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TANGENT WINDING NUMBERS AND BRANCHED MAPPINGS

J. R. QUINE

The notion of tangent winding number of a regular closed curve on a compact 2-manifold M is investigated, and related to the notion of obstruction to regular homotopy. The approach is via oriented intersection theory. For N, a 2-manifold with boundary and $F: N \to M$ a smooth branched mapping, a theorem is proved relating the total branch point multiplicity of F and the tangent winding number of $F|_{\partial N}$. The theorem is a generalization of the classical Riemann-Hurwitz theorem.

1. Introduction. Let M be a smooth, connected, oriented 2-manifold and let f and g be regular closed curves on M with the same initial point and tangent direction. An integer obstruction to regular homotopy $\gamma(f, g)$ is derived which is uniquely defined if $M \neq S^2$ and defined mod 2 if $M = S^2$. Let $F(t, \theta)$ be any homotopy such that $F(0, \theta) = f(\theta)$ and $F(1, \theta) = g(\theta)$ and F is smooth on the interior of the unit square. It is shown that $\gamma(f, g) = I(\partial F/\partial\theta, M_0)$, where M_0 is the zero section as a sub-manifold of TM, and I denotes the total number of oriented intersections. This is given interpretation as the number of loops acquired by curves $F(t, \,) = f_t$ in homotopy.

If M is compact and y is not on the image of f, then we define $\operatorname{twn}(f; y)$, a generalization of the tangent winding number. We show that $\gamma(f, g) = \operatorname{twn}(g; y) - \operatorname{twn}(f; y) + I(F, y)\chi(M)$, where is the Euler characteristic. If N is a 2-manifold with boundary and $F: N \to M$ is a smooth branched mapping and $\partial F = F|_{\partial N}$, we show that $\operatorname{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$, where r is the total branchpoint multiplicity and y is not in $F(\partial N)$. We show that the Riemann-Hurwitz theorem follows as a corrollary.

2. The obstruction to regular homotopy. Let M be a smooth, connected 2-manifold with Riemannian metric. Let TM be the tangent bundle and $\widetilde{T}M$ the unit tangent or sphere bundle. Let $f: R \to M$ with $f(\theta) = f(\theta + 1)$ for all $\theta \in R$ be a regular closed curve on M, that is, f has continuously turning, nonzero tangent vector at each point. Given $F: [0, 1] \times R \to M$ continuous with $F(t, \theta) = F(t, \theta + 1)$ for all $\theta \in R$, then F is said to be a regular homotopy if each closed curve $F(t, \,)$ is regular for $0 \leqq t \leqq 1$. We say the curves $f(\theta) = F(0, \theta)$ and $g(\theta) = F(1, \theta)$ are regularly homo-

topic.

Suppose now that f and g are regular closed curves with $f(0) = g(0) = y_0$. Let $\tilde{f}$ and $\tilde{g}$ be the closed curves on $\tilde{T}M$ obtained by taking the unit tangent vector at each point of f and g respectively. Suppose that $\tilde{f}(0) = \tilde{g}(0) = \tilde{y}_0$. Smale [9] has shown that f and g are regularly homotopic iff $\tilde{f}$ and $\tilde{g}$ are homotopic. Using this result we define the obstruction to regular homotopy, $\gamma(f, g)$, as follows.

Let S^1 be the fiber of $\tilde{T}M$ over y_0. Since $\Pi_2(\tilde{T}M) = 0$ for any 2-manifold M (this is clear if $\Pi_2(M) = 0$ and can be verified directly if M is S^2 or the projective plane), we have the following portion of the exact homotopy sequence of the bundle $\tilde{T}M$ over M

$$(1) \qquad 0 \longrightarrow \Pi_2(M) \xrightarrow{\phi} \Pi_1(S^1) \xrightarrow{\mu} \Pi_1(\tilde{T}M) \xrightarrow{\psi} \Pi_1(M) .$$

The sequence (1) induces an isomorphism

$$j\colon \ker \psi \longrightarrow \Pi_1(S^1)/\mathrm{im}\, \phi .$$

If f and g are homotopic, then the product $[\tilde{g}][\tilde{f}]^{-1}$ is in $\ker \psi$. Writing $\alpha = j([\tilde{g}][\tilde{f}]^{-1})$, Smale's theorem says that f and g are regularly homotopic iff $\alpha = 0$.

