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ON GENERALIZATIONS OF ALTERNATIVE ALGEBRAS

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Let A be a noncommutative Jordan algebra in which $([x, y], z, z) = 0$ for all x, y, z in A. In this paper the result of Block [4] and Shestakov [13] that a simple finite dimensional such algebra over a field of characteristic $\neq 2$ is either alternative or Jordan is extended to the infinite dimensional case with idempotent. In the case of a noncommutative Jordan algebra satisfying the weaker identity $([x, y], y, y) = 0$ for all x, y in the algebra, a simple finite dimensional such algebra is shown to be commutative, alternative, or an algebra of degree two.

In §2 we consider in the first case, power associative rings which satisfy $(w, x^2, z) = x\cdot(w, x, z)$ and $([x, y], y, y) = 0$, and in the second case, flexible rings satisfying $(w, x^2, z) = x\cdot(w, x, z) + (x, x, [w, z])$. Under certain conditions the rings are shown to be noncommutative Jordan or alternative respectively.

Throughout this paper all algebras considered are assumed to be algebras over a field of characteristic not two and all rings are assumed to be 2-torsion free (i.e., if $2a = 0$ for a in R then $a = 0$).

1. Nearly alternative algebras. Let A be a nonassociative algebra. As is usual for x, y, z in A we denote the associator $(xy)z - x(yz)$ by (x, y, z) and the commutator $xy - yx$ by $[x, y]$. A is flexible if $(x, y, x) = 0$, alternative if $(x, x, y) = (y, x, x) = 0$, and noncommutative Jordan if $(x, y, x) = (x^2, y, x) = 0$.

An algebra A is called simple if A is not a zero algebra, and the only ideals of A are the zero ideal and A itself. In case $A_K = A \otimes_F K$ is simple for every extension $K \supseteqq F$ then A over F is called central simple.

We shall call a noncommutative Jordan algebra A *nearly alternative* if A satisfies the following identity for all x, y, z in A:

$$([x, y], z, z) = 0 . \tag{1.1}$$

Shestakov [13] called such an algebra "almost alternative." However we choose not to use that terminology since Albert [2] had previously called other algebras by the name "almost alternative."

THEOREM 1.1. *If A is a simple nearly alternative algebra with an idempotent $e \neq 1$ then A is commutative or alternative.*

PROOF. It is shown by Shestakov [13] that if A is a noncommutative Jordan algebra with idempotent $e \neq 1$ satisfying $([x, y], z, z) = 0$ then A has the following Peirce decomposition:

$$A = A_1 + A_{10} + A_{1/2\,1/2} + A_{01} + A_0 ,$$

where

$$A_i = \{x \in A \mid ex = xe = ix\},\ i = 0, 1$$

and

$$A_{ij} = \{x \in A \mid ex = ix,\ xe = jx\},\ i + j = 1,\ i, j = 0, \frac{1}{2}, 1 .$$

Shestakov also showed that multiplication of elements of the different components is given in the following chart:

	A_1	A_{10}	$A_{1/2\,1/2}$	A_{01}	A_0
A_1	A_1	A_{10}	$A_{1/2\,1/2}$	0	0
A_{10}	0	A_{01}	A_{01}	A_1	A_{10}
$A_{1/2\,1/2}$	$A_{1/2\,1/2}$	A_{01}	$A_1 + A_{10} + A_{01} + A_0$	A_{10}	$A_{1/2\,1/2}$
A_{01}	A_{01}	A_0	A_{10}	A_{10}	0
A_0	0	0	$A_{1/2\,1/2}$	A_{01}	A_0

that $B = A_{10} + A_{01} + A_{10}A_{10} + A_{01}A_{10}$ is an ideal of A, and that $xy = -yx$ for any x, y in A_{ij} $(i \neq j)$. Furthermore, if $A_{10} = A_{01} = 0$ then $xy = yx$ for all x, y in $A_{1/2\,1/2}$.

Before proceeding to the proof of the theorem, we note the following:

LEMMA 1.1. *If $A_{1/2\,1/2} = 0$ then A is alternative.*

Proof. Since A is simple, the ideal $B = 0$ or $B = A$. If $B = 0$, then $A_{10} = A_{01} = 0$ and $A = A_1 + A_0$. This implies $e = 1$, a contradiction. Hence $B = A$, and $A_1 = A_{10}A_{01}$, $A_0 = A_{01}A_{10}$. We prove A is alternative by showing

$$(x, y, z) = \varepsilon(\sigma)(\sigma(x), \sigma(y), \sigma(z)) \tag{1.2}$$

for all permutations σ, with $\varepsilon(\sigma) = 1$ or -1 respectively for σ even or odd. It suffices to show that (1.2) holds for all possible choices of x, y, z in the component subspaces. Since A is noncommutative Jordan, it has been shown by Florey [5] that A satisfies the identity

$$(w, x^2, z) = x\cdot(w, x, z) \tag{1.3}$$

for all x, w, z in A where $x \cdot y = xy + yx$. A linearization of (1.3) yields

$$(w, x \cdot y, z) = x \cdot (w, y, z) + y \cdot (w, x, z) . \tag{1.4}$$

Now suppose $x_1, y_1, z_1 \in A_1$. Since $y_1 = w_{10}w_{01}$,

$$(x_1, y_1, z_1) = (x_1, w_{10} \cdot w_{01}, z_1) = w_{10} \cdot (x_1, w_{01}, z_1) + w_{01} \cdot (x_1, w_{10}, z_1) = 0 .$$

Hence (A_1, A_1, A_1) alternates. We show that the remaining thirty six associators with A_1 in any position alternate.

