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APPROXIMATE FIBRATIONS AND A MOVABILITY CONDITION FOR MAPS

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In a previous paper the authors defined the approximate homotopy lifting property and studied its implications. This property is a generalization of the homotopy lifting property of classical fiber space theory. Here a necessary and sufficient condition on point-inverses for a map to have the approximate homotopy lifting property for n-cells is given; and the approximate homotopy lifting property for n-cells is shown to imply the approximate homotopy lifting property for all spaces. A corollary is that, in a fairly general context, any two point-inverses of a Serre (weak) fibration have the same shape. By combining these results with results of L. Husch, some conditions are obtained under which a map between manifolds can be approximated by locally trivial fibrations.

1. Introduction and preliminaries. Throughout this paper, $p: E \rightarrow B$ will denote a surjective map between locally compact, separable metric ANR's E and B. We say that p has the *approximate homotopy lifting property* (AHLP) *with respect to the space* X if whenever $h: X \times I \rightarrow B$ and $\tilde{h}: X \times \{0\} \rightarrow E$ are maps such that $p\tilde{h} = h|X \times \{0\}$ and ε is a cover of B, h extends to a map $\tilde{h}: X \times I \rightarrow E$ such that h and $p\tilde{h}$ are ε-close. By a simple modification of [4; XX, 2.4], if p has the AHLP with respect to X, we may choose $\tilde{h}$ to be stationary when h is, i.e., if $h(p(x), t) = p(x)$ for all t, $\tilde{h}(x, t) = x$ for all t. If p has the AHLP for all spaces, we say that p is an *approximate fibration.* (It suffices to have the AHLP for metric spaces by [3, Prop. 1.4].)

Approximate fibrations and approximate lifting were introduced in [3] as an abstraction of the useful lifting properties possessed by UV^k-maps [9], [11], [12]. It is shown in [3] that approximate fibrations have shape theoretic properties analogous to the homotopy theoretic properties of Hurewicz fibrations. For example, under appropriate hypotheses on E and B any two point inverses of p have the same shape, and p induces an exact sequence involving the homotopy groups of E and B and the shape-theoretic homotopy groups of any point inverse of p.

In this paper, we study conditions which imply that a map is an approximate fibration. Section 2 is devoted to showing that, in the case of approximate liftings, the difference between Serre and Hurewicz fibrations disappears; that is, the AHLP for all cells is equivalent to the AHLP for all spaces. In §3 we define a movability condition for maps and study its relationship to approximate liftings. Finally, in §4 we give some applications.

We will use the following notation and terminology. *Map* means continuous function. All *covers* are open covers. If δ is a cover of a space X, and $V \in \delta$, the *star* of V is the union of all members of δ which intersect V. The *star* of δ is the cover whose elements are the stars of members of δ. δ is *star-finite* if each member of δ meets only finitely many members of δ. We say that δ *refines* the cover ε and write $\delta < \varepsilon$ if each member of δ is contained in a member of ε. Also δ *star-refines* ε if the star of δ refines ε. We will often use the fact that each open cover of a separable metric space has a star-finite star refinement [4, p. 167 and p. 255]. If ε is a cover of the space X and $f, g\colon Y \to X$ are maps, we say f and g are *ε-close* provided that for each $y \in Y$ there is $V \in \varepsilon$ such that $f(y), g(y) \in V$. Also f and g are *ε-homotopic* if there is a homotopy H between f and g such that for each $y \in Y$, there is a $V \in \varepsilon$ such that $H(\{y\} \times I) \subset V$. If ε is a cover of B and $f\colon X \to E$, $g\colon X \to B$ are maps such that pf and g are ε-close, we say that f is an *ε-lift of g*. If ε is a positive number, ε-close means close with respect to the cover by open ε-balls. Similar definitions hold for ε-homotopic, ε-lift. If C is a subset of a space X, a *neighborhood* of C is a set which contains C in its interior. If x is a point in a metric space X with metric d, $N(x, r) = \{y \in X \mid d(x, y) < r\}$. If x is a vertex of a complex K, $\mathring{st}(x, K) = \cup\{\operatorname{int}\sigma \mid x \text{ is a vertex of } \sigma\}$. For $b \in B$, F_b denotes $p^{-1}(b)$. In discussing homotopies defined on $X \times I$ we often identify $X \times \{0\}$ with X.

We conclude this section with several lemmas which are analogous to standard facts in the usual theory of fibrations. Since they will be used in a crucial way in the rest of the paper, we include indications of proofs for completeness.

LEMMA 1.1 (see [3], Lemma 1.2). *Suppose that p has the* AHLP *for the metric space X. Given a cover ε of B there is a refinement δ of ε such that if $h\colon X \times I \to B$ and $\tilde{h}\colon X \to E$ are maps such that $\tilde{h}$ is δ-lift of $h \mid X \times \{0\}$, $\tilde{h}$ extends to an ε-lift of h.*

Proof. Let ω be a twice star refinement of ε and let δ be a star refinement of ω such that any two δ-close maps into B are ω-homotopic. Let $q\colon X \to (0, 1)$ be a map such that for each $x \in X$, $h(x \times [0, q(x)])$ is contained in some member of δ. Let $\phi\colon X \times [-1, 0] \to B$ be an ω-homotopy between $p\tilde{h}$ and $h \mid X \times 0$. Let $g\colon X \times [-1, 1] \to B$ be given by $g(x, t) = \begin{cases} \phi(x, t), & -1 \leqq x \leqq 0 \\ h(x, t), & 0 \leqq x \leqq 1. \end{cases}$ There is a $\tilde{g}\colon X \times [-1, 1] \to E$ which extends $\tilde{h}$ and is an ω-lift of g by the AHLP. Define $\tilde{h}\colon X \times I \to E$ by

$$\tilde{h}(x, t) = \begin{cases} \tilde{g}\left(x, \dfrac{2t}{q(x)} - 1\right), & \text{if } 0 \leqq t \leqq \dfrac{q(x)}{2} \\ \tilde{g}(x, 2t - q(x)), & \text{if } \dfrac{q(x)}{2} \leqq t \leqq q(x) \\ \tilde{g}(x, t), & \text{if } q(x) \leqq t \leqq 1 . \end{cases}$$

