

## A NECESSARY CONDITION ON THE EXTREME POINTS OF A CLASS OF HOLOMORPHIC FUNCTIONS

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We prove that if  $f$  is an extreme point of the class  $N(G, p)$  of all holomorphic functions  $g$  on the region  $G$  in  $C^n$  such that  $\operatorname{Re} g > 0$  on  $G$  and  $g(p) = 1$ , and if three conditions on  $G$  hold, then the Cayley transform  $(f - 1)/(f + 1)$  is irreducible.

### 1. Introduction.

1.1. Let  $G$  be a nonempty open connected subset of  $C^n$ . We will denote by  $H(G)$  the class of all holomorphic functions on  $G$ , we will denote by  $N(G)$  the class of all  $f$  in  $H(G)$  such that  $\operatorname{Re} f > 0$ , and we will denote by  $W(G)$  the class of all  $f$  in  $H(G)$  such that  $|f| \leq 1$ . Thus  $W(G)$  is the closed unit ball of the Banach algebra

$$\{f: f \in H(G), \|f\| < \infty\}$$

where

$$\|f\| = \sup \{|f(z)|: z \in G\}.$$

Let  $p \in G$  and let

$$N(G, p) = \{f: f \in N(G), f(p) = 1\}.$$

Thus  $N(G, p)$  is convex (and compact with respect to the compact open topology). Furthermore let

$$W(G, p) = \{g: g \in W(G), g(p) = 0\}.$$

Thus

$$N(G, p) = \{(1 + g)/(1 - g): g \in W(G, p)\}.$$

We will denote (as is usual) by  $\operatorname{Aut}(G)$  the group of holomorphic homeomorphisms of  $G$ , and by  $H^1(G, \mathbf{Z})$  the first Čech cohomology group of  $G$  with integer coefficients. We recall [1, p. 769] that if  $g \in W(G)$ , if  $g$  is not constant, and if  $g$  is not the product of two nonconstant members of  $W(G)$ , then  $g$  is said to be irreducible. The purpose of this paper is to state and prove the following theorem.

**THEOREM 1.2.** *Let  $G$  be such that (a)  $\operatorname{Aut}(G)$  acts transitively on  $G$ , and (b)  $H^1(G, \mathbf{Z}) = 0$ . Furthermore let  $g \in W(G, p)$  and let  $f = (1 + g)/(1 - g)$ . If  $N(G, p) \neq \{1\}$  and if  $f$  is an extreme point of  $N(G, p)$ , then  $g$  is irreducible.*

1.3. With regard to Theorem 1.2 if

$$G = \{z: z \in \mathbf{C}^n, |z| < 1\}$$

and if  $n \geq 2$ , then the fact that  $g$  is irreducible does not imply that  $f$  is extreme. For example if  $g(z) = z_1$ , then (as is easily seen)  $g$  is irreducible. Whereas if  $h(z) = z_2^2/2$ , then  $g + e^{i\theta}h \in W(G, 0)$  and

$$f = \frac{1}{2\pi} \int_0^{2\pi} [(1 + g + e^{i\theta}h)/(1 - g - e^{i\theta}h)] d\theta .$$

If

$$G = \{z: z \in \mathbf{C}, |z| < 1\} ,$$

then (as is well known)  $f$  is extreme if and only if  $g$  is irreducible.

We do not know if Theorem 1.2 holds if we omit one or both of the hypotheses (a) and (b). With regard to this we remark that these hypotheses are not used in §§ 2 and 3. They are used only in § 4.

## 2. The beginning of the proof of Theorem 1.2.

**PROPOSITION 2.1.** *If  $N(G, p) \neq \{1\}$ , then 1 is not an extreme point of  $N(G, p)$ .*

*Proof.* We have  $W(G, p) \neq \{0\}$ .

Furthermore if  $g \in W(G, p)$ , then

$$\operatorname{Re}(1 + g) = 1 + \operatorname{Re} g > 0 ,$$

hence  $1 + g \in N(G, p)$ . Likewise  $1 - g \in N(G, p)$ , hence 1 is not an extreme point of  $N(G, p)$ .

