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A COMPARISON THEOREM AND OSCILLATION CRITERIA FOR SECOND ORDER DIFFERENTIAL SYSTEMS

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Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{B} = \mathscr{B}(\mathscr{H}, \mathscr{H})$ be the B^*-algebra of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$ with the uniform operator topology. Let $\mathscr{S}$ be the subset of $\mathscr{B}$ consisting of the selfadjoint operators. This paper is concerned with second order, selfadjoint differential equations of the form

$$(1) \qquad [P(x)Y']' + Q(x)Y = 0$$

on $\mathscr{R}^+ = [0, \infty)$, where P and Q are continuous mappings of $\mathscr{R}^+$ into $\mathscr{S}$ with $P(x)$ positive definite for all $x \in \mathscr{R}^+$. Let $\mathscr{G}$ be the set of positive linear functionals on $\mathscr{B}$. Positive functionals are used in deriving a generalization of Sturm's comparison theorem, and, in turn, the comparison theorem is used to obtain oscillation criteria for equation (1). These criteria are shown to include a large number of well-known oscillation criteria for (1) in the matrix and scalar case. Extensions of the results to nonlinear differential equations and differential inequalities are also discussed.

Appropriate discussions of the concepts of differentiation and integration of $\mathscr{B}$-valued functions, as well as treatments of the existence and uniqueness of solutions $Y: \mathscr{R}^+ \rightarrow \mathscr{B}$ of (1), can be found in a variety of texts. See, for example, E. Hille [8, Chapters 4, 6, and 9]. Studies of the behavior of solutions of second order equations in a B^*-algebra have been done by several authors, including Hille [8, Chapter 9], T. L. Hayden and H. C. Howard [7] and C. M. Williams [18]. Of course, if $\mathscr{H} = \mathscr{R}_n$, Euclidean n-space, then $\mathscr{B}$ is the B^*-algebra of $n \times n$ matrices, and equation (1) is the familiar second order, selfadjoint matrix differential equation which has been investigated in great detail by a large number of authors. In this regard we refer to the texts by A. Coppel [2], P. Hartman [6], Hille [8], W. T. Reid [13], and C. A. Swanson [15], all of which provide comprehensive bibliographies and extensive references to the research literature.

It is easy to verify by differentiation that if $Y = Y(x)$ is a solution of equation (1), then

$$(2) \qquad Y^*(x)[P(x)Y'(x)] - [P(x)Y'(x)]^*Y(x) \equiv C$$

on $\mathscr{R}^+$, $C \in \mathscr{B}$ a constant. The solution Y is *conjoined* (or *prepared*) if the constant operator C in (2) is 0, the zero operator. The term "conjoined" has its origins in the calculus of variations, and for amplifications of this concept, the reader is referred to Reid [13]. Conjoined solutions of (1) can be obtained by choosing conjoined initial conditions. In fact, it is easy to show that Y is a conjoined solution of (1) if and only if there is at least one point $a \in \mathscr{R}^+$ such that

$$Y^*(a)[P(a)Y'(a)] = [P(a)Y'(a)]^*Y(a) .$$

As noted by Noussair and Swanson [12], the conjoined hypothesis on solutions of (1) is needed in order that an analog of the classical theory of oscillation of (1) in the scalar case can be developed.

A solution $Y = Y(x)$ of equation (1) is *nonsingular at* $x = c$, $c \in \mathscr{R}^+$, provided $Y^{-1}(c) \in \mathscr{B}$. This is equivalent to the two conditions:

(i) the range of $Y(c)$ is $\mathscr{H}$, and

(ii) $Y(c)$ has a bounded inverse.

If either of these conditions fails to hold at $x = c$, then Y is *singular at* $x = c$. We note that in the special case $\mathscr{H} = \mathscr{R}_n$, conditions (i) and (ii) are equivalent, and that the nonsingularity of $Y(c)$ can be also be expressed in other terms, e.g., $\det Y(c) \neq 0$. Hayden and Howard [7, p. 384] give an example which illustrates why conditions (i) and (ii) are required in the definition of nonsingularity in the general B^*-algebra case.

