter school "Shape Theory and Pro-homotopy" held in Dubrovnik, January, 1976, R. B. Sher raised the following question:

Let $X$ be compact and let $X'$ and $Y'$ be locally compact. If $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(X')$ and $\mathrm{Sh}_p(R) = \mathrm{Sh}_p(Y')$, where $R$ is the real line, does the equality $\mathrm{Sh}_p(X \times R) = \mathrm{Sh}_p(X' \times Y')$ hold?

Theorem 5 solves this question.

*Proof of Theorem* 5. Since $X$ is compact and $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(X')$, $X'$ is compact. Hence $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ by Ball and Sher [1, 3.15]. We divide the proof to the following two cases: (A) $X = X'$, (B) $Y = Y'$.

First, let us prove the theorem under the hypothesis (A). Let $H = M - \{w\}$, where $M$ is the Hilbert cube and $w$ is a point of $M$. We consider $Y'$ and $Y$ as closed sets in $H$ and $X$ as a subset of $M$. Since $\mathrm{Sh}_p(Y) = \mathrm{Sh}_p(Y')$, there exist proper fundamental nets $\boldsymbol{f} = \{f_\lambda : \lambda \in \Lambda\}$ from $Y$ to $Y'$ in $H$ and $\boldsymbol{g} = \{g_\mu : \mu \in \Omega\}$ from $Y'$ to $Y$ in $H$ such that $\boldsymbol{gf} \underset{p}{\simeq} \boldsymbol{i}_Y$ and $\boldsymbol{fg} \underset{p}{\simeq} \boldsymbol{i}_{Y'}$, where $\boldsymbol{i}_{Y'}$ are and $\boldsymbol{i}_{X'}$ the fundamental nets generated by the identities $i_Y : Y \to Y$ and $i_{Y'} : Y' - Y'$. (See for the definitions Ball and Sher [1, p. 166].) Let $i : M \to M$ be the identity. Then $\tilde{\boldsymbol{f}} = \{i \times f_\lambda : \lambda \in \Lambda\}$ is a proper fundamental net of $X \times Y$ to $X \times Y'$ in $M \times H$. Similarly $\tilde{\boldsymbol{g}} = \{i \times g_\mu : \mu \in \Omega\}$ is a proper fundamental net of $X \times Y'$ to $X \times Y$. Obviously $\tilde{\boldsymbol{g}}\tilde{\boldsymbol{f}} \underset{p}{\simeq} \boldsymbol{i}_{X \times Y}$ and $\tilde{\boldsymbol{f}}\tilde{\boldsymbol{g}} \underset{p}{\simeq} \boldsymbol{i}_{X \times Y'}$. This means $\mathrm{Sh}_p(X \times Y) = \mathrm{Sh}_p(X \times Y')$.

Next, we shall prove the case (B). We consider $Y$ as a closed subset of $H$ and $X, X'$ as $Z$-sets in $M$. Since $X$ and $X'$ are $Z$-sets in $M$, by the proof of Lemma 4.1 of Chapman [7], there exist homotopies $\xi, \eta : M \times I \to M$ satisfying the following conditions:

(4.1) $$\xi(x, 0) = x = \eta(x, 0)\ , \quad x \in M\ ,$$

(4.2) for any $U \in \boldsymbol{U}(X, M)$ and $U' \in \boldsymbol{U}(X', M)$ there exists a $t' \in (0, 1]$ such that $\xi(x, t) \in U$ for $x \in U$ and $t \in [0, t']$, and $\eta(x, t) \in U'$ for $x \in U'$ and $t \in [0, t']$ ,

(4.3) for any $U \in \boldsymbol{U}(X, M)$ and $U' \in \boldsymbol{U}(X', M)$ there exists a $t' \in (0, 1]$ such that $\xi(x, t) = x$ for $x \in X - U$ and $t \in [0, t']$, and $\eta(x, t) = x$ for $x \in X - U'$ and $t \in [0, t']$ ,

(4.4) $\xi(x, t) \notin X$ and $\eta(x, t) \notin X'$ for $x \in M$ and $t \in (0, 1]$ .