We need to show that $\tilde{h}$ is an ε-lift of h. If (x, t) is a point in $X \times I$ with $t \leqq q(x)/2$, then $p\tilde{h}(x, t) = p\tilde{g}(x, 2t/q(x) - 1)$ which is ω-close to $g(x, 2t/q(x) - 1) = \phi(x, 2t/q(x) - 1)$. $\phi(x, 2t/q(x) - 1)$ is ω-close to $\phi(x, 0) = h(x, 0)$, and $h(x, 0)$ is ω-close to $h(x, t)$ by our choice of $q(x)$. Since ω twice star refines ε, $p\tilde{h}(x, t)$ is ε-close to $h(x, t)$.

If $q(x)/2 \leqq t \leqq q(x)$, then $y = 2t - q(x) \in [0, q(x)]$, so $p\tilde{h}(x, t) = p\tilde{g}(x, y)$ which is ω-close to $g(x, y) = h(x, y)$, which is ω-close to $h(x, t)$ by our choice of $q(x)$, so $p\tilde{h}(x, t)$ is ε-close to $h(x, t)$. If $t > q(x)$, $p\tilde{h}(x, t)$ is ω-close to $g(x, t) = h(x, t)$.

LEMMA 1.2. *Suppose that p has the* AHLP *for I^q, $q \leqq k$ and let (X, A) be a polyhedral pair with dimension $X \leqq k$. Then given a cover ε of B there is a cover δ of B such that if $h\colon X \times I \to B$ and $\tilde{h}\colon X \times \{0\} \cup A \times I \to E$ are maps such that $p\tilde{h} | X \times \{0\} = h | X \times \{0\}$ and $\tilde{h} | A \times I$ is δ-lift of $h | A \times I$, $\tilde{h}$ extends to an ε-lift of h.*

Proof. Given ε, let $\delta = \delta_k < \delta_{k-1} < \cdots < \delta_0 = \varepsilon$ be a collection of covers so that δ_i plays the role of δ in 1.1 for $\varepsilon = \delta_{i-1}$, $i > 0$. Triangulate X so that A is a subcomplex and build the extension over the cells of $X - A$ as in [19, 7.2.6], using Lemma 1.1 to extend.

COROLLARY 1.3. *If p has the* AHLP *for I^q, $q \leqq k$, then p has the* AHLP *for all polyhedra of dimension k or less.*

Note that the δ in Lemma 1.1 depends only on the cover ε, and on the dimension of X and ε in Lemma 1.2.

Now let B^I denote the space of paths in B with the compact open topology, and let $\Delta_p = \{(\alpha, e) \in B^I \times E \mid p(e) = \alpha(0)\}$. Just as in the case of the usual theory of fibrations [4, XX], we have the following characterization for approximate fibrations.

LEMMA 1.4. *p is an approximate fibration if and only if p has the* AHLP *for Δ_p.*

Sketch of proof. Necessity is clear. For the converse, observe that approximate fibrations can be described in terms of approximate path lifting functions [3], whose existence is equivalent to the AHLP for Δ_p.

II. The equivalence of weak and strong approximate fibrations. In this section, we will show that the AHLP for finite polyhedra implies the AHLP for all spaces. For the time being, let us say that p is a *weak approximate fibration* if p satisfies the AHLP for I^q for all $q < \infty$.

THEOREM 2.1. *Suppose that p is a weak approximate fibration. Then p has the* AHLP *for all countable locally finite polyhedra.*

Proof. Let X be a countable (noncompact) locally finite polyhedron, and let $h: X \times I \to B$, $\tilde{h}: X \to E$ be maps such that $p\tilde{h} = h|X$. Write $X = (\bigcup_{i=0}^{\infty} A_i) \cup (\bigcup_{i=0}^{\infty} B_i)$ such that

(1) $B_0 = \emptyset$, $B_i \neq \emptyset$, $A_j \neq \emptyset$ for $i \geqq 1$, $j \geqq 0$,

(2) each A_i, B_i is a compact polyhedron,

(3) $(B_i \cap \bigcup_{j=0}^{\infty} A_j) \subset A_{i-1} \cup A_i$ for $i \geqq 1$,

(4) $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$ if $i \neq j$.

Given a cover ε of B, let δ_i be the cover of B promised in Lemma 1.2 for ε and the pair $(B_i, B_i \cap (A_{i-1} \cup A_i))$ for each i, and let η_i be a common refinement of δ_i and δ_{i+i}. There is an extension of $\tilde{h}$ to $(\bigcup_{i=0}^{\infty} A_i) \times I$ such that $\tilde{h}|A_i \times I$ is an η_i-lift of $h|A_i \times I$. By Lemma 1.2, we can now extend $\tilde{h}$ over each of the B_i's to an ε-lift of h defined on all of $X \times I$.

THEOREM 2.2. *Suppose that p has the* AHLP *for I^q for all $q \leqq k \leqq \infty$. Then p has the* AHLP *for all separable metric spaces of dimension k or less.*

Before proving 2.2, we need to develop some terminology and several lemmas. The strategy of proof is clear. Since we are only concerned with approximate liftings, we can use nerves of covers and canonical maps to translate a lifting problem for metric spaces into one involving polyhedra. For a discussion of nerves and canonical maps, see [1] or [7]. Our first lemma is a restatement of [7, Theorem 8.1].