By the Peirce multiplication chart and flexibility,

$$\begin{aligned}(A_1, A_1, A_{01}) &= (A_{01}, A_1, A_1) = (A_1, A_{01}, A_1) = (A_1, A_0, A_1) = (A_1, A_0, A_0) \\ &= (A_0, A_0, A_1) = (A_0, A_1, A_0) = (A_0, A_1, A_1) = (A_1, A_1, A_0) \\ &= 0 .\end{aligned}$$

Again from flexibility and the multiplication chart each of the associators (A_1, A_1, A_{10}), (A_1, A_{01}, A_{10}), (A_1, A_0, A_{10}), and (A_0, A_1, A_{01}) alternates.

Now suppose $x_1 \in A_1$, $y_{10}, z_{10} \in A_{10}$. Linearizing (1.1) and the flexible law $(x, y, x) = 0$, we obtain

$$([x, y], z, w) + ([x, y], w, z) = 0 \tag{1.5}$$

and

$$(x, y, z) + (z, y, x) = 0 . \tag{1.6}$$

Then

$$\begin{aligned}(x_1, y_{10}, z_{10}) &= (x_1, y_{10}, [e, z_{10}]) - (y_{10}, x_1, z_{10}) = (z_{10}, x_1, y_{10}) \\ &= ([e, z_{10}], x_1, y_{10}) = -(z_{10}, y_{10}, x_1).\end{aligned}$$

Also $-(z_{10}, y_{10}, x_1) = -(y_{10}, x_1, z_{10}) = -([e, y_{10}], x_1, z_{10}) = (y_{10}, z_{10}, x_1) = -(x_1, z_{10}, y_{10})$ by (1.5) and (1.6). This shows (A_1, A_{10}, A_{10}) alternates. In the same manner (A_1, A_{01}, A_{01}) alternates. Therefore every associator with A_1 or in an analogous manner with A_0 in any position alternates.

We have reduced the proof to the case in which $x, y, z \in A_{ij} + A_{ji}$, $i, j = 0, 1$, $i + j = 1$. Again using (1.5) and (1.6),

$$\begin{aligned}(x_{ij}, y_{ij}, z_{ij}) &= i(x_{ij}, y_{ij}, [e, z_{ij}]) - j(x_{ij}, y_{ij}, [e, z_{ij}]) \\ &= -i(y_{ij}, x_{ij}, [e, z_{ij}]) + j(y_{ij}, x_{ij}, [e, z_{ij}]) \\ &= i([e, z_{ij}], x_{ij}, y_{ij}) - j([e, z_{ij}], x_{ij}, y_{ij}) \\ &= -i(x_{ij}, z_{ij}, [e, y_{ij}]) + j(x_{ij}, z_{ij}, [e, y_{ij}]) \\ &= -(x_{ij}, z_{ij}, y_{ij}) = (y_{ij}, z_{ij}, x_{ij}) .\end{aligned}$$

Also

$$\begin{aligned}(z_{ij}, x_{ij}, y_{ij}) &= i([e, z_{ij}], x_{ij}, y_{ij}) - j([e, z_{ij}], x_{ij}, y_{ij}) \\ &= -i([e, z_{ij}], y_{ij}, x_{ij}) + j([e, z_{ij}], y_{ij}, x_{ij}) \\ &= -(z_{ij}, y_{ij}, x_{ij}) .\end{aligned}$$

Combining these results yields (A_{ij}, A_{ij}, A_{ij}) alternates. The case $x, y \in A_{ij}$ and $z \in A_{ji}$ is proved in a similar manner. Thus (A_{ij}, A_{ij}, A_{ji}) alternates and the lemma is proved.

We are now in a position to complete our main result. Assume A is not alternative. By the simplicity of A, the ideal B must be A or the zero ideal. We found by Lemma 1.1 that $B = A$ implied A was alternative. Thus we are left with $B = 0$ from which it follows that $A_{10} = A_{01} = 0$ and $A = A_1 + A_{1/2\,1/2} + A_0$ We next observe that $[x, A_{1/2\,1/2}] = 0$ for all x in A. For if $x \in A_i$, $i = 0, 1$, $y \in A_{1/2\,1/2}$, then $(x, e, y) = -(y, e, x)$ by flexibility implies $(xe)y - x(ey) = -(ye)x + y(ex)$ so that $xy = yx$. Shestakov [13] has proved $xy = yx$ for x, y in $A_{1/2\,1/2}$.