Since $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ and $X, X'$ are $Z$-sets in $M$, by Chapman [7, Theorem 2] there exists a homeomorphism $h: M - X \to M - X'$. Define $\tilde{h}: (M - X) \times H \to (M - X') \times H$ by $\tilde{h}(x, y) = (h(x), y)$ for $(x, y) \in (M - H) \times H$. Obviously $\tilde{h}$ is a homeomorphism. Denote by $\Lambda$ the set of all maps from $H$ into $(0, 1]$. We define an order $\leqq$ in $\Lambda$ as follows: If $\alpha, \beta \in \Lambda$ and $\alpha(y) \leqq \beta(y)$ for each $y \in H$, then $\alpha \geqq \beta$. Obviously $\Lambda$ forms a directed set by this order. Consider the sets $\boldsymbol{f} = \{f_\alpha: \alpha \in \Lambda\}$ and $\boldsymbol{g} = \{g_\alpha: \alpha \in \Lambda\}$ of maps $f_\alpha, g_\alpha: M \times H \to M \times H$ defined as follows.

$$(4.5) \qquad \begin{matrix} f_\alpha(x, y) = \tilde{h}(\xi(x, \alpha(y)), y) \\ g_\alpha(x, y) = \tilde{h}^{-1}(\eta(x, \alpha(y)), y) \end{matrix}, \quad (x, y) \in M \times H .$$

By (4.4) $f_\alpha$ and $g_\alpha$ are well defined. We shall prove that $\boldsymbol{f}: X \times Y \to X' \times Y$ and $\boldsymbol{g}: X' \times Y \to X \times Y$ are proper fundamental nets in $M \times H$. Obviously the *maps* $f_\alpha$ and $g_\alpha$, $\alpha \in \Lambda$, are closed. Since $f_\alpha^{-1}(x', y') = \{(x, y'): h^{-1}(x') = \xi(x, \alpha(y')), x \in M\}$ and $g_\alpha^{-1}(x', y') = \{(x, y'): h(x') = \eta(x, \alpha(y')), x \in M\}$ for $(x', y') \in M \times H$, both $f_\alpha^{-1}(x', y')$ and $g_\alpha^{-1}(x', y')$ are compact. Therefore $f_\alpha$ and $g_\alpha$ are proper maps. Let $W$ be any closed neighborhood of $X' \times Y$ in $M \times H$. We can assume by Lemma 3 that $W$ is a basic neighborhood. Therefore there exist a locally finite closed cover $\{W_\alpha: \alpha \in \Omega\}$ of the base $B(W)$ of $W$ and a collection $\{V_\alpha: \alpha \in \Omega\}$ of closed neighborhoods of $X'$ in $M$ such that $W = \bigcup_{\alpha \in \Omega} V_\alpha \times W_\alpha$. Set $W' = \tilde{h}^{-1}(W - X' \times H) \cup X \times B(W)$. We claim

$$(4.6) \qquad W' \in \boldsymbol{U}(X \times Y, M \times H) .$$

To prove (4.6), it is enough to prove that $\overline{M \times B(W) - W'}$ is disjoint from $X \times B(W)$, where the closure is taken in $M \times H$. Since $M \times B(W) - W' \subset \bigcup_{\alpha \in \Omega} \tilde{h}^{-1}((M - V_\alpha) \times W_\alpha)$ and $\{W_\alpha\}$ is locally finite in $H$, $\overline{M \times B(W) - W'} \cap X \times B(W) \subset \bigcup_{\alpha \in \Omega}(\overline{\tilde{h}^{-1}((M - V_\alpha) \times W_\alpha)} \cap X \times B(W)) = \varnothing$. Hence (4.6) is true.