LEMMA 2.3. *Let Y be an* ANR, *ω a cover of Y, and $f: X \to Y$ a map, where X is metric. Then X has a cover π such that if ξ is any locally finite refinement of π and N_ξ is the nerve of ξ, there is a map $\psi_\xi: |N_\xi| \to Y$ such that f and $\psi_\xi\phi_\xi$ are ω-homotopic in Y for any canonical map $\phi_\xi: X \to |N_\xi|$.*

The map ψ_ξ is called a *bridge map for f relative to ω.*

LEMMA 2.4. *Let X be a metric space and let ω be a cover of $X \times I$. Then there is a star-finite refinement ξ of ω whose nerve is a triangulation of $P \times I$, where P is the carrier of the nerve of $\xi|X$. Furthermore, there is a canonical map $\phi: X \times I \to P \times I$ for ξ such that $\phi|X: X \to P$ is a canonical map for $\xi|X$.*

Proof. By [6, IX, 5.6], we may assume that ω is a cover whose elements are of the form $V \times J$ where the V are the elements of a star-finite cover γ of X and J is a subinterval of I. Let K be the nerve of γ, let $P = |K|$, let $g: X \to P$ be the barycentric map with respect to γ [1, p. 76], and let $\phi: X \times I \to P \times I$ be defined by $\phi(x, t) = (g(x), t)$. Define the cover ρ of $P \times I$ by $\overset{\circ}{st}(q, K) \times J \in \rho$ if and only if $V \times J \in \omega$ and q is the vertex of K corresponding to V. Choose a subdivision L of $P \times I$ such that $\{\overset{\circ}{st}(q, L) | q$ a vertex of $L\}$ refines ρ. Then it is easy to check that ϕ and $\xi = \{\phi^{-1}(st(q, L)) | q$ is a vertex of $L\}$ satisfy the conclusions of lemma.

The next lemma is proved by a slight alteration of the proof of 1.1. We omit the proof.

LEMMA 2.5. *Suppose X is a metric space and that p satisfies that following condition: given covers δ and ε of E and B and maps $h: X \times I \to B$ and $g: X \to E$ with $pg = h|X$, there is an ε-lift $\tilde{h}$ of h such that $\tilde{h}|X$ is δ-close to g. Then p has the* AHLP *with respect to X.*

Proof of 2.2. We will apply Lemma 2.5. Suppose that ε, δ, g, and h are given as above. Let η' be a star refinement of ε. Let η be a star refinement of η' which is fine enough to play the role of δ in Lemma 1.1 for η'. Let ω_B be a star refinement of η. Refine the cover $p^{-1}\omega_B$ by a cover ω_E which is also a star refinement of δ. Let $\pi_E(\pi_B)$ be a cover of $X(X \times I)$ obtained from applying Lemma 2.3 to ω_E and $g(\omega_B$ and $h)$, let $\pi_E \times I$ be the cover of $X \times I$ defined by $\pi_E \times I = \{V \times I | V \in \pi_E\}$, and let ξ, ϕ, P be as promised be Lemma 2.4 for some common refinement of $\pi_E \times I$ and π_B. We have the diagram

$$\begin{tikzcd} P \arrow[dr, "\beta_0"] & \\ X \arrow[u, "\phi|X"] \arrow[r, "g"] & E \arrow[dd, "p"] \\ P \times I \arrow[dr, "\beta"] & \\ X \times I \arrow[u, "\phi"] \arrow[r, "h"'] & B \end{tikzcd}$$

where β_0, β are bridge maps. By the construction, $p\beta_0$ and $\beta|P$ are η-close, so β has an η'-lift $\tilde{\beta}$ extending β_0 by Lemma 1.1. Then $\tilde{\beta}\phi$ is an ε-lift of h such that g and $\tilde{\beta}\phi|X$ are δ-close, so the proof is complete.

The situation is now as follows. The proof of Theorem 2.2 says that when we have the AHLP for the nerves of a cofinal family of covers of a metric space X, we have the AHLP for X itself. Theorem 2.1 says that the AHLP for cells implies the AHLP for countable polyhedra. We can put these together to get

THEOREM 2.6. *p is a weak approximate fibration if and only if it is an approximate fibration.*

Proof. Suppose p is weak approximate fibration. Since E, B are second countable, $B^I \times E$ is second countable [4], so Δ_p is a second countable metric space. By 2.1 and 2.2 p has the AHLP for Δ_p, so the theorem follows by 1.4.

We now obtain a corollary about Serre fibrations (weak fibrations in [19]). It is known that any two fibers of a Serre fibration over a path connected base have the same weak homotopy type [19, Cor. 7.8.5]. However, the $\sin(1/x)$-circle has the same weak homotopy type as a point. Could these sets be fibers of a Serre fibration? The corollary answers this question negatively in our setting by showing that a stronger relationship holds between the fibers.

COROLLARY 2.7. *If p is a Serre fibration, p is an approximate fibration. If, in addition, B is path connected and p is a proper map, any two fibers of p have the same shape.*

Proof. The first conclusion is immediate from 2.6. The second follows from [3].

III. Movability for maps and approximate lifting. If p is a cell-like map, it follows from [12] that p is an approximate fibration with no further assumptions. As soon as the fibers of p have nontrivial shapes, however, it is clear that some consistency conditions must be placed on the fibers if p is to be an approximate fibration. In this section, we characterize approximate fibrations in terms of a movability condition which is reminiscent of the notions of regularity that have been used to advantage in the study of fibrations [5], [13], [17], [18].