Next we show that $xy = yx$ for $x, y \in A_i$, $i = 0, 1$. McCrimmon [8] has shown that $D = (A_{1/2}A_{1/2})_0 + A_{1/2} + (A_{1/2}A_{1/2})_1$ is an ideal of A where A is a noncommutative Jordan algebra and $A = A_0 + A_{1/2} + A_1$. In our case $A_{1/2} = A_{1/2\,1/2}$. If $D = 0$ then $A_{1/2\,1/2} = 0$ and $e = 1$, a contradiction. If $D = A$ then $A_1 = (A_{1/2}A_{1/2})_1$ and $A_0 = (A_{1/2}A_{1/2})_0$. Let $x, y \in A_i$, $i = 0,1$. Then $x = (uw)_i$, $y = (zt)_i$ where $u, w, z, t \in A_{1/2}$. In a flexible ring the equation

$$[x\cdot y, z] = x\cdot[y, z] + [x, z]\cdot y \tag{1.7}$$

holds [13]. Thus $2[uw, zt] = [u\cdot w, zt] = u\cdot[w, zt] + [u, zt]\cdot w$. But $zt \in A_1 + A_0$ implies $[w, zt] = 0$ and $[u, zt] = 0$ since $[A_{1/2\,1/2}, A] = 0$. Hence $2[uw, zt] = 0$ and $xy = yx$ in A_i, $i = 0, 1$. A is therefore commutative, and the theorem is proved.

We next consider a noncommutative Jordan algebra A which satisfies the following identity for all x, y in A:

$$([x, y], y, y) = 0\ . \tag{1.8}$$

LEMMA 1.2. *If A is a noncommutative Jordan algebra which satisfies* (1.8) *then the identity*

$$\begin{aligned}(x\cdot y, z, w) + (x, y, z\cdot w) = x\cdot(y, z, w) + y\cdot(x, z, w)\\ + z\cdot(x, y, w) + w\cdot(x, y, z)\end{aligned} \tag{1.9}$$

holds in A.

Proof. We use the Teichmüller identity

$$(x, yz, w) = (xy, z, w) + (x, y, zw) - x(y, z, w) - (x, y, z)w \tag{1.10}$$

and flexibility to obtain $(x, y\cdot z, w) = (x, yz, w) + (x, zy, w) = (x, yz, w) - (w, zy, x) = (xy, z, w) + (x, y, zw) - x(y, z, w) - (x, y, z)w - (wz, y, x) - (w, z, yx) + w(z, y, x) + (w, z, y)x = (x\cdot y, z, w) + (x, y, z\cdot w) - w\cdot(x, y, z) -$

$x \cdot (y, z, w)$. Next we apply (1.4) to $(x, y, z \cdot w)$ to get $y \cdot (x, z, w) + z \cdot (x, y, w) = (x \cdot y, z, w) + (x, y, z \cdot w) = x \cdot (y, z, w) + y \cdot (x, z, w) + z \cdot (x, y, w) + w \cdot (x, y, z)$ and the lemma is proved.

We now follow a process similar to that of Shestakov [13] to classify a central simple finite dimensional noncommutative Jordan algebra satisfying (1.8).

THEOREM 1.2. *If A is a simple finite dimensional noncommutative Jordan algebra satisfying identity* (1.8) *then A is alternative, commutative, or an algebra of degree two.*

Proof. By considering A over its centroid and taking a scalar extension, we see that it is enough to prove the theorem when the base field F is algebraically closed. Then by the known classification of central simple noncommutative Jordan algebras [8] A has one of the following forms:

(1) A is a Jordan algebra;

(2) A is a quasiassociative algebra, i.e., A is isomorphic to B as vector spaces, where B is a complete matrix algebra over F, $\lambda \neq 1/2$ in F, with multiplication $(xy)_A = (x \cdot y)_B + (1-\lambda)(y \cdot x)_B$;

(3) A is an algebra of degree one or two.

Assume A is not commutative, i.e., Case 1 does not hold. Suppose Case 2 holds. The identity $([x, y], y, y) = 0$ implies

$$([x, y]y)y - [x, y]y^2 = 0$$

in A. Then

$$\begin{aligned}[x, y]_A &= (xy)_A - (yx)_A = \lambda x \cdot y + (1-\lambda) y \cdot x - \lambda y \cdot x - (1-\lambda) x \cdot y \\ &= (2\lambda - 1)[x, y]_B .\end{aligned}$$

We have in B,

$$\begin{aligned}(2\lambda - 1)\{\lambda(\lambda[x, y]_B \cdot y + (1-\lambda) y \cdot [x, y]_B) \cdot y + (1-\lambda) y \cdot [\lambda[x, y]_B \cdot y \\ + (1-\lambda) y \cdot [x, y]_B] - \lambda[x, y]_B \cdot y^2 - (1-\lambda) y^2 \cdot [x, y]_B\} = 0 .\end{aligned}$$