A solution $Y = Y(x)$ of equation (i) is *nontrivial* if there exists at least one point $c \in \mathscr{R}^+$ such that $Y(c)$ is nonsingular. In the finite dimensional case $\mathscr{H} = \mathscr{R}_n$ it is known that the condition $Y^*Y + [PY']^*[PY'] > 0$ (i.e., positive definite) on $\mathscr{R}^+$ is equivalent to the nontriviality of Y (see, e.g., Etgen [4]). However, like nonsingularity, there are difficulties with this characterization in the general B^*-algebra case. The following example, communicated to us by R. T. Lewis and S. C. Tefteller, relates to both nonsingularity and nontriviality.

EXAMPLE. Consider the differential equation

$$Y'' + AY = 0$$

on $\mathscr{R}^+$, where A is the infinite diagonal matrix $A = \mathrm{diag}\,[1, 1/4, 1/9, \cdots]$. It is easy to show that $Y(x) = \mathrm{diag}\,[\sin x, \sin x/2, \sin x/3, \cdots]$ is a conjoined solution of the equation, and that $Y^*Y + Y^{*\prime}Y' > 0$ on $\mathscr{R}^+$. For any $c \in \mathscr{R}^+$, $c \neq n\pi$, n a nonnegative integer, $Y(c)$ is one-to-one, but it does not have a bounded inverse because 0 is a limit point of the eigenvalues of $Y(c)$, or, equivalently, there does not exist a positive number m such that $\|Y(c)\alpha\| \geqq m\|\alpha\|$ for all $\alpha \in \mathscr{H}$. When c is an integral multiple of π, $Y(c)$ is not one-to-one. Thus we can conclude that Y is identically singular on $\mathscr{R}^+$, and so Y is not a nontrivial solution of the equation. The solution $Z(x) = \operatorname{diag}[\cos x, \cos x/2, \cos x/3, \cdots]$ is a nontrivial solution of the equation.

Let $Y = Y(x)$ be a nontrivial conjoined solution of equation (1). The solution Y is *oscillatory* if for each $a \in \mathscr{R}^+$ there is a point b, $b \geqq a$, such that $Y(b)$ is singular. The solution Y is *nonoscillatory* if it is not oscillatory. Equivalently, Y is nonoscillatory if there is a point $c \in \mathscr{R}^+$ such that Y is nonsingular on $[c, \infty)$. The differential equation (1) is *oscillatory* if it has at least one nontrivial conjoined oscillatory solution, otherwise equation (1) is *nonoscillatory*.

The following lemma and camparison theorem have been established by K. Kreith [10, Lemmas 1 and 2] in the finite dimensional case. His proofs of these results extend without modification to the general B^*-algebra case.

LEMMA 1.1. *Given the differential equations* (1) *and*

$$[F(x)Y']' + G(x)Y = 0 \tag{3}$$

where $F, G\colon \mathscr{R}^+ \rightarrow \mathscr{S}$ *are continuous and* $F(x) > 0$ *for each* $x \in \mathscr{R}^+$. *Let* $Y = Y(x)$ *be a conjoined solution of* (1) *such that* Y *is nonsingular on some interval* $[a, b] \subset \mathscr{R}^+$, *and let* $S(x) = [P(x)Y'(x)]Y^{-1}(x)$ *on* $[a, b]$. *If* $V = V(x)$ *is a solution of* (3), *then*

$$\begin{aligned}[V^*FV' - V^*SV]_{x=a}^{x=b} = \int_a^b V^*(Q - G)Vdx + \int_a^b V^{*\prime}(F - P)V'dx \\ + \int_a^b (V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)dx\,.\end{aligned} \tag{4}$$

LEMMA 1.2. *Let* $[a, b] \subset \mathscr{R}^+$ *and suppose* $V = V(x)$ *is a solution of equation* (3), $V \not\equiv 0$, *which satisfies*

(i) $V^*(x)[Q(x) - G(x)]V(x) \geqq 0$ (i.e., nonnegative definite) *on* $[a, b]$,

(ii) $V^{*\prime}(x)[F(x) - P(x)]V'(x) \geqq 0$ *on* $[a, b]$,

(iii) $V(a) = V(b) = 0$.

If $Y = Y(x)$ *is a conjoined solution of equation* (1), *then* $Y(x)$ *is singular for at least one* $x \in [a, b]$.

2. The comparison theorem. In this section we introduce the set of positive functionals on $\mathscr{B}$, and use them to obtain a comparison theorem which generalizes Lemma 1.2.