Now in what follows suppose M is oriented. This gives us a natural choice of orientation on S^1 as the fiber of $\tilde{T}M$ at y_0, which in turn determines a "positively oriented" generator of $\Pi_1(S^1)$. This generator determines an isomorphism of $\Pi_1(S^1)$ with the integers Z. Now $\Pi_2(M) = 0$ unless $M = S^2$. Identifying $\Pi_1(S^1)$ with Z, we see that $\mathrm{im}\, \phi = 2Z$ in case $M = S^2$. (Since the Euler characteristic of S^2 is 2, the fundamental 2-cycle is mapped into 2 by $\varnothing$.) Thus for $M \neq S^?$, α is an integer which we denote $\gamma(f, g)$. If $M = S^2$, α is an element of Z_2. In this case we write $n = \gamma(f, g)$ if the integer n determines the class α in Z_2. We will refer to $\gamma(f, g)$ as the obstruction to regular homotopy. We remark that $\gamma(f, g)$ is only defined if f and g are homotopic. In the next section we will show how to characterize $\gamma(f, g)$ using intersection theory and in a later section we explain its relationship to tangent winding numbers on surfaces as in Reinhart [8] and Chillingworth [1].

3. A characterization of $\gamma(f, g)$. Let f, g and M be as in the previous section. We will continue to assume that M is oriented. Suppose $F(\theta, t)$ is a homotopy, not necessarily regular, with $F(0, \theta) = f(\theta)$ and $F(1, \theta) = g(\theta)$ for all θ. Let K be the square $[0, 1] \times [0, 1]$ and write $F\colon K \rightarrow M$. Now the pullback bundle $F^*(TM)$ is trivial over K, so we can find vector valued functions $v_1, v_2\colon K \rightarrow TM$ such that the ordered pair $(v_1(x), v_2(x))$ is positively oriented in $TM_{F(x)}$

for all $x \in K$. Now consider $\partial/\partial\theta$ as a section of TK and write $F_* \circ (\partial/\partial\theta) = (\partial F/\partial\theta)\colon K \longrightarrow TM$. Write $(\partial F/\partial\theta)(x) = p_1(x)v_1(x) + p_2(x)v_2(x)$ where $p = (p_1, p_2)\colon K \longrightarrow R^2$. By the definition of the map μ in the exact sequence (1), we see that the preimage of $[\tilde{g}][\tilde{f}]^{-1}$ under is just $\deg(p/|p|)|_{\partial K}$, where ∂K is the positively oriented boundary of K, $|\ |$ is the usual Euclidean norm in R^2, and deg is topological degree. Thus $\gamma(f, g) = \deg(p/|p|)|_{\partial K}$. If $M = R^2$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$ then $\gamma(f, g) = \operatorname{twn} g - \operatorname{twn} f$, where twn denotes tangent winding number.

Now suppose x is an isolated zero of $\partial F/\partial\theta$ and D is a closed coordinate disc containing x, but no other zeros of $\partial F/\partial\theta$. We define

$$\operatorname{ind}_x \frac{\partial F}{\partial\theta} = \deg \frac{p}{|p|}\bigg|_{\partial D} .$$

This is easily verified to be independent of the choice of v_1 and v_2. Thus if all the zeros of $\partial F/\partial\theta$ are isolated, then

$$\gamma(f, g) = \sum_{x \in S} \operatorname{ind}_x \frac{\partial F}{\partial\theta}$$

where S is the set of zeros of $\partial F/\partial\theta$.