This yields $(2\lambda-1)[\lambda^2[x, y]_B \cdot y^2 + \lambda(1-\lambda) y \cdot [x, y]_B \cdot y + (1-\lambda)\lambda y \cdot [x, y]_B \cdot y + (1-\lambda)^2 y^2 \cdot [x, y]_B - \lambda[x, y]_B \cdot y^2 - (1-\lambda) y^2 \cdot [x, y]_B] = 0$ which becomes $(2\lambda - 1)[\lambda(\lambda - 1)[x, y]_B \cdot y^2 + (1-\lambda)(-\lambda) y^2 \cdot [x, y]_B + 2\lambda(1-\lambda) y \cdot [x, y]_B \cdot y] = 0$ or $(2\lambda - 1)\lambda(\lambda - 1)[[x, y]_B \cdot y^2 + y^2 \cdot [x, y]_B - 2y \cdot [x, y]_B \cdot y] = 0$. If $\lambda \neq 0, 1$ then $[x, y]_B \cdot y^2 + y^2 \cdot [x, y]_B - 2y \cdot [x, y]_B \cdot y = 0$. With the elements $x = e_{12}$, $y = e_{11}$, $z = e_{22}$ from the usual matrix basis we have $[e_{12}, e_{11}] \cdot e_{22}^2 + e_{22}^2 \cdot [e_{12}, e_{11}] - 2e_{22} \cdot [e_{12}, e_{11}] \cdot e_{22} = 0$ and $(-e_{12}) \cdot e_{22} - 2e_{22} \cdot (-e_{12}) \cdot e_{22} + e_{22} \cdot (-e_{12}) = 0$ implies $e_{12} = 0$, a contradiction.

Kleinfeld and Kokoris [6] have shown there are no simple noncommutative Jordan algebras of degree one over a field F of characteristic 0. Kokoris has classified the nodal noncommutative Jordan algebras over a field of characteristic $p \neq 2$ [7]. Block's proof that there are no nearly alternative such algebras [4] applies to our case as well.

2. Generalizations of nearly alternative rings. In this section we consider rings more general than nearly alternative rings. We shall call a power associative ring R an *F ring* if R satisfies the following identities:

$$(w, x^2, z) = x\cdot(w, x, z) \tag{2.1}$$

$$([x, y], y, y) = 0 . \tag{2.2}$$

That an F ring is a weaker concept than a nearly alternative ring is shown by an example due to Anderson [3] of a power associative algebra satisfying (2.1) and (2.2) which is not flexible; hence not noncommutative Jordan. We are able to prove, however, that a flexible F ring is noncommutative Jordan.

LEMMA 2.1. *In a flexible F ring the following equations hold:*

$$\begin{aligned}(x\cdot y, z, w) + (x, y, z\cdot w) = x\cdot(y, z, w) + y\cdot(x, z, w)\\ + z\cdot(x, y, w) + w\cdot(x, y, z)\end{aligned} \tag{2.3}$$

$$[x, (x, x, y)] = 0 \tag{2.4}$$

$$(x^2, y, x) = x(x, x, y) - (x, x, xy) . \tag{2.5}$$

Proof. Property (2.3) is proved in Lemma 1.2 using only (2.1) and flexibility. For property (2.4) we use the Teichmüller identity (1.10) twice to get $(xy, z, w) + (x, y, zw) - (x, yz, w) - x(y, z, w) - (x, y, z)w = 0$ and $(wz, y, x) + (w, z, yx) - (w, zy, x) - w(z, y, x) - (w, z, y)x = 0$. Add these equations to obtain by flexibility

$$\begin{aligned}(x, y, [z, w]) - (w, [z, y], x) + ([x, y], z, w) - [x, (y, z, w)]\\ + [w, (x, y, w)] = 0 .\end{aligned} \tag{2.6}$$

Let $z = x, y = x, w = y$ in (2.6). Then it follows that $(x, x, [x, y]) - (y, [x, x], x) + ([x, x], x, y) - [x, (x, x, y)] + [y, (x, x, x)] = 0,$ and $[x, (x, x, y)] = 0.$

To prove property (2.5), let $y = x, w = x, z = y$ in (2.3). Then $(x\cdot x, y, x) + (x, x, y\cdot x) = x\cdot(x, y, x) + x\cdot(x, y, x) + y\cdot(x, x, x) + x\cdot(x, x, y)$ becomes

$$(2.7) \qquad 2(x^2, y, x) + (x, x, y \cdot x) - x \cdot (x, x, y) = 0 .$$

But property (2.4) implies $x \cdot (x, x, y) = 2x(x, x, y)$, and $(x, x, [x, y]) = 0$ implies $(x, x, xy) = (x, x, yx)$. Hence (2.7) becomes $2(x^2, y, x) + 2(x, x, xy) - 2x(x, x, y) = 0$. Since R is 2-torsion free, $(x^2, y, x) = x(x, x, y) - (x, x, xy)$.