Next, note that there exists a map $\alpha \in \Lambda$ such that

$$(4.7) \qquad (\xi(x, t), y) \in W' \text{ for } (x, y) \in W' \text{ and } t \in [0, \alpha(y)] .$$

This is proved by making use of the properties (4.1), (4.2), (4.3) of the homotopy $\xi$, the paracompactness of $B(W)$ and Tietz's extension theorem. By (4.7) and the definition of $W'$ we have $f_\alpha | W': W' \to W$. Suppose that $\beta \in \Lambda$ and $\alpha \leqq \beta$. Define a homotopy $\varphi$ on the set $W' \times I$ by $\varphi((x, y), t) = \tilde{h}(\xi(x, t\alpha(y) + (1 - t)\beta(y)), y)$ for $(x, y) \in W'$ and $t \in I$. Since $\alpha \leqq \beta$, we have $\varphi(W' \times I) \subset W$ by (4.7). Obviously $\varphi$ is a proper map. This means that $f_\alpha | W' \underset{p}{\simeq} f_\beta | W'$ in $W$. Thus we have proved that for a given closed neighborhood $W$ of $X' \times Y$ in $M \times H$, there exist a closed neighborhood $W'$ of $X \times Y$ in $M \times H$ and an index $\alpha \in \Lambda$ such that if $\beta \geqq \alpha$ then $f_\alpha | W' \underset{p}{\simeq} f_\beta | W'$ in $W$. This implies that $\boldsymbol{f} = \{f_\alpha\}$ is a proper fundamental net. Similarly it is seen that $\boldsymbol{g} = \{g_\alpha\}$ is a proper fundamental net.

Finally let us prove

$$(4.8) \qquad \boldsymbol{gf} \underset{p}{\simeq} \boldsymbol{i}_{X\times Y} \quad \text{and} \quad \boldsymbol{fg} \underset{p}{\simeq} \boldsymbol{i}_{X'\times Y} .$$

Take a closed basic neighborhood $W \in \boldsymbol{U}(X \times Y, M \times H)$. There exists an $\alpha \in \Lambda$ such that

$$(4.9) \qquad (\xi(x, t), y) \in W \quad \text{for} \quad (x, y) \in W \quad \text{and} \quad t \in [0, \alpha(y)]$$

Consider the map $f_\alpha: M \times H \to M \times H, f_\alpha \in \boldsymbol{f}$ (cf. (4.5)). By the same way as the proof of (4.6) it is proved that

$$W' = M \times B(W) - f_\alpha(M \times B(W)) \in \boldsymbol{U}(X' \times Y, M \times H) .$$

From (4.3) it follows that there exists a $\beta \in \Lambda$ such that

$$(4.10) \quad \text{if } (x, y) \notin W' \text{ then } (\eta(x, t), y) = (x, y) \text{ for each } t \in [0, \beta(y)] .$$

Consider the map $g_\beta f_\alpha: M \times H \to M \times H$. If $(x, y) \in X$, then $g_\beta f_\alpha(x, y) = (\eta(\xi(x, \alpha(y)), \beta(y)), y)$. Since $(\xi(x, \alpha(y)), y) \notin W'$, $g_\beta f_\alpha(x, y) = (\xi(x, \alpha(y)), y)$ by (4.10). Define a homotopy $H$ on the set $W \times I$ by $H((x, y), t) = (\xi(x, t\alpha(y)), y)$, $(x, y) \in W$ and $t \in I$. From (4.9) it follows that $H(W \times I) \subset W$. Therefore we have $g_\beta f_\alpha | W \underset{p}{\simeq} i_W$ in $W$. By the choice of $\alpha$ and $\beta$, we can see that if $\alpha' \geqq \alpha$ and $\beta' \geqq \beta$ then $g_{\beta'} f_{\alpha'} | W \underset{p}{\simeq} i_W$ in $W$. Thus we have proved that if $W$ is a closed basic neighborhood of $X \times H$ in $M \times H$, then there exist $\alpha, \beta \in \Lambda$ such that if $\alpha' \geqq \alpha$ and $\beta \geqq \beta$ then $g_{\beta'} f_{\alpha'} | W \underset{p}{\simeq} i_W$ in $W$. This proves the first relation of (4.8). The second relation of (4.8) is proved similarly. This completes the proof.