Let F be a compactum in the space E, $x \in F$, and ω the inverse system of neighborhoods of F in E with inclusions as bonding maps. Define $\underline{\pi}_k(F, x) = \varprojlim \{\pi_k(U, x) \mid U \in \omega\}$ for each $k \geqq 0$. If $V \subset U$ are neighborhoods of F and the projection $\underline{\pi}_k(F, x) \to \pi_k(U, x)$ is an isomorphism onto the image of the inclusion induced map $\pi_k(V, x) \to \pi_k(U, x)$ for every $x \in F$, we say that $\underline{\pi}_k F$ is *realized as the image* of $\pi_k V$ in $\pi_k U$. This would occur for example if $\pi_k(V, x)$ and $\pi_k(U, x)$ were the first two terms in a constant inverse sequence with inverse limit $\underline{\pi}_k(F, x)$ for each $x \in F$. A proper map $p: E \to B$ is a *k-movable* map provided that for each $b \in B$ and each neighborhood U_0 of the fiber F_b there are neighborhoods U and V of F_b such that $V \subset U \subset U_0$ and $\underline{\pi}_i F_c$ is realized as the image of $\pi_i V$ in $\pi_i U$, $0 \leqq i \leqq k$, for every fiber $F_c \subset V$. The next lemma gives a useful consequence of k-movability.

LEMMA 3.1. *Suppose $p: E \to B$ is a k-movable map, $k \geqq 1$. Let $b \in B$, U_0 be an arbitrary neighborhood of F_b, and U and V be neighborhoods of F_b in U_0 such that, for any $F_c \subset V$, $\underline{\pi}_i F_c$ is realized as the image of $\pi_i V$ in $\pi_i U$, $0 \leqq i \leqq k$. Then for any fiber $F_c \subset V$ and any neighborhood W_0 of F_c in V, there exist neighborhoods W and Z of F_c in W_0 such that $\alpha_\#: \pi_i(V, Z) \to \pi_i(U, W)$, $1 \leqq i \leqq k$, is the zero homomorphism where $\alpha: (V, Z) \to (U, W)$ is inclusion.*

Proof. Since p is a k-movable map, there are neighborhoods W and Z of F_c in W_0 such that $\underline{\pi}_i F_c$ is realized as the image of $\pi_i Z$ in $\pi_i W$, $0 \leqq i \leqq k$. Let $\phi: Z \to W$, $\chi: W \to V$, and $\psi: V \to U$ be the inclusion maps. By the choice of Z and V, $(\psi\chi)_\#: \operatorname{im} \phi_\# \to \operatorname{im} \psi_\#$ is an isomorphism on π_i for $0 \leqq i \leqq k$. In particular $\chi_\#: \pi_{i-1} W \to \pi_{i-1} V$ is monic on $\operatorname{im} \phi_\#$, and image $(\psi\chi)_\# =$ image $\psi_\#$. Consider the following diagram in which the vertical arrows are inclusion induced and the horizontal arrows are part of the exact homotopy sequence of a pair.

$$\begin{tikzcd}
 & & \pi_i(V, Z) \arrow[r] \arrow[d] & \pi_{i-1} Z \arrow[d] & \\
 & \pi_i V \arrow[r] \arrow[d, "\psi_\#"] & \pi_i(V, W) \arrow[r] \arrow[d] & \pi_{i-1} W \arrow[r, "\chi_\#"] & \pi_{i-1} V \\
\pi_i W \arrow[r, "\psi_\# \chi_\#"] & \pi_i U \arrow[r] & \pi_i(U, W) & &
\end{tikzcd}$$

It is an easy "diagram chasing" argument to prove that $\alpha_\#$ is the zero homomorphism.

Following Kozlowski and Segal [10] we define a metric compactum F to be *k-movable* ($k \geqq 0$) if for any ANR sequence $\{X_i, p_{ij}\}$

associated with F given i there exists $j \geqq i$ such that for any mapping $f_0: K \rightarrow X_j$ from a k-complex and for any $l \geqq j$, there is a map $f_1: K \rightarrow X_l$ such that $p_{ij}f_0 \simeq p_{il}f_1$.

LEMMA 3.2. *Let $p: E \rightarrow B$ be a proper map. If p is a k-movable map, then each fiber F is a k-movable compactum.*

Proof. We choose the ANR sequence $\{X_i\}$ associated with F to be a nested sequence of compact ANR neighborhoods of $F \times \{q\}$ in $E \times Q$ with the inclusions as bonding maps. (See [3], Q denotes the Hilbert cube.) The proof is by induction on k. Let $k = 0$. Given i, choose open sets U and V in E and a connected open set $Q_i \subset Q$ such that

$$F \times \{q\} \subset V \times Q_i \subset U \times Q_i \subset X_i$$

and $\underline{\pi}_0 F$ is realized as the image of $\pi_0 V$ in $\pi_0 U$. Choose j so that $X_j \subset V \times Q_i$. Suppose $f: K \rightarrow X_j$ is a map from a 0-complex K and $l \geqq j$. Select open sets W and Z in E and Q_l in Q such that

$$F \times \{q\} \subset Z \times Q_l \subset W \times Q_l \subset X_l$$

and $\pi_0 F$ is realized as the image of $\pi_0 Z$ in $\pi_0 W$. For each x in K the first coordinate of $f(x)$ can be connected by a path in U to a point in Z by the π_0 realization statements, and the second coordinate of $f(x)$ can be connected to q by the path connectivity of Q_i. Since $U \times Q_i \subset X_i$, f extends to $\hat{f}: K \times I \rightarrow X_i$ such that $\hat{f}(x, 1) \in Z \times Q_l \subset X_l$.