THEOREM 2.1. *A flexible F ring is a noncommutative Jordan ring.*

Proof. Since R is power associative $(x, x, x^2) = 0$. Partially linearize $(x, x, x^2) = 0$ to get

$$(2.8) \qquad (x, x, xy) + (x, x, yx) + (x, y, x^2) + (y, x, x^2) = 0 .$$

This implies

$$(2.9) \qquad 2(x, x, xy) + (y, x, x^2) = (x^2, y, x) .$$

Subtracting (2.5) from (2.9) gives $3(x, x, xy) - x(x, x, y) + (y, x, x^2) = 0$ or

$$(2.10) \qquad (x^2, x, y) = 3(x, x, xy) - x(x, x, y) .$$

Now property (2.3) with $z = y = x$, $w = y$ gives $2(x^2, x, y) + 2(x, x, xy) = 6x(x, x, y)$ which becomes

$$(2.11) \qquad (x^2, x, y) = 3x(x, x, y) - (x, x, xy) .$$

Subtracting (2.11) from (2.10) gives $4x(x, x, y) - 4(x, x, xy) = 0$ or

$$(2.12) \qquad x(x, x, y) = (x, x, xy) .$$

Substitute (2.12) in (2.5) to get $(x^2, y, x) = 0$. The theorem is thus proved.

We next consider flexible rings which satisfy the identity

$$(2.13) \qquad (w, x^2, z) = x \cdot (w, x, z) + (x, x, [w, z]) .$$

THEOREM 2.2. *If R is a simple flexible ring which satisfies identity* (2.13) *and $e \neq 1$ is an idempotent of R such that $(e, e, R) = 0$ then R is alternative.*

Proof. Since $(e, e, R) = (R, e, e) = (e, R, e) = 0$, R has Peirce decomposition into the direct sum $R = R_1 + R_{10} + R_{01} + R_0$ where $R_i = \{x \in R \mid ex = xe = ix\}$ for $i = 0, 1$, and $R_{ij} = \{x \in R \mid ex = ix, xe = jx\}$ for $i, j = 0, 1$, $i \neq j$. We first determine the multiplication table of the decomposition as

	R_1	R_{10}	R_{01}	R_0
R_1	R_1	R_{10}	0	0
R_{10}	0	R_{01}	R_1	R_{10}
R_{01}	R_{01}	R_0	R_{10}	0
R_0	0	0	R_{01}	R_0

Linearize identity (2.13) to get

$$(2.14)\qquad \begin{aligned}(w, x\cdot y, z) &= x\cdot(w, y, z) + y\cdot(w, x, z) + (x, y, [w, z]) \\ &\quad + (y, x, [w, z])\,.\end{aligned}$$

