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AN OPERATOR ALGEBRA WHICH IS NOT CLOSED IN THE CALKIN ALGEBRA

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An example is constructed of an operator algebra which is closed in the weak operator topology but has a nonclosed image in the Calkin algebra.

Given a norm closed algebra of operators on Hilbert space, is the sum of this algebra with the ideal of all compact operators a closed set? The answer is well known to be yes for C^*-algebras [2]. It was shown by Arveson [1] to be closed for the discrete triangular algebra, and has recently been shown for all nest algebras by Muhly using his methods of [3]. Also, it is easily established for algebras of subnormal operators and Toeplitz operators. In this note, we will construct an algebra for which the sum is not closed. In fact, this algebra will be closed in the weak operator topology.

Let $\mathscr{A}$ be a norm closed algebra on a Hilbert space $\mathscr{H}$, let $\mathscr{L}\mathscr{C}$ denote the compact operators, and let π denote the quotient map of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{L}\mathscr{C}$. It is apparent that $\mathscr{A} + \mathscr{L}\mathscr{C}$ is closed if and only if the image $\pi(\mathscr{A})$ is closed in the Calkin algebra. We will require the following elementary observation.

PROPOSITION 1. *If $\mathscr{A}$ is a norm closed subalgebra of $\mathscr{L}(\mathscr{H})$, then $\mathscr{A} + \mathscr{L}\mathscr{C}$ is closed if and only if $\mathscr{A}/\mathscr{A} \cap \mathscr{L}\mathscr{C}$ and $\pi(\mathscr{A}) = \mathscr{A} + \mathscr{L}\mathscr{C}/\mathscr{L}\mathscr{C}$ are isomorphic as Banach algebras.*

COROLLARY 1. *If $\mathscr{A}$ is a norm closed algebra such that $\mathscr{A}$ contains no compact operators except 0, then $\mathscr{A} + \mathscr{L}\mathscr{C}$ is closed if and only if the quotient map π is bounded below on $\mathscr{A}$.*

The proof is a simple application of the Closed Graph Theorem.

PROPOSITION 2. *There is a* (commutative) *operator algebra $\mathscr{A}$, closed in the weak operator topology, such that $\mathscr{A} + \mathscr{L}\mathscr{C}$ is not closed.*

Proof. Let $\{e_n\}$ indexed on the integers be a basis for $\mathscr{H}$. Let U be the bilateral shift defined by $Ue_n = e_{n+1}$, and let P be the one dimensional projection onto the span of e_0. Let $A = U + P$, and let $\mathscr{A}$ be the weakly closed algebra generated by A.

We will show that $\mathscr{A}$ does not contain any nonzero compact operators. Suppose $\mathscr{A}$ contains a compact operator K. Then we can find polynomials p_n so that $p_n(A)$ tends to K in the weak operator topology. The restriction of $p_n(A)$ to $\mathscr{K}$, the span of $\{e_n; n \geqq 1\}$ is equal to the restriction of $p_n(U)$ to $\mathscr{K}$ which is unitarily equivalent to the Toeplitz operator of multiplication by p_n on H^2 of the circle. Since the algebra of analytic Toeplitz operators is weakly closed, and no nonzero Toeplitz operator is compact, we conclude that $p_n(A)|_{\mathscr{K}}$ tends to zero in the weak operator topology. In particular we get that $(Ke_i, e_j) = 0$ for all $i \geqq 1$ and all j. Since the same is true for coanalytic Toeplitz operators and $p_n(A^*)|_{\mathscr{K}^\perp}$, we note also that $(Ke_i, e_j) = 0$ for all i and all $j \leqq 0$.

Furthermore, if we write $p_n(z) = \sum_k a_k^{(n)} z^k$, the fact that the Toeplitz operators for p_n tend weakly to zero implies that for each k, $\lim_{n\to\infty} a_k^{(n)} = 0$. So, without loss of generality, we can assume that $a_k^{(n)} = 0$ for $1 \leqq k \leqq N$ and all n. We notice that for (i, j) belonging to the triangle T_N:

$$0 \geqq i \geqq -N\,, \qquad 0 \leqq j \leqq N\,, \qquad j - i \leqq N$$

and for $m \geqq N$, we have $(A^m e_i, e_j) = 1$. Hence $(p_n(A)e_i, e_j)$ is constant for (i, j) in T_N. This must also hold in the limit, so (Ke_i, e_j) is constant for (i, j) in T_N. But this N was arbitrary, so (Ke_i, e_j) is constant on $i \leqq 0$, $j \geqq 0$. Since K is a bounded operator, this must be zero. Together with the previous paragraph, we see that K is zero.

However, $\|A^n\| \geqq \sqrt{n}$ and $\|\pi(A^n)\| = \|\pi(U^n)\| = 1$, so π is not bounded below on $\mathscr{A}$. Thus by Corollary 1, $\mathscr{A} + \mathscr{L}\mathscr{C}$ is not closed.

COROLLARY 2. *There is a separable* (commutative) *norm closed algebra* $\mathscr{B}$ *such that* $\mathscr{B} + \mathscr{L}\mathscr{C}$ *is not closed.*

Proof. Take $\mathscr{B}$ to be the norm closed algebra generated by A.

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