A linear functional g on $\mathscr{B}$ is a *positive functional* if $g(A^*A) \geqq 0$ for all $A \in \mathscr{B}$. Equivalently, the linear functional g is a positive functional if $g(B) \geqq 0$ for all $B \in \mathscr{S}$ such that $B \geqq 0$. C. E. Rickart [14] has shown that each positive functional g on $\mathscr{B}$ is bounded (i.e., continuous) with $||g|| = g(I)$ (I denotes the identity operator in $\mathscr{B}$). Also, each positive functional g satisfies a generalized Cauchy-Schwarz inequality

$$(5) \qquad [g(A^*B)]^2 \leqq g(A^*A)g(B^*B)$$

for all $A, B \in \mathscr{B}$. It follows from (5) that the positive functional g is the zero functional if and only if $g(I) = 0$. If g is not the zero functional, then $g(I) > 0$ and, in general, $g(A) > 0$ whenever $A \in \mathscr{S}$, $A > 0$. Finally, since a positive functional g is continuous,

$$g\left[\int_a^x A(t)dt\right] = \int_a^x g[A(t)]dt$$

whenever $A: \mathscr{R}^+ \to \mathscr{B}$ is integrable, and

$$g[B'(x)] = \{g[B(x)]\}'$$

whenever $B: \mathscr{R}^+ \to \mathscr{B}$ is differentiable.

Let $\mathscr{G}$ be the set of positive functionals on $\mathscr{B}$. The fact that $\mathscr{G}$ contains elements in addition to the zero functional can be verified by associating with each nonzero vector $\alpha \in \mathscr{H}$ the functional g_α defined on $\mathscr{B}$ by

$$(6) \qquad g_\alpha(A) = (A\alpha, \alpha)\ , \qquad A \in \mathscr{B}\ ,$$

where $(\cdot,\cdot)$ is the inner product on $\mathscr{H}$. It is easy to show that g_α is a positive functional on $\mathscr{B}$, with $g_\alpha(I) = ||\alpha||^2 > 0$. In fact, it can be verified that $\mathscr{G}$ is a positive cone in the space of continuous linear functionals on $\mathscr{B}$.

THEOREM 2.1. *Suppose there is a* $g \in \mathscr{G}$ *and a solution* $V = V(x)$ *of* (3) *such that:*

(i) $g[V^*(Q - G)V] \geqq 0$ *on* $[a, b]$,
(ii) $g[V^{*\prime}(F - P)V'] \geqq 0$ *on* $[a, b]$,
(iii) $g[V^*(a)V(a)] = g[V^*(b)V(b)] = 0$,
(iv) *for any* $c \in [a, b]$, $g[V^*(c)V(c)] = 0$ *implies that*

$$g[V^{*\prime}(c)P(c)V'(c)] > 0\ .$$

If $Y = Y(x)$ *is a conjoined solution of* (1), *then* $Y(x)$ *is singular for at least one* $x \in [a, b]$.

Proof. Suppose that Y is a conjoined solution of (1) which is nonsingular on $[a, b]$. Then $S = PY'Y^{-1}$ exists on $[a, b]$ and (4) holds. By applying the functional g to (4), and using the linearity and continuity of g, we get the equation

$$\begin{aligned} &g[V^*FV'](b) - g[V^*SV](b) - g[V^*FV'](a) + g[V^*SV](a) \\ &\quad = \int_a^b g[V^*(Q-G)V]dx + \int_a^b g[V^{*\prime}(F-P)V']dx \\ &\qquad + \int_a^b g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)]dx \,. \end{aligned} \tag{7}$$

It follows from the generalized Cauchy-Schwarz inequality (5) that

$$\{g[V^*(x)F(x)V'(x)]\}^2 \leqq g[V^*(x)V(x)]g[(F(x)V'(x))^*F(x)V'(x)]$$

and

$$\{g[V^*(x)S(x)V(x)]\}^2 \leqq g[V^*(x)V(x)]g[(S(x)V(x))^*S(x)V(x)] \,.$$

Thus, by hypothesis (iii), $g[V^*(x)F(x)V'(x)] = g[V^*(x)S(x)V(x)] = 0$ when $x = a$, or $x = b$. Therefore equation (7) reduces to

$$\begin{aligned} 0 &= \int_a^b g[V^*(Q-G)V]dx + \int_a^b g[V^{*\prime}(F-P)V']dx \\ &\quad + \int_a^b g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)]dx \,. \end{aligned} \tag{8}$$

From hypotheses (i) and (ii), and from the fact that

$$(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)$$

is nonnegative definite on $[a, b]$, the integrand in each term on the right side of (8) is nonnegative and, consequently, each term on the right side of (8) is nonnegative. Thus it suffices to show that at least one term is positive in order to obtain the desired contradiction.