Flexibility clearly implies $R_{10}R_1 = R_{01}R_0 = R_0R_{10} = 0$ and $R_{ij}R_j \subseteqq R_{ij}$, $R_iR_{ij} \subseteqq R_{ij}$ for $i, j = 0, 1, i \neq j$. For $x_1, y_1 \in R_1$, $(x_1, y_1, e) = -(e, y_1, x_1)$ implies $(x_1y_1)_{10} = 0$ and $(y_1x_1)_{01} = 0$ or $R_1R_1 \subseteqq R_0 + R_1$. But $(x_1, y_1\cdot e, e) = y_1\cdot(x_1, e, e)+e\cdot(x_1, y_1, e)+(y_1, e, [x_1, e])+(e, y_1, [x_1, e])$ implies $2(x_1, y_1, e) = e\cdot(x_1, y_1, e)$ or $2(x_1y_1)e - 2x_1y_1 = e\cdot[(x_1y_1)e - x_1y_1]$. Hence $(x_1y_1)_0 = 0$ and $R_1R_1 \subseteqq R_1$. In a similar manner $R_0R_0 \subseteqq R_0$. Again by flexibility, $(x_1, y_0, e) = -(e, y_0, x_1)$ and $x_1y_0 \in R_1 + R_0$. Also $(x_1, e, y_0) = -(y_0, e, x_1)$ implies $x_1y_0 = y_0x_1$. Applying (2.14) yields $(x_1, e\cdot y_0, e) = e\cdot(x_1, y_0, e) + y_0\cdot(x_1, e, e) + (e, y_0, [x_1, e]) + (y_0, e, [x_1, e])$ or $e\cdot(x_1, y_0, e) = 0$. This gives $(x_1y_0)_1 = 0$. Again by (2.14), $(y_0, e\cdot x_1, e) = e\cdot(y_0, x_1, e) + x_1\cdot(y_0, e, e) + (e, x_1, [y_0, e]) + (x_1, e, [y_0, e])$ which implies $2(y_0, x_1, e) = e\cdot(y_0, x_1, e)$ or $2(y_0x_1)e - 2y_0x_1 = e\cdot[(y_0x_1)e - y_0x_1]$. This gives $(y_0x_1)_0 = (x_1y_0)_0 = 0$ and $R_1R_0 = R_0R_1 = 0$. Therefore R_0, R_1 are orthogonal subrings. Now by identity (2.14), $(e, x_{10}\cdot e, y_{10}) = x_{10}\cdot(e, e, y_{10})+e\cdot(e, x_{10}, y_{10})+(x_{10}, e, [e, y_{10}])+(e, x_{10}, [e, y_{10}])$ or $x_{10}y_{10} - e(x_{10}y_{10}) = e\cdot[x_{10}y_{10} - e(x_{10}y_{10})] - x_{10}y_{10} + x_{10}y_{10} - e(x_{10}y_{10})$. This becomes $x_{10}y_{10} = e\cdot[x_{10}y_{10} - e(x_{10}y_{10})]$ and $x_{10}y_{10} \in R_{01}$. We have $R_{10}R_{10} \subseteqq R_{01}$. Similarly $R_{01}R_{01} \subseteqq R_{10}$. In the case $R_{10}R_{01}$, apply (2.14) to obtain $(e, x_{10}\cdot e, y_{01}) = x_{10}\cdot(e, e, y_{01})+e\cdot(e, x_{10}, y_{01})+(x_{10}, e, [e, y_{01}])+(e, x_{10}, [e, y_{01}])$ or $(x_{10}y_{01}) - e(x_{10}y_{01}) = e\cdot[x_{10}y_{01} - e(x_{10}y_{01})] - x_{10}y_{01} + e(x_{10}y_{01})$. This becomes $2(x_{10}y_{01}) - 2e(x_{10}y_{01}) = e\cdot[x_{10}y_{01} - e(x_{10}y_{01})]$, and $x_{10}y_{01} \in R_1 + R_{10}$. Apply (2.14) once more to obtain $(e, y_{01}\cdot e, x_{10}) = y_{01}\cdot(e, e, x_{10}) + e\cdot(e, y_{01}, x_{10})+(y_{01}, e, [e, x_{10}])+(e, y_{01}, [e, x_{10}])$ which becomes $e\cdot(e, y_{01}, x_{10}) = 0$ since $(y_{01}, e, x_{10}) = -(x_{10}, e, y_{01}) = 0$. Thus $e\cdot(x_{10}, y_{01}, e) = e\cdot[(x_{10}y_{01})e - x_{10}y_{01}] = 0$ and $(x_{10}y_{01})_{10} = 0$. It follows that $R_{10}R_{01} \subseteqq R_1$ and $R_{01}R_{10} \subseteqq R_0$. In a similar manner using flexibility and identity (2.14) the multiplication chart is verified.

That $B = R_{10} + R_{01} + R_{10}R_{01} + R_{01}R_{10}$ is an ideal of R follows from flexiblity and the multiplicative properties of the subrings. If $B = 0$, $R_{10} = R_{01} = 0$ and $R = R_0 + R_1$, a contradiction. $B = R$ implies $R_{10}R_{01} = R_1$, $R_{01}R_{10} = R_0$, and $(R_1, R_1, R_1) = 0$ since $(x_1, y_1, z_1) = (x_1, y_{10}\cdot y_{01}, z_1) = y_{10}\cdot(x_1, y_{01}, z_1) + y_{01}\cdot(x_1, y_{10}, z_1) + (y_{10}, y_{01}, [x_1, z_1]) + (y_{01}, y_{10}, [x_1, z_1])$ or $(x_1, y_1, z_1) = -([x_1, z_1], y_{01}, y_{10}) = 0$. Similarly $(R_0, R_0, R_0) = 0$.

For alternativity, we first consider (R_1, R_{10}, R_{10}). We observe that for $x_{10} \in R_{10}$, $(x_{10}, x_{10}, e) = -(e, x_{10}, x_{10})$ implies $x_{10}^2 = 0$ and $(x_{10} + y_{10})^2 = 0$ implies $x_{10}y_{10} = -y_{10}x_{10}$. Therefore $(x_1, y_{10}, z_{10}) = -(z_{10}, y_{10}, x_1)$ implies $(x_1y_{10})z_{10} = -(z_{10}y_{10})x_1$ and $(x_1, z_{10}, y_{10}) = (x_1z_{10})y_{10} = -(y_{10}z_{10})x_1 = (z_{10}y_{10})x_1 = (x_1, y_{10}, z_{10})$. Also $(z_{10}, x_1, y_{10}) = -z_{10}(x_1y_{10}) = (x_1y_{10})z_{10} = (x_1, y_{10}, z_{10}) = -(z_{10}, y_{10}, x_1)$. Therefore we have $(x_1, y_{10}, z_{10}) = -(z_{10}, y_{10}, x_1) = (z_{10}, x_1, y_{10}) = -(y_{10}, x_1, z_{10}) = (y_{10}, z_{10}, x_1) = -(x_1, z_{10}, y_{10})$, and (R_1, R_{10}, R_{10}) alternates. Similarly (R_1, R_{01}, R_{01}) alternates.