REMARK 3. Theore 5 implies that if $X$ is compact then $\mathrm{Sh}_p(X \times Y)$ is uniquely determined by $\mathrm{Sh}_p(X)$ and $\mathrm{Sh}_p(Y)$. However $\mathrm{Sh}_p(X \times Y)$ is not generally the product of $\mathrm{Sh}_p(X)$ and $\mathrm{Sh}_p(Y)$. Because, if $Y$ is a locally compact, non compact and 0-dimensional space then $\mathrm{Sh}_p(X) \times \mathrm{Sh}_p(Y)$ need not exist (cf. Example 3).

Following Ball [2, p. 185] a locally compact separable metric space $X$ is said to be an *absolute proper shape retract* (APSR) if $X$ is a proper shape retract of every locally compact separable metric space $Y$ in which $X$ is properly embedded. Here $X$ is said to be *properly embedded* in $Y$ if $X$ is a closed set of $Y$ and the injection $i: X \to Y$ is end preserving, that is, $F(i) | E(X): E(X) \to E(Y)$ is injective, where $E(X) = F(X) - X$ is the remainder of the Freudenthal compactification $F(X)$ of $X$ (cf. [2, p. 180]). Sher [19] defined an *absolute neighborhood proper shape retract* (ANPSR) as follows: $X \in$

ANPSR if and only if for each locally compact separable metric space $X'$ containing $X$ as a closed set there exists a closed neighborhood $X''$ of $X$ in $X'$ such that $X$ is a proper shape retract of $X''$. We shall prove that

THEOREM 6. *If $X$ is a pointed* FANR (resp. FAR) *and $Y$ is an* ANPSR (resp. APSR), *then $X \times Y$ is an* ANPSR (resp. APSR).

*Proof*. We give the proof in case $X$ is a pointed FANR and $Y$ is an ANPSR. The other case is proved similarly and we omit the proof.

Let $X$ be a pointed FANR and $Y \in$ ANPSR. Since $Y \in$ ANPSR, by [19, Corollary (6.3)] there exist a locally compact ANR $H$ in which $Y$ is properly embedded, an AR $N$ containing $H$ as a closed set and a proper fundamental net $\boldsymbol{r} = \{r_\lambda: \lambda \in \Lambda\}: H \to Y$ such that $r_\lambda: N \to N$ and $r_\lambda(y) = y$ for $y \in Y$ and $\lambda \in \Lambda$. Consider $X$ as a $Z$-set of the Hilbert cube $M$. By Lemma 2 and Remark 1 there exist a decreasing sequence $\{M_k: k = 0, 1, 2, \cdots\}$ of neighborhoods of $X$ in $M$ and a map $\varphi: M \times [0, \infty) \to M$ satisfying the properties (3.7), (3.8), (3.9) and (3.10). ((3.6) is not required.) We can assume that $M_1$ is a compact ANR. By $\Omega$ denote the set of all maps $\alpha$ of $H$ into $[0, \infty)$. We define an order $\leqq$ in $\Omega$ as follows: If $\alpha, \beta \in \Omega$ and $\alpha(y) \leqq \beta(y)$ for each $y \in H$, then $\alpha \leqq \beta$. Then $\Omega$ becomes a directed set. For a pair $(\alpha, \lambda) \in \Omega \times \Lambda$, define a map $f_{(\alpha,\lambda)}: M \times H \to M \times H$ by

$$(4.11) \qquad f_{(\alpha,\lambda)}(x, y) = (\varphi(x, \alpha r_\lambda(y)), r_\lambda(y)) \text{ for } (x, y) \in M \times H .$$

Set $\boldsymbol{f} = \{f_{(\alpha,\lambda)}: (\alpha, \lambda) \in \Omega \times \Lambda\}$. For each $(\alpha, \lambda) \in \Omega \times \Lambda$, if $(x, y) \in X \times Y$, $f_{(\alpha,\lambda)}(x, y) = (x, y)$ because $\varphi(x, r) = x$ for $x \in X$ and $r \in [0, \infty)$ by (3.8) and $r_\lambda(y) = y$ for $y \in Y$. Thus, to prove the theorem it is enough to show that $\boldsymbol{f}$ is a proper fundamental net of $M_1 \times H$ to $X \times Y$, because this means that $X \times Y$ is a proper shape retract of a locally compact ANR $M_1 \times H$ and the theorem follows from ]19, Corollary (63)]. Let $W$ be a basic neighborhood of $X \times Y$ in $M \times H$. There exists an $\alpha \in \Omega$ such that

$$(4.12) \qquad \begin{array}{l} \text{if } (x, y) \in M_1 \times B(W) \text{ then } (\varphi(x, r), y) \in W \text{ for each} \\ r \in [\alpha(y), \infty) . \end{array}$$