We expand the integrand in the third term on the right side of (8) to obtain

$$\begin{aligned} &g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)] \\ &\quad = g[V^{*\prime}PV'] + g[(Y'Y^{-1}V)^*P(Y'Y^{-1}V)] \\ &\qquad - g[(Y'Y^{-1}V)^*PV^{*\prime}] - g[V'P(Y'Y^{-1}V)] \,. \end{aligned}$$

By evaluating this expression at $x = a$, and by using hypothesis (iii) and the generalized Cauchy-Schwarz inequality (5) in the manner suggested above, we have

$$g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)](a) = g[V^{*\prime}(a)P(a)V'(a)] \,.$$

Thus, by hypothesis (iv) there is a subinterval $[a, a')$ of $[a, b]$ on which $g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)] > 0$. (The same reasoning shows that there is also a subinterval $(b', b]$ of $[a, b]$ such that $g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)] > 0$ on $(b', b]$.) It now follows that

$$\int_a^b g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)]dx$$
$$\geqq \int_a^{a'} g[(V' - Y'Y^{-1}V)^*P(V' - Y'Y^{-1}V)]dx > 0$$

which contradicts (8) and completes the proof of the theorem.

3. Oscillation criteria. In this section we use Theorem 2.1 to develop oscillation criteria for equation (1), and we show how our criteria include a number of well-known oscillation criteria as special cases.

Our oscillation criteria will be developed by "comparing" equation (1) with second order scalar equations of the form

$$(p(x)y')' + q(x)y = 0 \tag{9}$$

on $\mathscr{R}^+$, where $p, q: \mathscr{R}^+ \to \mathscr{R}$ (the reals) are continuous and $p(x) > 0$ for all $x \in \mathscr{R}^+$. We assume that the reader is familiar with the appropriate definitions concerning the oscillation of solutions of (9).

THEOREM 3.1. *Suppose equation* (9) *is oscillatory. If there is a* $g \in \mathscr{G}$ *such that*
(i) $g[Q - q(x)I] \geqq 0$
(ii) $g[p(x)I - P] \geqq 0$
on $[c, \infty)$ *for some* $c \in \mathscr{R}^+$, *then equation* (1) *is oscillatory.*

Proof. Let $Y = Y(x)$ be a nontrivial conjoined solution of (1), and let $v = v(x)$ be a nontrivial solution of (9). Since (9) is oscillatory, for each point d, $d \geqq c$, there are points a and b, $d \leqq a < b$, such that $v(a) = v(b) = 0$. Let $V = V(x)$ be the $\mathscr{B}$-valued function on $\mathscr{R}^+$ defined by $V(x) = v(x)I$. Then it is clear that V is a solution of the $\mathscr{B}$-valued equation.

$$(F(x)Y')' + G(x)Y = 0 \tag{3'}$$

where $F(x) = p(x)I$ and $G(x) = q(x)I$. By using the hypotheses of this theorem, it is easy to verify that the positive functional g and the solution V satisfy the hypotheses of Theorem 2.1. Thus $Y(x)$ is singular for at least one $x \in [a, b]$, and equation (1) is oscillatory.

The following three corollaries list some oscillation criteria for (1) in the finite dimensional case, i.e., in the case where P and Q are $n \times n$ continuous, symmetric matrices on $\mathscr{R}^+$ with P positive definite. These criteria are well-known, and they are demonstrated to be special cases of Theorem 3.1.