That all other associators with at least one R_1 in any position alternate follows from the chart, flexibility, and $(R_1, R_1, R_1) = 0$. Likewise we can see that all associators involving at least one R_0 in any position alternate.

It remains to verify (R_{ij}, R_{ij}, R_{ij}) and (R_{ji}, R_{ij}, R_{ij}) with $i, j = 0, 1$, $i \neq j$ alternate. Letting $x_{10}, y_{10}, z_{10} \in R_{10}$ and applying (2.14) we obtain $(x_{10}, y_{10} \cdot e, z_{10}) = y_{10} \cdot (x_{10}, e, z_{10}) + e \cdot (x_{10}, y_{10}, z_{10}) + (y_{10}, e, [x_{10}, z_{10}]) + (e, y_{10}, [x_{10}, z_{10}])$ which becomes $(x_{10}y_{10})z_{10} - x_{10}(y_{10}z_{10}) = y_{10} \cdot (-x_{10}z_{10}) + e \cdot [(x_{10}y_{10})z_{10} - x_{10}(y_{10}z_{10})] + 2y_{10}(x_{10}z_{10}) - 2e[y_{10}(x_{10}z_{10})]$ or $(x_{10}y_{10})z_{10} - x_{10}(y_{10}z_{10}) = 2x_{10}(y_{10}z_{10}) - y_{10} \cdot (x_{10}z_{10})$. Since R is a direct sum, $(x_{10}y_{10})z_{10} = -(x_{10}z_{10})y_{10}$ and $x_{10}(y_{10}z_{10}) = -y_{10}(x_{10}z_{10})$. This implies $(x_{10}, z_{10}, y_{10}) = (x_{10}z_{10})y_{10} - x_{10}(z_{10}y_{10}) = -(z_{10}x_{10})y_{10} + z_{10}(x_{10}y_{10}) = -(z_{10}, x_{10}, y_{10})$. Also $(y_{10}, x_{10}, z_{10}) = (y_{10}x_{10})z_{10} - y_{10}(x_{10}z_{10}) = -(x_{10}y_{10})z_{10} + x_{10}(y_{10}z_{10}) = -(x_{10}, y_{10}, z_{10})$. We have $(x_{10}, y_{10}, z_{10}) = -(z_{10}, y_{10}, x_{10}) = -(y_{10}, x_{10}, z_{10}) = (z_{10}, x_{10}, y_{10}) = -(x_{10}, z_{10}, y_{10}) = (y_{10}, z_{10}, x_{10})$. This proves (R_{10}, R_{10}, R_{10}) and similarly (R_{01}, R_{01}, R_{01}) alternate. Lastly, let $x_{10}, y_{10} \in R_{10}$, $z_{01} \in R_{01}$. It follows that $(z_{01}, y_{10}, x_{10}) = -(x_{10}, y_{10}, z_{01}) = -(x_{10}y_{10})z_{01} = (y_{10}x_{10})z_{01} = (y_{10}, x_{10}, z_{01}) = -(z_{01}, x_{10}, y_{10})$. Also by (2.14), $(e, x_{10} \cdot z_{01}, y_{10}) = x_{10} \cdot (e, z_{01}, y_{10}) + z_{01} \cdot (e, x_{10}, y_{10}) + (x_{10}, z_{01}, [e, y_{10}]) + (z_{01}, x_{10}, [e, y_{10}])$ which becomes $0 = z_{01} \cdot (x_{10}y_{10}) + (x_{10}, z_{01}, y_{10}) + (z_{01}, x_{10}, y_{10})$. But $z_{01} \cdot (x_{10}y_{10}) = 0$ since $x_{10}y_{10} \in R_{01}$ and $(x_{10}, z_{01}, y_{10}) = -(z_{01}, x_{10}, y_{10})$. We therefore have $(x_{10}, y_{10}, z_{01}) = -(z_{01}, y_{10}, x_{10}) = (y_{10}, z_{01}, x_{10}) = -(x_{10}, z_{01}, y_{10}) = (z_{01}, x_{10}, y_{10}) = -(y_{10}, x_{10}, z_{01})$. This shows that (R_{01}, R_{10}, R_{10}) and by a reversal of subscripts that (R_{10}, R_{01}, R_{01}) alternate. The theorem is proved.