Now suppose $k > 0$. Given i, choose open sets U and V in E and a contractible open set Q_i in Q such that

$$F \times \{q\} \subset V \times Q_i \subset U \times Q_i \subset X_i$$

and $\underline{\pi}_k F$ is realized as the image of $\pi_k V$ in $\pi_k U$. Choose i' such that $X_{i'} \subset V \times Q_{i'}$, and choose $j \geqq i'$ by induction so that for $l' \geqq j$ each map of a $(k-1)$-complex into X_j is homotopic in $X_{i'}$ to a map of the complex into $X_{l'}$. Suppose $f: K \rightarrow X_j$ is a map from a k-complex K and $l \geqq j$. Select open sets Z and W in E and a contractible open set Q_l in Q such that

$$F \times \{q\} \subset Z \times Q_l \subset W \times Q_l \subset X_l$$

and the inclusion induced homomorphism $\alpha_\#: \pi_k(V, Z) \rightarrow \pi_k(U, W)$ is zero. Choose l' such that $X_{l'} \subset Z \times Q_l$. Then f extends to $f: K \times \{0\} \cup K^{k-1} \times I \rightarrow X_{i'}$ such that $f(x, 1) \in X_{l'}$ for each $x \in K^{k-1}$. For each simplex A in K, $f|(A \times \{0\} \cup \operatorname{Bd} A \times I)$ defines an element of

$\pi_k(V \times Q_i, Z \times Q_l)$ which maps to zero in $\pi_k(U \times Q_i, W \times Q_l)$ since $\alpha_\#$ is zero and Q_i and Q_l are contractible. Hence f extends to $\hat{f}: K \times I \to U \times Q_i \subset X_i$ and $f(x, 1) \in W \times Q_l \subset X_l$ for each $x \in K$ as desired to complete the proof.

On the other hand, the converse of Lemma 3.2 is false. For example define $f: S^k \times B^{k+1} \to B^{k+1}$ by $f(x, y) = |y|x$ where $x \in S^k$, $y \in B^{k+1}$ and $|y|$ denotes the norm of y. Each point-inverse of f is homeomorphic to S^k which is k-movable compactum, but f fails to be a k-movable map at $f^{-1}(0)$.

THEOREM 3.3. *If $p: E \to B$ is a k-movable map, then p has the approximate homotopy lifting property for I^i, $0 \leqq i \leqq k$.*

Proof. The proof is by induction on k. We thus first take $k = 0$. Let $g: \{0\} \to E$ and $G: \{0\} \times I \to B$ be given maps such that $pg(0) = G(0, 0)$, and let $\varepsilon > 0$ be a given number. For each $b \in B$, choose $U_b \subset p^{-1}(N(b, \varepsilon/2))$ and an open set V_b such that $F_b \subset V_b \subset U_b$ and $\underline{\pi}_0 F_c$ is realized as the image of $\pi_0 V_b$ in $\pi_0 U_b$ for every fiber $F_c \subset V_b$. Choose a finite subcollection ω of $\{V_b | b \in B\}$ such that ω covers $p^{-1}G(I)$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of I such that the image of each subinterval $p^{-1}G[t_i, t_{i+1}]$ lies in some element of ω which we denote V_i. (These need not be distinct.) Suppose g has been extended to $G_j: [0, t_j] \to E$, $0 \leqq j \leqq n-1$ such that $G_j([t_i, t_{i+1}]) \subset U_i$ for $0 \leqq i \leqq j-1$, and $G_j(t_j) \in V_j$. ($G_0 = g$ begins this induction.) If $j < n-1$, let $W_j = V_j \cap V_{j+1}$; and if $j = n-1$ let $W_j = V_j$. Choose an open set Z_j such that $p^{-1}(G(t_{j+1})) \subset Z_j \subset W_j$ and $\underline{\pi}_0 p^{-1}(G(t_{j+1}))$ is realized as the image of $\pi_0(Z_j)$ in $\pi_0(W_j)$. Hence, every point in V_j can be joined by a path in U_j to a point in Z_j. Thus, G_j can be extended to $G_{j+1}: [0, t_{j+1}] \to E$ such that

$$G_{j+1}([t_j, t_{j+1}]) \subset U_j$$

and

$$G_{j+1}(t_{j+1}) \in V_{j+1} .$$

Let $\widetilde{G} = G_n: [0, 1] \to E$. Then $\widetilde{G}$ extends g; and if $t \in [t_i, t_{i+1}]$, then $G(t) \in p(V_i) \subset p(U_i)$ and $p\widetilde{G}(t) \in p(U_i)$, so $d(p\widetilde{G}, G) < \varepsilon$ by the choice of U_i.

Now assume $k > 0$ and that the theorem is true for integers less than k. Let $g: I^k \to E$ and $G: I^k \times I \to B$ be given maps such that $pg(t) = G(t, 0)$, and let $\varepsilon > 0$ be a given number. For each $b \in B$ choose $U_b \subset p^{-1}(N(b, \varepsilon/2))$ and an open set V_b such that $F_b \subset V_b \subset U_b$ and $\underline{\pi}_k F_c$ is realized as the image of $\pi_k V_b$ in $\pi_k U_b$ for every fiber $F_c \subset V_b$. Choose a finite subcollection ω of $\{V_b | b \in B\}$ which covers $p^{-1}G(I^k \times I)$. Subdivide $I^k \times I$ into rectangles each one of whose images is contained in some $V_b \in \omega$. We order the collection of rectangles lexicographically and denote the result as $\{R_i \mid i = 1, 2, \cdots, n\}$. For each rectangle R_i choose an element of ω denoted V_i such that $p^{-1}G(R_i) \subset V_i$. (These need not be distinct.) Each $R_i = J_i \times [s_i, t_i]$ where J_i is a rectangle in I^k and $[s_i, t_i] \subset I$. Let $m_i = (m_i', t_i)$ where m_i' is the center of J_i. If $t_i \neq 1$, $m_i \in R_i \cap R_j$ for some $j > i$. In this case define $W_i = V_i \cap V_j$. If $t_i = 1$, let $W_i = V_i$. Now choose a neighborhood Z_i of $p^{-1}G(m_i)$ in W_i such that the conclusion of Lemma 3.1 holds. There is a number $\zeta > 0$ such that if $e \in E$ and $d[p(e), G(J_i \times \{t_i\})] < \zeta$ for some i then $e \in W_i$. Also, there is another number $\eta > 0$ such that if $e \in E$ and $d[p(e), G(m_i)] < \eta$ for some i then $e \in Z_i$. Corresponding to $\varepsilon_1 = \min\{\eta/2, \zeta\}$, there is a δ_1 satisfying the conclusion to Lemma 1.1.