COROLLARY 1 (cf. Kreith [10, Theorem 1]). *Let* J *be an* $n \times n$

nonzero matrix with zeros and ones on the main diagonal and zeros elsewhere. If equation (9) *is oscillatory, and if*

$$\text{(a)} \qquad J[Q - q(x)I]J \geqq 0$$

$$\text{(b)} \qquad J[p(x)I - P]J \geqq 0$$

on $[c, \infty)$ *for some* $c \in \mathscr{R}^+$, *then equation* (1) *is oscillatory.*

Proof. Let α be the vector whose ith component is j_{ii}, the ith entry on the main diagonal of J, $i = 1, 2, \cdots, n$. Let g_α be the positive functional defined by $g_\alpha(A) = \alpha^* A \alpha$ (here* denotes transpose) for all $n \times n$ matrices A (see (6)). Since $\alpha^* J = \alpha^*$ and $J\alpha = \alpha$,

$$g_\alpha(Q - q(x)I) = g_\alpha[J(Q - q(x)I)J]$$

and

$$g_\alpha(p(x)I - P) = g_\alpha[J(p(x)I - P)J] \,.$$

Therefore hypotheses (a) and (b) imply that g_α satisfies (i) and (ii) of the theorem, and we can conclude that (1) is oscillatory.

COROLLARY 2 (cf. E. S. Noussair and C. A. Swanson [12, Theorem 2]). *If there exist diagonal elements* p_{ii} *and* q_{ii} *of* P *and* Q, *respectively, such that*

$$\text{(c)} \qquad \int_0^\infty \frac{dx}{p_{ii}(x)} = \int_0^\infty q_{ii}(x)dx = \infty \,,$$

then equation (1) *is oscillatory.*

Proof. By letting $p(x) = p_{ii}(x)$ and $q(x) = q_{ii}(x)$ in (9), it follows that hypothesis (c) implies (9) is oscillatory. Let ε_i be the vector whose ith component is 1 with all other components being zero. Let g_{ε_i} be the positive functional associated with the vector ε_i as defined by (6). Then $g_{\varepsilon_i}[Q - q(x)I] = g_{\varepsilon_i}[p(x)I - P] = 0$ on $\mathscr{R}^+$. Thus the hypotheses of the theorem are satisfied and (1) is oscillatory.

Before stating the next corollary, it is necessary to introduce some notation. Let $S_{k,n}$ denote the collection of strictly increasing sequences of k integers chosen from the set $\{1, 2, \cdots, n\}$. For any $n \times n$ matrix A and any $\sigma(k) = (i_1, i_2, \cdots, i_n) \in S_{k,n}$, let $\sum_\sigma A$ denote the sum of the entries of the $k \times k$ submatrix of A obtained by deleting all rows and columns of A except the i_j, $j = 1, 2, \cdots, k$, rows and columns.

COROLLARY 3 (cf. W. Allegretto and L. Erbe [1, Corollary 1]). *If there exists an element* $\sigma(k) \in S_{k,n}$ *such that the equation*

$$\{[\sum_{\sigma} P(x)]y'\}' + [\sum_{\sigma} Q(x)]y = 0 \tag{d}$$

is oscillatory, then equation (1) *is oscillatory.*

Proof. Let α be the n-component vector with ones in the $i_1, i_2, \cdots, i_k$ positions and zeros elsewhere, and let g_α be the positive functional associated with α as defined by (6). Put $g = (1/k)g_\alpha$. Then g is a positive functional, and

$$g[P(x)] = (1/k)\sum_{\sigma} P(x), \quad g[Q(x)] = (1/k)\sum_{\sigma} Q(x), \quad \text{and} \quad g(I) = 1\,.$$

Finally, let $p(x) = g[P(x)]$ and $q(x) = g[Q(x)]$. Then hypothesis (d) implies that equation (9) (with this p and q) is oscillatory. Since $g[Q - q(x)I] = g[p(x)I - P] = 0$ on $\mathscr{R}^+$, the hypotheses of the theorem are satisfied. Thus (1) is oscillatory.

Additional oscillation criteria have been obtained by H. C. Howard [9], T. L. Hayden and H. C. Howard [7], and Etgen [5], and since the results of these authors are included in the criteria of Corollaries 1, 2, and 3, these results are also special cases of our theorem.

The next result holds in the general $\mathscr{B}$-valued case.