Now suppose that F is smooth on int K, and let M_0 be the zero section of TM considered as a smooth, oriented 2-submanifold of TM. If $\partial F/\partial\theta$ intersects M_0 transversely at $x \in K$, then $\operatorname{ind}_x \partial F/\partial\theta$ is the same as the oriented intersection number of $\partial F/\partial\theta$ with M_0 at x. (For an explanation of intersection numbers see Guillemin and Pollack [3].) Thus $\gamma(f, g) = I(\partial F/\partial\theta, M_0)$, the total number of oriented intersections of $\partial F/\partial\theta$ with M_0. We remark that $I(\partial F/\partial\theta, M_0)$ is defined even if $\partial F/\partial\theta$ does not intersect M_0 transversely: we simply count the transverse intersections for a "nearby" map. Since $\partial F/\partial\theta(\partial K) \cap M_0 = \varnothing$, the total number of intersections is the same for every "nearby" map. We summarize our results in

THEOREM 1. *Let f and g be regular closed curves on M with the same initial points and initial tangent directions. Suppose f and g are homotopic and $F\colon K \longrightarrow M$ is a homotopy, smooth on* int K, *with $F(0, \theta) = f(\theta)$ and $F(1, \theta) = g(\theta)$, then the obstruction to regular homotopy $\gamma(f, g)$ is equal to $I(\partial F/\partial\theta, M_0)$, the total number of oriented intersections of $\partial F/\partial\theta$ with the zero section M_0.*

We give the following interpretation of Theorem 1. Suppose $\partial F/\partial\theta(x) = 0$ where $x = (t_0, \theta_0)$ and suppose $\partial F/\partial\theta$ intersects M_0 transversely at x. The curve $F(t_0, \theta)$ has a cusp at $\theta = \theta_0$. As t increases, if this cusp represents the appearance of a positively oriented loop or the disappearance of a negatively oriented loop, then the intersection number at x is 1. If it represents the appearance of a negatively oriented loop or the disappearance of a positively oriented loop, then the intersection number is -1. Thus $I(\partial F/\partial\theta, M_0)$ counts the null homotopic loops lost or gained in the homotopy.

4. Tangent winding numbers. We now wish to show the relationship between $\gamma(f, g)$ as defined in the previous section and the notion of tangent winding number of a regular curve with respect to a vector field v on a compact 2-manifold M as in Reinhart [8] and Chillingworth [1]. Suppose f is a regular closed curve on M and v is a vector field on M which vanishes at a single point y not on the image of f. The order that v vanishes at y is clearly $\chi(M)$. We define $\operatorname{twn}_v f$ to be the number of times the tangent of f rotates in relation to v. More specifically, suppose $v = v_1$ and choose vector field v_2 such that (v_1, v_2) is a positively oriented basis except at y, where both vanish to the order $\chi(M)$. Write $df/d\theta = p_1v_1 + p_2v_2$ where $p = (p_1, p_2)\colon S^1 \to R^2$. We then define $\operatorname{twn}_v f$ to be $\deg p/|p|$. It is straightforward to show that $\operatorname{twn}_v f$ depends only upon the choice of y, in fact, it depends only upon the component of $M - f(R)$ in which y lies. Thus, we write $\operatorname{twn}(f; y)$ in place of $\operatorname{twn}_v f$.

THEOREM 2. *Suppose M is compact and let f, g, and F be as in Theorem* 1. *Let $y \in M - f(R) \cup g(R)$, then $\gamma(f, g) = I(\partial F/\partial\theta, M_0) =$ $\operatorname{twn}(g; y) - \operatorname{twn}(f; y) + I(F, y)\chi(M)$.*

Proof. Let v_1 and v_2 be as in the definition of $\operatorname{twn}(f; y)$. Without loss of generality, suppose y is a regular value of F, $(\partial F/\partial\theta) \neq 0$ on $F^{-1}(y)$, and $\partial F/\partial\theta$ has only isolated zeros. Let $x_1, \cdots, x_m$ be the zeros of $\partial F/\partial\theta$ and $\{x_{m+1}, \cdots, x_l\} = F^{-1}(y)$. Write $(\partial F/\partial\theta)(x) = q_1(x)v_1(F(x)) + q_2(x)v_2(F(x))$ for $x \notin F^{-1}(y)$. Let $T_1, \cdots, T_l$ be closed disjoint coordinate discs on M such that $x_k \in T_k$ for $k =$ $1, \cdots, l$. Since v_1 and v_2 vanish of order $\chi(M)$ at y, we have