Let $A = I^k \times \{0\} \cup \{\operatorname{Bd} J_i \times I \mid i = 1, 2, \cdots, n\}$. By the inductive hypothesis g can be extended to $G'\colon A \to E$ such that $G'(R_i \cap A) \subset V_i$ for each i and $d(pG', G \mid A) < \min\{\delta_1, \zeta\}$. Now suppose G' has been extended to $G_j'\colon A \cup \{R_i i = 1, 2, \cdots, j-1\}$ such that $G_j'(R_i) \subset U_i$ and $G_j'(J_i \times \{t_i\}) \subset W_i$ for $1 \leqq i \leqq j-1$. Let S_j be a rectangle in $J_j \times \{t_j\}$ containing m_j such that diam $G(S_j) < \eta/2$. By the inductive hypothesis and the choice of δ_1, G_j' extends to $G_j''\colon A \cup \{R_i = 1, 2, \cdots, j-1\} \cup (\operatorname{Bd} R_j - \operatorname{Int} S_j) \to E$ such that $d[pG_j''(x), G(x)] < \varepsilon_1$ for $x \in J_j \times \{t_j\} - \operatorname{Int} S_j$. Now $G_j'' \mid \operatorname{Bd} R_j - \operatorname{Int} S_j$ defines an element of $\pi_k(V_j, Z_j)$ since if $x \in \operatorname{Bd} S_j$ then

$$d[pG_j''(x), G(m_j)] \leqq d[pG_j''(x), G(x)] + d[G(x), G(m_j)] < \eta\,.$$

Hence by Lemma 3.2, G_j'' extends to $G_{j+1}'\colon A \cup \{R_i i = 1, 2, \cdots, j\} \to E$ such that $G_{j+1}'(R_j) \subset U_j$ and $G_{j+1}'(J_j \times \{t_j\}) \subset W_j$.

The map $\tilde{G} = G_{n+1}'\colon I^k \times I \to E$ is the desired map since $\tilde{G}$ extends g and if $t \in R_i$ then both $G(t)$ and $p\tilde{G}(t)$ are elements of $p(U_i)$ so $d(p\tilde{G}, G) < \varepsilon$.

COROLLARY 3.4. *If $p\colon E \to B$ is a k-movable map for all k, then p is an approximate fibration.*

Proof. By Theorem 3.3, p has the approximate homotopy lifting property for cells of all dimensions. Hence by Theorem 2.6, p is an approximate fibration.

Using the argument of [3, Theorem 2.4] we also have the following proposition.

PROPOSITION 3.5. *If $p\colon E \to B$ is a proper map with the approximate homotopy lifting property for cells of dimension $\leqq k+1$,* *then p is a k-movable map.*

REMARK. If the definition of k-movability for a map were changed to require only an epimorphism in dimension k rather than an isomorphism, then Theorem 3.3 would still be true as stated and Proposition 3.5 would be true without assuming the approximate homotopy lifting property for cells of dimension $k+1$. This added generality is not however worth the added complication.

We now give another condition on a map, similar to k-movability for all k but more geometric, which also implies that the map is an approximate fibration. A proper map $p: E \to B$ is a *completely movable* map provided that for each $b \in B$ and each neighborhood U of the fiber F_b there is a neighborhood V of F_b in U such that if F_c is any fiber in V and W is any neighborhood of F_c in V then there is a homotopy $H: V \times I \to U$ such that $H(x, 0) = x$ and $H(x, 1) \in W$ for each $x \in V$ and $H(x, t) = x$ for each $x \in F_c$. We say that V is a *movability choice for U and b*.

PROPOSITION 3.6. *Let $p: E \to B$ be a proper map. Then p is completely movable if and only if p is an approximate fibration.*

Proof. If p is an approximate fibration, the argument of Theorem 2.4 of [3] shows that p is completely movable. Suppose that p is completely movable. We shall show that p is k-movable for all k and apply 3.4. Let $b \in B$ be given and let U_0 be a neighborhood of F_b. Let U_1 be a movability choice for U_0 and b and let U_2 be a movability choice for U_1 and b. We claim $U = U_0$ and $V = U_1$ satisfy the definition of k-movability for U_0 and b. To show this, let c be such that $F_c \subset U_2$. Construct a sequence $\{U_i\}_{i=0}^{\infty}$ of neighborhoods of F_c as follows. U_0, U_1, U_2 have been chosen. U_i, $i > 2$ is chosen as a movability choice for U_{i-1} and c such that $\bigcap_{i=0}^{\infty} U_i = F_c$. Note that U_i is also a movability choice for U_{i-1} and c, $i = 1, 2$. Let $j_i: U_i \to U_{i-1}$ be the inclusion. First we show that for $i > 1$, $j_{i*}: \pi_k(U_i) \to \pi_k(U_{i-1})$ takes $\operatorname{im} j_{i+1*}$ isomorphically onto $\operatorname{im} j_{i*}$ for each base point in F_c. The proof that $\operatorname{im} j_{i+1*}$ maps onto $\operatorname{im} j_{i*}$ is an immediate consequence of the movability choices, so we only show that j_{i*} is 1-1 on image j_{i+1*}. To this end, let $\alpha: S^k \to U_{i+1}$ be a pointed map which represents the zero element in $\pi_k(U_{i-1})$. Let h_t be a homotopy of U_{i-1} in U_{i-2} which is fixed on F_c and is such that $h_0 =$ inclusion, $h_1(U_{i-1}) \subset U_{i+1}$. There is a neighborhood W of F_c such that $h_t(W) \subset U_{i+1}$ for all t. Using the movability choice of U_{i+1}, we can find a pointed map $\beta: S^k \to W$ such that $[\alpha] = [\beta]$ in U_i. There is a pointed homotopy $g_t: S^k \to U_{i-1}$ such that $g_0 = \beta$,