THEOREM 3.2. *If there exists a* $g \in \mathscr{G}$ *such that the scalar equation*

$$(g[P(x)]Y')' + g[Q(x)]y = 0 \tag{10}$$

is oscillatory, then equation (1) *is oscillatory.*

Proof. Suppose $g \in \mathscr{G}$ satisfies the hypotheses of the theorem. Let $p(x) = g[P(x)]$ and $q(x) = g[Q(x)]$. Since (10) is oscillatory, g is not the zero functional, i.e., $\|g\| = g(I) > 0$. We can assume $g(I) = 1$, for if $g(I) \neq 1$, then $\bar{g} = g/g(I)$ does have norm 1, and

$$(\bar{g}[P(x)]y')' + \bar{g}[Q(x)]y = 0$$

is oscillatory if and only if (10) is. Now,

$$g[Q - q(x)I] = q(x)[1 - g(I)] = 0 = p(x)[g(I) - 1] = g[p(x)I - P]$$

and the hypotheses of Theorem 3.1 are satisfied. Therefore equation (1) is oscillatory, and the theorem is established.

Theorem 3.2 enables one to consider the question of the oscillation of equation (1) in terms of the oscillation of a corresponding second order selfadjoint scalar differential equation of the form (9). Thus the very large number of well-known oscillation criteria for equation (9) can be used to determine associated oscillation criteria for (1). The following corollary is a simple example of the type of oscillation criteria which can be obtained for equation (1) through Theorem 3.2.

COROLLARY (cf. W. Leighton [11] and A. Wintner [19]). *If there exists a* $g \in \mathscr{G}$ *such that*

$$\int_0^\infty \frac{dx}{g[P(x)]} = \int_0^\infty g[Q(x)]dx = \infty ,$$

then equation (1) *is oscillatory.*

4. Nonlinear systems. Let $P, Q; \mathscr{R}^+ \times \mathscr{B} \times \mathscr{B} \rightarrow \mathscr{S}$ be continuous with $P(x, A, B) > 0$ for all $x \in \mathscr{R}^+$, $A, B \in \mathscr{B}$. Let Γ denote the collection of functions $Y: \mathscr{R}^+ \rightarrow \mathscr{B}$ such that Y and $P(x, Y, Y')Y'$ are differentiable and

$$Y^*[P(x, Y, Y')Y'] \equiv [P(x, Y, Y')Y']^*Y$$

on $\mathscr{R}^+$. Consider the second order "nonlinear" differential operator L defined by

$$L[Y] = [P(x, Y, Y')Y']' + Q(x, Y, Y')Y . \tag{11}$$

Nonlinear differential systems of the form

$$L[Y] = 0 , \tag{12}$$

as well as nonlinear differential inequalities of the form

$$Y^*L[Y] \leqq 0 \tag{13}$$

have been considered by a number of authors. See, for example, [1], [3], [5], [10], [12], [16], and [17]. An examination of these results shows that the nonlinear systems are defined in a manner such that the methods developed for linear differential systems of the form (1) are applicable. In this sense, then, the results presented in this paper can be extended to both (12) and (13).

We conclude this paper with the analogue of Theorem 2.1 for the nonlinear differential system (12). The proof of this result depends upon the fact that Lemma 1.1 also holds in the nonlinear case. Since the proof is virtually identical to the proof of Theorem 2.1, it will be omitted.

THEOREM 4.1. *Suppose there is a $g \in \mathscr{G}$ and a solution $V = V(x)$ of*

$$[F(x, Y, Y')Y']' + G[x, Y, Y']Y = 0$$

such that

(i) $g\{V^*[Q(x, Y, Y') - G(x, V, V')]V\} \geqq 0$ *on* $[a, b]$ *for all* $Y \in \Gamma$,

(ii) $g\{V^{*\prime}[F(x, V, V') - P(x, Y, Y')]V'\} \geqq 0$ *on* $[a, b]$ *for all* $Y \in \Gamma$,

(iii) $g[V^*(a)V(a)] = g[V^*(b)V(b)] = 0$,

(iv) *for any* $c \in [a, b]$, $g[V^*(c)V(c)] = 0$ *implies* $g[V^{*\prime}(c)P(c, Y, Y')V'(c)] > 0$ *for all* $Y \in \Gamma$.

If $Y = Y(x)$ *is solution of* (12) *and* $Y \in \Gamma$, *then* $Y(x)$ *is singular for at least one* $x \in [a, b]$.

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