**LEMMA 2.2.** *Let  $g \in W(G, p)$  and let  $f = (1 + g)/(1 - g)$ . If  $g = h^2$  where  $h \in H(G)$  and if  $N(G, p) \neq \{1\}$ , then  $f$  is not an extreme point of  $N(G, p)$ .*

*Proof.* If  $j = 2h/(1 - g)$ , then  $f + j = (1 + h)/(1 - h)$  and  $f - j = (1 - h)/(1 + h)$ , hence  $f + j, f - j \in N(G, p)$ . Thus if  $h \neq 0$ , then  $f$  is not an extreme point of  $N(G, p)$ . If  $h = 0$ , then by Proposition 2.1  $f$  is not an extreme point of  $N(G, p)$ .

2.3. We recall the following theorem of Ahern and Rudin (which is proved by means of the theory of normal families) [1, Lemma 3.3 and the postscript on p. 777].

**PROPOSITION 2.4.** *If  $g \in W(G, p)$  and if  $g \neq 0$ , then  $g = ab$  where*

$a \in W(G, p)$ ,  $a$  is irreducible, and  $b \in W(G)$ .

2.5. We will denote (as is usual) by  $D$  the class of all  $z$  in  $C$  such that  $|z| < 1$ , by  $\bar{D}$  the class of all  $z$  in  $C$  such that  $|z| \leq 1$ , and by  $T$  the class of all  $z$  in  $C$  such that  $|z| = 1$ .

PROPOSITION 2.6. *If  $z, w \in \bar{D}$ , then*

$$\left| zw + \frac{z+w}{2} \right| \leq \left| 1 + \frac{z+w}{2} \right|.$$

*Proof.* Let  $2 < t < \infty$ . If we define  $f: \bar{D} \times \bar{D} \rightarrow C$  by

$$f(z, w) = \left( zw + \frac{z+w}{t} \right) / \left( 1 + \frac{z+w}{t} \right),$$

then  $|f| = 1$  on  $T \times T$ , hence by the principle of maximum  $|f| \leq 1$  which will suffice to prove Proposition 2.6.

2.7. We recall that if  $z, w \in C$  and if  $z \neq 1$ , then

$$(2.1) \quad \operatorname{Re} [(1+z+2w)/(1-z)] = (|1+w|^2 - |z+w|^2)/|1-z|^2.$$

LEMMA 2.8. *Let  $g \in W(G, p)$  and let  $f = (1+g)/(1-g)$ . If  $N(G, p) \neq \{1\}$  and if  $f$  is an extreme point of  $N(G, p)$ , then  $g = a(s-a)/(1-\bar{s}a)$  where  $a \in W(G, p)$ ,  $a$  is irreducible,  $s \in C$ ,  $s + \bar{s} = 0$ , and  $0 < |s| \leq 1$ .*

*Proof.* By Proposition 2.1  $g \neq 0$ . Let  $g = ab$  where  $a \in W(G, p)$  and  $b \in W(G)$ .

Let  $\lambda$  in  $T$  be such that  $\operatorname{Re} [\bar{\lambda}b(p)] = 0$ , let  $c = \lambda a$ , and let  $d = \bar{\lambda}b$ . Thus  $g = cd$ . Let  $s = d(p)$ . Thus  $s + \bar{s} = 0$  and  $|s| \leq 1$ . If  $h = [(c+d)/(1-g)] - s$ , then  $h(p) = 0$ . Furthermore

$$\operatorname{Re}(f+h) = \operatorname{Re}(f+h+s) = \operatorname{Re} [(1+g+c+d)/(1-g)],$$

hence by (2.1)

$$\operatorname{Re}(f+h) = \left( \left| 1 + \frac{c+d}{2} \right|^2 - \left| cd + \frac{c+d}{2} \right|^2 \right) / |1-g|^2,$$

hence by Proposition 2.6  $f+h \in N(G, p)$ . Likewise  $f-h \in N(G, p)$ , thus  $h = 0$ . We have

$$c+d = s(1-g) = s(1-cd),$$

hence

$$d = (s-c)/(1+sc) = (s-c)/(1-\bar{s}c).$$

If  $s = 0$ , then  $d = -c$ , hence  $g = (ic)^2$ . Thus by Lemma 2.2  $s \neq 0$ . Lemma 2.8 now follows from Proposition 2.4.