(a) For $k = m + 1, \cdots$, $\deg(p/|p|)|_{\partial T_k} = \pm\chi(M)$ where the sign is negative if F preserves orientation at x_k, and positive if F reverses orientation at x_k.

(b) For $k = 1, \cdots, m$, $\deg(p/|p|)|_{\partial T_k} = \operatorname{ind}_{x_k}(\partial F/\partial\theta)$.

Now since $p\colon K - \bigcup_{k=1}^m T_k \to R^2$, we have that

$$\deg(p/|p|)|_{\partial K} = \sum_{k=1}^m \deg(p/|p|)|_{\partial T_k}\,.$$

Since by definition $\deg(p/|p|)|_{\partial K} = \operatorname{twn}(g; y) - \operatorname{twn}(f; y)$, the theorem follows from Remarks (a) and (b).

Thus we see that $\operatorname{twn}(g; y) - \operatorname{twn}(f; y)$ determines mod $\chi(M)$ the obstruction to regular homotopy.

5. Branched mappings. Let $\tilde{N}$ be a compact oriented 2-manifold and let $D_1, \cdots, D_n$ be n disjoint copies of the closed unit disc on $\tilde{N}$. Let

$$N = \tilde{N} - \bigcup_{k=1}^{n} \operatorname{int} D_k .$$

Let M be a compact oriented 2-manifold. Let $F: N \rightarrow M$ be smooth. Say F is a branched mapping if F is nonsingular and orientation preserving except at a finite number of points in int N where F behaves locally like the complex analytic mapping z^l, for l an integer $\geqq 2$. The multiplicity of this branch point is defined to be $l-1$.

If $F: N \rightarrow M$ is smooth, we define $\partial F = F|_{\partial N}$. We say ∂F is regular if $F|_{\partial D_k}$ is regular for $k = 1, \cdots, n$. If $y \in M$ is not on the image of ∂F, we define $\operatorname{twn}(\partial F; y) = \sum_{k=1}^{n} \operatorname{twn}(F|_{\partial D_k}; y)$. We wish to investigate the relationship between $\operatorname{twn}(\partial F; y)$ and the total branchpoint multiplicity at branchpoints of F, if F is a branched mapping.

LEMMA 1. *Let $F: \boldsymbol{C} \rightarrow \boldsymbol{C}$ be the complex map z^l, $l \geqq 2$ and let v be a nonzero vector field on $\boldsymbol{C}$, then* $\operatorname{ind}_0 F_* v = l - 1$.

Proof. Let $\tau = \tau(z)$ be a complex valued function giving the vector field v. Identifying $T\boldsymbol{C}$ with $\boldsymbol{C} \times \boldsymbol{C}$, the map $F_* v$ is given by $z \rightarrow (z^l, lz^{l-1}\tau)$. Now $\operatorname{ind}_0 F_* v = (1/2\pi)\int_{|z|=1} d \arg lz^{l-1}\tau$. Since $\tau(z) \neq 0$ for $z \in \boldsymbol{C}$, $\int_{|z|=1} d \arg \tau = 0$. Therefore

$$\operatorname{ind}_0 F_* v = (1/2\pi)\int_{|z|=1} d \arg lz^{l-1} = l - 1 ,$$

which completes the proof of the lemma.

THEOREM 3. *Suppose $F: N \rightarrow M$ is a branched mapping, ∂F is regular, and $y \in M - F(\partial N)$, then*

$$\operatorname{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$$

where r is the total branchpoint multiplicity at branchpoints of F.