$g_1 =$ constant map. Then $h_1 g_t$ is a homotopy of $h_1\beta$ to a constant map in U_{i+1} and $h_t\beta$ is a homotopy of β to $h_1\beta$ in U_{i+1}. Thus $[\beta] = [\alpha] = 0$ in $\pi_k(U_i)$. To complete the proof of the proposition, note that we have shown that the sequence $\cdots \to \pi_k(U_{i+1}) \to \pi_k(U_i) \to$ is constant, so that the inverse limit $\underset{\leftarrow}{\pi}_k(F_c)$ projects isomorphically onto each image j_{i*} for each $i > 1$.

Next we prove that k-movability up to the dimension of the fibers plus one is sufficient to get an approximate fibration.

THEOREM 3.7. *If $p: E \to B$ is a $(k+1)$-movable map and $\dim F_b \leqq k$ for each fiber F_b, then p is an approximate fibration.*

Proof. We will show that p is completely movable. Let $b \in B$. Since p is a $(k+1)$-movable map, F_b is a $(k+1)$-movable compactum by Lemma 3.2. Thus by [10, Theorem 4], F_b is movable. Given a neighborhood $\tilde{U}$ of F_b, we may assume that $\tilde{U} = p^{-1}(U)$ for some neighborhood U of b. By standard homotopy constructions, there is a neighborhood U_0 of b in U such that if $b' \in U_0$, U_0 can be contracted to b' in U keeping b' fixed. Let $\tilde{U}_0 = p^{-1}(U_0)$. By the movability of F_b there is a neighborhood $\tilde{V}$ of F_b in $\tilde{U}_0$ so that $\tilde{V}$ deforms into each neighborhood of F_b staying in $\tilde{U}_0$. Again we may assume that $\tilde{V} = p^{-1}(V)$, where V is a compact connected neighborhood of b in U_0. Let $b' \in V$ be any point and let W be any neighborhood of b' in V. It suffices to show that $\tilde{V}$ deforms into $\tilde{W} = p^{-1}(W)$ in $\tilde{U}$ keeping $F_{b'}$ fixed. By our choice of U_0, there is a homotopy $K: U_0 \times I \to U$ which fixes b' and such that $K_0 = 1_{U_0}$, $K_1(U_0) = b'$. Let W_0 be a neighborhood of b' such that $K_t(W_0) \subset W$ for all t, and let $\tilde{W}_0 = p^{-1}W_0$. Let $\alpha: I \to V$ be a path from b to b'. Let $h: F_b \times I \to B$ be the homotopy given by $h(x, t) = \alpha(t)$, let $g: F_b \times \{0\} \to E$ be given by $g(x, 0) = x$, and let $G: F_b \times I \to \tilde{V}$ be an approximate lifting of h extending g such that $G(F_b \times \{1\}) \subset \tilde{W}_0$. By [1, IV, 8.1], there is a neighborhood W' of F_b and an extension $G: W' \times I \to \tilde{V}$ such that $G(W' \times 1) \subset \tilde{W}_0$. Let $H: \tilde{V} \times I \to \tilde{U}_0$ be a homotopy such that $H_0 = 1_{\tilde{V}}$ and $H_1(\tilde{V}) \subset W'$. Then $\phi: \tilde{V} \times I \to \tilde{U}_0$ given by

$$\phi(x, t) = \begin{cases} H(x, 2t)\,, & 0 \leqq t \leqq \dfrac{1}{2}\,; \quad \text{and} \\[2ex] G(H_1(x), 2t-1)\,, & \dfrac{1}{2} \leqq t \leqq 1 \end{cases}$$

is a homotopy of $\tilde{V}$ which deforms $\tilde{V}$ into $\tilde{W}$ in $\tilde{U}_0$. It remains to alter ϕ so as to hold $F_{b'}$ fixed. Define $f: F_{b'} \times I \times I \to U$ by

$$f(x, t, s) = K(p\phi(x, t), s)$$

and let $\tilde{f}: F_{b'} \times I \to E$ be defined by $\tilde{f} = \phi|(F_{b'} \times I)$. By our choice of $\tilde{W}_0$, we can find an approximate lifting $\tilde{K}: F_{b'} \times I \times I \to \tilde{U}$ of f which extends $\tilde{f}$ and such that $\tilde{K}(F_{b'} \times I \times \{1\}) \subset \tilde{W}$ and $\tilde{K}(F_{b'} \times \{1\} \times I) \subset \tilde{W}$. By stationary lifting we may make $\tilde{K}(x, 0, s) = x$ for all s.

By reparameterizing $I \times I$, we can use $\tilde{K}$ to get a map $K'': F_{b'} \times I \times I \to \tilde{U}$ such that $K''(x, t, 0) = \phi(x, t)$, $K''(x, t, s) = x$ if $t = 0$ or $s = 1$, and $K''(x, 1, s) \in \tilde{W}$ for all t, $x \in F_{b'}$.