### 3. The action of $\text{Aut}(G)$ on $N(G, p)$ .

3.1. We define  $\alpha: \text{Aut}(G) \times N(G) \rightarrow N(G, p)$  by

$$\alpha(Z, f) = (f \circ Z - \lambda)/\mu$$

where  $\lambda = i \text{Im } f(Z(p))$  and  $\mu = \text{Re } f(Z(p))$ . We will omit the proof (which is straightforward) of the following proposition.

**PROPOSITION 3.2.** *If  $Y, Z \in \text{Aut}(G)$ , if  $f, g \in N(G)$ , and if  $t > 0$ , then*

$$(3.1) \quad \alpha(Z, f + g) = a\alpha(Z, f) + b\alpha(Z, g)$$

where  $a, b > 0$  and  $a + b = 1$ ,

$$(3.2) \quad \alpha(Z, tf) = \alpha(Z, f),$$

and

$$(3.3) \quad \alpha(YZ, f) = \alpha(Z, \alpha(Y, f)).$$

Furthermore if  $h \in N(G, p)$ , then

$$(3.4) \quad \alpha(I, h) = h.$$

**PROPOSITION 3.3.** *Let  $f \in N(G, p)$  and let  $Z \in \text{Aut}(G)$ . If  $f$  is an extreme point of  $N(G, p)$ , then  $\alpha(Z, f)$  is an extreme point of  $N(G, p)$ .*

*Proof.* Let  $Y = Z^{-1}$ . If  $\alpha(Z, f) = ag + bh$  where  $a, b > 0$  and  $g, h \in N(G, p)$ , then by (3.1) and (3.2)

$$\alpha(Y, \alpha(Z, f)) = c\alpha(Y, g) + d\alpha(Y, h)$$

where  $c, d > 0$  and  $c + d = 1$ . Furthermore by (3.3) and (3.4)

$$\alpha(Y, \alpha(Z, f)) = f,$$

hence

$$f = c\alpha(Y, g) + d\alpha(Y, h).$$

Thus  $f = \alpha(Y, g)$ , hence

$$\alpha(Z, f) = \alpha(Z, \alpha(Y, g)) = g$$

which completes the proof of Proposition 3.3.

3.4. Although a paraphrase of Proposition 3.3 is proved in [2], it seemed worthwhile to give a proof here.

We define  $\beta: \text{Aut}(G) \times W(G, p) \rightarrow W(G, p)$  by

$$\beta(Z, g) = \frac{1 - \bar{\mu}}{1 - \mu} \frac{g \circ Z - \mu}{1 - \bar{\mu}g \circ Z}$$

where  $\mu = g(Z(p))$ . We will omit the proof (which is straightforward) of the following proposition.

PROPOSITION 3.5. *Let  $g \in W(G, p)$  and let  $f = (1 + g)/(1 - g)$ . If  $Z \in \text{Aut}(G)$ , then*

$$\alpha(Z, f) = [1 + \beta(Z, g)]/[1 - \beta(Z, g)].$$

4. The end of the proof of Theorem 1.2.

PROPOSITION 4.1. *Let  $a \in W(G)$ . If  $a$  is irreducible and if  $H^1(G, Z) = 0$ , then  $a(G) = D$ .*

*Proof.* If  $t \in D - a(G)$ , then since  $H^1(G, Z) = 0$ ,  $(a - t)/(1 - \bar{t}a) = b^2$  where  $b \in W(G)$ . We have  $a = (b^2 + t)/(1 + \bar{t}b^2)$ . Thus if  $t = -s^2$ , then

$$a = \frac{b - s}{1 - \bar{s}b} \frac{b + s}{1 + \bar{s}b}.$$

Since  $a$  is irreducible either  $(b - s)/(1 - \bar{s}b) \in C$  or  $(b + s)/(1 + \bar{s}b) \in C$ , hence  $b \in C$ , hence  $a \in C$  which contradicts the fact that  $a$  is irreducible.