In the case of a finite dimensional algebra A we can prove the following:

THEOREM. *If A is a simple flexible finite dimensional power associative algebra over an algebraically closed field F of characteristic $\neq 2, 3$ which satisfies $(w, x^2, z) = x \cdot (w, x, z) + (x, x, [w, z])$ and $e \neq 1$ is an idempotent of A then A is noncommutative Jordan.*

Proof. Oehmke [9, 10] has shown that a simple, flexible, stable, finite dimensional power associative algebra over an algebraically closed field of characteristic $\neq 2, 3$ is a noncommutative Jordan algebra. We show A is stable, i.e., $A_iA_{1/2} \subseteq A_{1/2}$ and $A_{1/2}A_i \subseteq A_{1/2}$ for $i = 0, 1$.

Since A is power associative, $A = A_1 + A_{1/2} + A_0$ where $A_i = \{x \in A \mid ex + xe = ix\}$. Also $A_1A_0 = A_0A_1 = 0$, $A_iA_i \subseteq A_i$ for $i = 0, 1$, $A_{1/2}A_{1/2} \subseteq A_1 + A_0$, and $A_iA_{1/2} \subseteq A_{1/2} + A_{1-i}$, $A_{1/2}A_i \subseteq A_{1/2} + A_{1-i}$ for $i = 0, 1$.

By flexibility $(e, x_{1/2}, e) = 0$ implies $e(x_{1/2}e) = (ex_{1/2})e$, and $x_{1/2} = ex_{1/2} + x_{1/2}e$ implies $ex_{1/2} = e(ex_{1/2}) + e(x_{1/2}e)$ or $ex_{1/2} = e(ex_{1/2}) + (ex_{1/2})e$. Hence $ex_{1/2} \in A_{1/2}$ and $x_{1/2}e \in A_{1/2}$.

Next we consider $A_1A_{1/2}$. By identity (2.14), for $x \in A_1$, $y \in A_{1/2}$, $(x, y\cdot e, e) = y\cdot(x, e, e) + e\cdot(x, y, e) + (y, e, [x, e]) + (e, y, [x, e])$ and $(x, y, e) = e\cdot(x, y, e)$, i.e., $(x, y, e) \in A_{1/2}$. We also have $(x, e\cdot e, y) = 2(x, e, y) = 2e\cdot(x, e, y) + 2(e, e, [x, y])$. Therefore $(x, e, y)_1 = (x, e, y)_0 = 0$ and $(x, e, y) \in A_{1/2}$. Again by (2.14), $(e, x\cdot e, y) = x\cdot(e, e, y) + e\cdot(e, x, y) + (x, e, [e, y]) + (e, x, [e, y])$. Since A is a direct sum and $(x, e, [e, y]) \in A_{1/2}$, it follows that

$$(2.15) \qquad 2(e, x, y)_0 = [x\cdot(e, e, y)]_0 + (e, x, [e, y])_0 \,.$$

Apply the Teichmüller identity (1.10) to get

$$(2.16) \qquad (e, x, ey) = -(ex, e, y) + (e, xe, y) + e(x, e, y) + (e, x, e)y$$

and

$$(2.17) \qquad (e, x, ye) = -(ex, y, e) + (e, xy, e) + e(x, y, e) + (e, x, y)e \,.$$

But the A_0 components of (2.16) give $(e, x, ey)_0 = (e, x, y)_0$ and those of (2.17) give $(e, x, ye)_0 = [(e, x, y)e]_0 = 0$. Substituting these results in (2.15) yields $2(e, x, y)_0 = [x\cdot(e, e, y)]_0 + (e, x, ey)_0 - (e, x, ye)_0$. This becomes

$$(2.18) \qquad (e, x, y)_0 = [x\cdot(e, e, y)]_0 \,.$$

Now consider $[x\cdot(e, e, y)]_0$. As in [4], $x\cdot(e, e, y) = xy - (x, e, y) - x(ey) + (x, e, ey) + e(yx) + (e, y, x) - e[(ey)x] - (e, ey, x)$. All terms on the right side except the first and third are in $A_{1/2}$. Therefore

$$(2.19) \qquad [x\cdot(e, e, y)]_0 = (xy)_0 - [x(ey)]_0 \,.$$

Substitute (2.19) into (2.18) to get

$$(2.20) \qquad (e, x, y)_0 = (xy)_0 - [x(ey)]_0 \,.$$

Identity (2.20) expanded becomes $(xy)_0 - [e(xy)]_0 = (xy)_0 - [x(ey)]_0$. Since $[e(xy)]_0 = 0$ it follows that $[x(ey)]_0 = 0$, and since $(x, e, y) \in A_{1/2}$, it follows that $(xy)_0 - [x(ey)]_0 = 0$. We have therefore $(xy)_0 = 0$ or $A_1A_{1/2} \subseteq A_{1/2}$. Identity (2.20) becomes $(e, x, y)_0 = 0$ and by flexibility $(y, x, e)_0 = 0$. Thus $[(yx)e]_0 - (yx)_0 = 0$. This gives $A_{1/2}A_1 \subseteq A_{1/2}$. In a similar manner $A_0A_{1/2} \subseteq A_{1/2}$, $A_{1/2}A_0 \subseteq A_{1/2}$ and A is stable. The theorem is therefore proved.

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