Consider the map $\phi': \tilde{V} \times \{1\} \times \{0\} \cup F_{b'} \times \{1\} \times I \to \tilde{W}$ given by $\phi'(x, 1, s) = \begin{cases} \phi(x, 1), & s = 0 \\ K''(x, 1, s), & x \in F_{b'}. \end{cases}$ By [1, IV, 8.1] ϕ' extends to a map $\phi': \tilde{V} \times \{1\} \times I \to \tilde{W}$. Now define

$$\phi'': (\tilde{V} \times I \times \{0\}) \cup (\tilde{V} \times \{0, 1\} \times I) \cup (F'_{b'} \times I \times I) \longrightarrow \tilde{U}$$

by

$$\phi''(x, t, s) = \begin{cases} \phi(x, t), & s = 0; \\ \phi'(x, 1, s) & t = 1; \\ x, & t = 0; \quad \text{and} \\ K''(x, t, s)\,, & x \in F_{b'}\,. \end{cases}$$

By [1] again, ϕ'' extends to $\phi'': \tilde{V} \times I \times I \to \tilde{U}$. Then $\psi: \tilde{V} \times I \to \tilde{U}$, given by $\psi(x, t) = \phi''(x, t, 1)$ is a homotopy which deforms $\tilde{V}$ into $\tilde{W}$ in $\tilde{U}$ keeping $F_{b'}$ fixed.

PROPOSITION 3.8. *Let $p: E \to B$ be a proper map. If p is a k-movable map and a $(k + 1)$-UV map, $k \geqq -1$, then p is a $(k + 1)$-movable map.*

Proof. Given a neighborhood U_0 of some fiber F_b choose neighborhoods U and V of F_b in U_0 such that each singular $(k + 1)$-sphere in V is null-homotopic in U. Then $\underline{\pi}_{k+1}F_c$ is realized as the image of $\pi_{k+1}(V)$ in $\pi_{k+1}(U)$ for any fiber $F_c \subset V$ since both are zero.

Finally, we summarize the results of this and the previous section:

THEOREM 3.9. *For $p: E \to B$ a proper map between locally compact, separable* ANR'*s, the following are equivalent*:

(1) *p is completely movable,*

(2) *p is an approximate fibration,*

(3) *p is k-movable for all k, and*
(4) *p has the* AHLP *for I^q, $0 \leqq q < \infty$.*

IV. An application. In this section, we combine the results of §§2 and 3 with the geometric results of L. Husch in [8] to give some conditions under which a map between manifolds can be approximated by locally trivial fibrations.

THEOREM 4.1. *Suppose E is a closed connected 3-manifold such that each innessential tame 2-sphere in E bounds a 3-cell and let B be a connected 2-manifold. If $f: E \to B$ is a surjective map such that f is 1-movable and each fiber of f has fundamental dimension less than or equal to 1, then f is the uniform limit of locally trivial fiber maps.*

Proof. Each fiber has property k-UV for $k \geqq 2$ since each fiber has a shape representative X in the Menger universal curve and we can use [15, L. 2] to write X as the intersection of cubes-with handles. Hence, f is an approximate fibration by 3.9, 3.3, and 3.4. The theorem thus follows from [8, Theorem A].

THEOREM 4.2. *Let $f: E \to S^1$ be an $n+1$-movable map where E is a closed, connected n-manifold, $n = 3$ or $n \geqq 6$. If $n = 3$ suppose that each tame innessential 2-sphere in E bounds a 3-cell and that for some $b \in S^1$, $\pi_1(F_b) \neq Z_2$; if $n \geqq 6$, suppose that for some $b \in S^1$, the Whitehead and projective class groups of $\pi_1(F_b)$ are trivial. Then f can be uniformly approximated by locally trivial fibrations.*

Proof. By 3.7 and 3.4 f is an approximate fibration, so the theorem follows by [8, Theorems A and B].

Finally, we give a characterization of the n-sphere, $n \geqq 5$, similar to McAuley's version of the Reeb-Milnor theorem [14].

THEOREM 4.3. *Suppose M^n is a closed* (connected) *manifold, $n \geqq 5$, and $f: M^n \to [0, 1]$ is a surjective map such that $f|f^{-1}(0, 1)$ is completely movable and $f^{-1}(i)$ is UV^∞ for $i = 0, 1$. Then M^n is homeomorphic to S^n.*

Proof. Let $A = f^{-1}(0)$, $B = f^{-1}(1)$. We first show that $M - A$ and $M - B$ are contractible. Since A is UV^∞, there exists an $\varepsilon > 0$ so that the inclusion of $f^{-1}([0, \varepsilon))$ into $M - B$ is null-homotopic. By Proposition 3.6, $f|M - (A \cup B)$ is an approximate fibration. Let h_t be a homotopy of $(0, 1)$ such that $h_0 =$ identity, $h_1(0, 1) \subset (0, \varepsilon)$, and h_t is stationary on some interval $(0, \delta) \subset (0, \varepsilon)$. Let

$$g: (M - (A \cup B)) \times \{0\} \longrightarrow M - (A \cup B)$$

be given by $g(x, 0) = x$, $G: (M - (A \cup B)) \times I \to (0, 1)$ by $G(x, t) = h_t f(x)$. By choosing a suitable stationary approximate lift of G, we get a homotopy H_t of $M - (A \cup B)$ which deforms $M - (A \cup B)$ into $f^{-1}(0, \varepsilon)$ and is stationary on some neighborhood of A. Thus H_t extends to a deformation of $M - B$ into $f^{-1}([0, \varepsilon))$ so $M - B$ is contractible. A similar argument shows that $M - A$ is contractible. Since $M - (A \cup B)$ is connected by duality, M is 1-connected by the Van Kampen theorem. To complete the proof, we need only show that M has the homology of S^n. By duality,

$$H_k(M - A, M - (A \cup B)) \cong \check{H}^{n-k}(A \cup B, A) \cong \begin{cases} Z, & k = n \\ 0, & k \neq n . \end{cases}$$

Then $M - (A \cup B)$ has the homology of S^{n-1} by the homology sequence of the pair $(M - A, M - (A \cup B))$, and M has the homology of S^n by the Mayer-Vietoris sequence of $(M, M - A, M - B)$.

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