Proof. Let $\{x_1, \cdots, x_m\} = B$ be the set of branchpoints of F. Let $\{x_{m+1}, \cdots, x_l\} = F^{-1}(y)$. Note that $l - m = I(F, y)$.

Without loss of generality, assume that y is a regular value of F and $B \cap F^{-1}(y) = \varnothing$. Let v_1 and v_2 be vector fields on M such that (v_1, v_2) is positively oriented on M except at y, where both vector fields vanish to the order $\chi(M)$. Let w be a vector field on N which defines positive orientation on ∂N. Suppose that w vanishes only at $x_0 \notin B \cup F^{-1}(y)$. Write

$$F_* w(x) = p_1(x) v_1(f(x)) + p_2(x) v_2(f(x))$$

where $p = (p_1, p_2)\colon N - F^{-1}(y) \to R^2$. Choose disjoint closed coordinate discs $T_0, \cdots, T_l$ with $x_k \in T_k$ for $k = 0, 1, \cdots, l$.

Since F is regular and preserves orientation except at $x_1, \cdots, x_m$, we have

(a) $\deg(p/|p|)|_{\partial T_0} = \chi(N)$.

(b) For $k = m+1, \cdots, l$, $\deg(p/|p|)|_{\partial T_k} = -\chi(M)$.

Also by Lemma 1 we have

(c) For $k = 1, \cdots, m$, $\deg(p/|p|)|_{\partial T_k} = r_k - 1$ where r_k is the branchpoint multiplicity at x_k.

Finally, by definition

(d) $\deg(p/|p|)|_{\partial N} = \operatorname{twn}(\partial f; y)$.

Since p is a smooth map from $N - \bigcup_{k=0}^{l} T_k$ into R^2, we have also $\deg(p/|p|)|_{\partial N} = \sum_{k=0}^{l} \deg(p/|p|)|_{\partial T_k}$. The theorem now follows from Remarks (a), (b), (c), and (d).

Theorem 3 is intended to be a generalization of results of the type stated by Titus [10], Haefliger [4], and Francis [2]. This is illustrated by the following corollaries.

COROLLARY 1. *If* $F\colon N \to R^2$ *is a branched mapping and* ∂F *is regular, then* $\operatorname{twn} \partial F = \chi(N) + r$ *where* r *is the total multiplicity at branchpoints of* F, *and* twn *is the usual tangent winding number for regular curves in the plane.*

Proof. Let $M = S^2$ in Theorem 3 and identify R^2 with $S^2 - \{y\}$. Then $I(F, y) = 0$, $\operatorname{twn} \partial F = \operatorname{twn}(\partial F; y)$, and the theorem follows.

COROLLARY 2. *If* $F\colon N \to R^2$ *is a sense-preserving immersion and* ∂F *is regular, then* $\operatorname{twn} \partial F = \chi(N)$.

For information on assembling branched mappings see Francis [2] and Marx [5].

To show how the classical Riemann-Hurwitz theorem follows from Theorem 3, we prove

COROLLARY 3 (Riemann-Hurwitz). *If* $\tilde{F}\colon \tilde{N} \to M$ *is a branched* *mapping, where* $\tilde{N}$ *and* M *are compact oriented 2-manifolds, then* $\chi(\tilde{N}) + r = (\deg \tilde{F})\chi(M)$.

Proof. Let y be a regular value of $\tilde{F}$ and D a sufficiently small open disc containing y such that $\tilde{F}^{-1}(D)$ consists of $\deg \tilde{F}$ disjoint discs D_j. Let $N = \tilde{N} - \cup D_j$ and $F = \tilde{F}|_N$. Now $\mathrm{twn}\,(F|_{\partial D_j}; y) = \chi(M) - 1$ for $j = 1, \cdots, \deg \tilde{F}$ and $I(F; y) = 0$. Therefore Theorem 3 gives

$$(\deg \tilde{F})(\chi(M) - 1) = \chi(N) + r = \chi(\tilde{N}) - \deg \tilde{F} + r$$

and the conclusion follows.

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FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306