PROPOSITION 4.2. *Let  $g \in W(G, p)$  and let  $f = (1 + g)/(1 - g)$ . If  $N(G, p) \neq \{1\}$ , if  $f$  is an extreme point of  $N(G, p)$ , and if  $H^1(G, Z) = 0$ , then  $g(G) = D$ .*

*Proof.* By Lemma 2.8

$$(4.1) \quad g = a(s - a)/(1 - \bar{s}a)$$

where  $a \in W(G, p)$ ,  $a$  is irreducible, and  $s \in \bar{D}$ . By Proposition 4.1  $a(G) = D$ , hence by (4.1)  $g(G) = D$ .

4.3. We will now prove Theorem 1.2. By Lemma 2.8

$$(4.2) \quad g = a(s - a)/(1 - \bar{s}a)$$

where  $a \in W(G, p)$ ,  $a$  is irreducible,  $s \in \bar{D}$ , and  $s \neq 0$ .

The roots of  $\bar{s}x^2 + 2ix - s$  are  $-i[1 \pm (1 - s\bar{s})^{1/2}]/\bar{s}$ . Thus if  $\lambda = -i[1 - (1 - s\bar{s})^{1/2}]/\bar{s}$ , then

$$(4.3) \quad \bar{s}\lambda^2 + 2i\lambda - s = 0.$$

Furthermore if  $|s| < 1$ , then  $|\lambda| < 1$ . Let  $\mu = -\lambda^2$ , let  $z$  in  $G$  be such that  $g(z) = \mu$  (Proposition 4.2), and let  $Z$  in  $\text{Aut}(G)$  be such that  $Z(p) = z$ . Thus  $g(Z(p)) = \mu$ . If  $b = a \circ Z$ , then by (4.2)

$$\begin{aligned} (g \circ Z - \mu)/(1 - \bar{\mu}g \circ Z) &= \left( b \frac{s-b}{1-\bar{s}b} + \lambda^2 \right) / \left( 1 + \bar{\lambda}^2 b \frac{s-b}{1-\bar{s}b} \right) \\ &= [\lambda^2 + (s - \bar{s}\lambda^2)b - b^2] / [1 - (\bar{s} - s\bar{\lambda}^2)b - \bar{\lambda}^2 b^2]. \end{aligned}$$

Furthermore by (4.3)  $s - \bar{s}\lambda^2 = 2i\lambda$ , hence

$$\begin{aligned} (g \circ Z - \mu)/(1 - \bar{\mu}g \circ Z) &= (\lambda^2 + 2i\lambda b - b^2)/(1 + 2i\bar{\lambda}b - \bar{\lambda}^2 b^2) \\ &= [(\lambda + ib)/(1 + i\bar{\lambda}b)]^2. \end{aligned}$$

Thus

$$\beta(Z, g) = \left( \frac{1 - \bar{\mu}}{1 - \mu} \right) \left( \frac{\lambda + ib}{1 + i\bar{\lambda}b} \right)^2 = h^2$$

where  $h \in W(G, p)$ . Thus by Lemma 2.2 and Proposition 3.5  $\alpha(Z, f)$  is not an extreme point of  $N(G, p)$  which contradicts Proposition 3.3. Thus  $|s| = 1$ .

We have

$$g = as(1 - \bar{s}a)/(1 - \bar{s}a) = sa$$

which completes the proof of Theorem 1.2.

#### REFERENCES

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