Since $\boldsymbol{r}$ is a proper fundamental net, there exist a $\lambda \in \Lambda$ and a closed neighborhood $F$ of $Y$ in $B(W)$ such that if $\lambda \leqq \mu$ then $r_\lambda | F \underset{p}{\simeq} r_\mu | F$ in $B(W)$. We claim

$$(4.13) \quad \text{if } (\beta, \mu) \geqq (\alpha, \lambda) \text{ then } f_{(\alpha,\lambda)} | M_1 \times F \underset{p}{\simeq} f_{(\beta,\mu)} | M_1 \times F \text{ in } W .$$

To prove (4.13), let $\xi: F \times I \to B(W)$ be a proper homotopy such that $\xi(y, 0) = r_\lambda(y)$ and $\xi(y, 1) = r_\mu(y)$ for $y \in F$. Define proper homotopies $h_1$ and $h_2$ on the set $M_1 \times F \times I$ by

$$(4.14) \quad \begin{aligned} h_1(x, y, t) &= (\varphi(x, (1-t)\alpha r_\lambda(y) + t\beta r_\lambda(y)), r_\lambda(y)) , \\ & \qquad\qquad (x, y, t) \in M_1 \times F \times I , \\ h_2(x, y, t) &= (\varphi(x, \beta\xi(y, t)), \xi(y, t)) , \end{aligned}$$

By (4.12) we have $h_1(M_1 \times F \times I) \cup h_2(M_1 \times F \times I) \subset W$. Since, by (4.11) and (4.14), $h_1(x, y, 0) = (\varphi(x, \alpha r_\lambda(y)), r_\lambda(y)) = f_{(\alpha,\lambda)}(x, y)$, $h_1(x, y, 1) = (\varphi(x, \beta r_\lambda(y)), r_\lambda(y)) = h_2(x, y, 0)$ and $h_2(x, y, 1) = (\varphi(x, \beta\xi(y, 1)), \xi(y, 1)) = (\varphi(x, \beta r_\mu(r)), r_\mu(y)) = f_{(\beta,\mu)}(x, y)$ for $(x, y) \in M_1 \times F$, the relation (4.13) holds. Thus $f$ is a proper fundamental net. This completes the proof.

REMARK 4. In [3] Ball defined four proper shape categories $\mathscr{S}_p^0$, $\mathscr{S}_p^1$, $\mathscr{S}_p^2$ and $\mathscr{S}_p^3$. Here $\mathscr{S}_p^0$ is our category $\mathscr{C}_p$. He proved that $\mathscr{S}_p^i$, $i = 1, 2, 3$, are isomorphic to each other. Denote by $\mathrm{Sh}_p^i(X)$ the shape of a locally compact space $X$ in the category $\mathscr{S}_p^i$, $i = 0, 1, 2, 3$. It is known that we can replace the category $\mathscr{C}_p$ by the category $\mathscr{S}_p^i$, $i = 0, 1, 2, 3$, throughout this paper. For example, the following theorem is proved.

THEOREM 4′ *Let $i = 0, 1, 2, 3$. If $X$ is compact, $X'$, $Y$ and $Y'$ are locally compact, and* $\mathrm{Sh}_p^i(X) = \mathrm{Sh}_p^i(X')$ *and* $\mathrm{Sh}_p^i(Y) = \mathrm{Sh}_p^i(Y')$, *then* $\mathrm{Sh}_p^i(X \times Y) = \mathrm{Sh}_p^i(X' \times Y')$.

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UNIVERSITY OF TSUKUBA
IBARAKI, JAPAN