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MERCERIAN THEOREMS VIA SPECTRAL THEORY

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Given a regular matrix A, Mercerian theorems are concerned with determining the real or complex values of α for which $\alpha I + (1-\alpha)A$ is equivalent to convergence. For $\alpha \neq 1$, the problem is equivalent to determining the resolvent set for A, or, determining the spectrum $\sigma(A)$ of A, where $\sigma(A) = \{\lambda \mid A - \lambda I$ is not invertible$\}$. This paper treats the problem of determining the spectra of weighted mean methods; i.e., triangular matrices $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geqq 0$, $\sum_{k=0}^{n} p_k = P_n$. It is shown that the spectrum of every weighted mean method is contained in the disc $\{\lambda \mid |\lambda - 1/2| \leqq 1/2\}$ (Theorem 1), and, if $\lim p_n/P_n$ exists,

$$\sigma(A) = \{\lambda \mid |\lambda - (2-\varepsilon)^{-1}| \leqq (1-\varepsilon)/(2-\varepsilon)\} \cup \{p_n/P_n \mid p_n/P_n < \varepsilon/(2-\varepsilon)\}\ ,$$

where $\varepsilon = \lim p_n/P_n$.

Let $\gamma = \underline{\lim}\, p_n/P_n$, $\delta = \overline{\lim}\, p_n/P_n$, $S = \overline{\{p_n/P_n \mid n \geqq 0\}}$. When $\gamma < \delta$, some examples are provided to indicate the difficulty of determining the spectrum explicitly. It is shown that $\{\lambda \mid |\lambda - (2-\delta)^{-1}| \leqq (1-\delta/(2-\delta)\} \cup S \subseteqq \sigma(A)$ and

$$\sigma(A) \subseteqq \{\lambda \mid |\lambda - (2-\gamma)^{-1}| \leqq (1-\gamma)/(2-\gamma)\} \cup S\ .$$

Theorem 1 is a generalization of the corresponding theorems of: S. Aljancic, L. N. Cakalov, K. Knopp, M. E. Landau, J. Mercer, Y. Okada, W. Sierpinski, and G. Sunouchi.

Using spectral theory we obtain the best possible Mercerian theorems for certain classes of weighted mean methods of summability.

The weighted mean method is a triangular matrix $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geqq 0$, $n \geqq 0$, $P_n = \sum_{k=0}^{n} p_k$ and A is a bounded linear operator on c, the space of convergent sequences.

For $\alpha \neq 0$ we may write $\alpha I + (1-\alpha)A = \alpha(I + qA)$, where $q = (1-\alpha)/\alpha$. Mercer's original theorem [9] states the following: Let $\{x_n\}$ be a sequence such that $x_{n+1} - x_n + \mu n^{-1} x_n \rightarrow \lambda$ as $n \rightarrow \infty$. (i) If λ is finite and $\mu > -1$, then $x_{n+1} - x_n$ and $n^{-1}x_n$ both tend to $\lambda/(\mu+1)$ as $n \rightarrow \infty$. (ii) If λ is infinite and $\mu > -1$, then $n^{-1}x_n \rightarrow \lambda$ and $x_{n+1} - x_n \rightarrow \lambda$ only if $0 \geqq \mu > -1$.

Landau [8] showed that, if $\{x_n\}$ is a complex sequence, q a positive integer, then $\lim_n(x_n + (q/n)\sum_{k=1}^{n} x_k) = 0$ implies $\lim_n x_n = 0$. Sierpinski [14] extended Landau's result to real numbers $q > -1$ and showed it could not be extended to q $\leqq -1$. Sierpinski's result for $q > -1$ was reproved in [3].

Let $\sum_{n=2}^{\infty} p_n/(p_1 + p_2 + \cdots + p_{n-1})$ be a divergent series of positive terms, $\{x_n\}$ a complex sequence. Okada [10] showed that if $q > -1$, then $\lim_n (x_n + q(\sum_{k=1}^{n} p_k x_k / \sum_{k=1}^{n} p_k)) = l$, l finite, implies $\lim_n x_n = l/(1+q)$. He also verified that the theorem does not hold for $\underline{\lim}_n \sum_{k=1}^{n-1} p_k/p_n > -(1+q) \geqq 0$.

Using a different technique, Knopp [6] reproved Okada's result. Beekman [2] showed that, if A is a conservative triangle with inverse satisfying $a_{nn}^{-1} > 0$, $a_{nk}^{-1} \leqq 0$ for $n > k$, then $I + qA$ is equivalent to convergence for $\operatorname{Re}(q) > -1$.

We determine the spectrum of A, $\sigma(A)$, in every case in which $\lim p_n/P_n$ exists (Corollaries 1 and 2). When $\{p_n/P_n\}$ does not converge, in which case A is necessarily regular, the situation seems pathological: Theorems 2 and 3 do give set inclusions for $\sigma(A)$, but, as we show by examples, $\sigma(A)$ can be disconnected and is very difficult to describe explicitly.

Let $B = A - \lambda I$. Our first task is to compute the entries of B^{-1}. Except for Theorem 1, we shall restrict our attention to regular weighted mean methods; i.e., those for which $P_n \to \infty$. For, if P_n tends to a finite limit, then A is compact and $\sigma(A) = \{p_k/P_k : k \geqq 0\} \cup \{0\}$. (See, e.g. [13, Theorem 1].)

LEMMA 1. *Let A be a weighted mean matrix, $B = A - \lambda I$, λ a scalar such that $b_{nn} \neq 0$ for each n. Then $D = B^{-1}$ has entries*

$$(1)\qquad \begin{aligned} d_{nn} &= \frac{P_n}{p_n - \lambda P_n}, \\ d_{nk} &= (-1)^{n+k} \frac{\lambda^{n-k-1} p_k}{P_n} \prod_{j=k}^{n} \frac{P_j}{p_j - \lambda P_j}, \quad k < n. \end{aligned}$$

Proof. A direct computation verifies d_{nn} and $d_{n,n-1}$. By induction one can show that

$$(2)\qquad \sum_{j=0}^{k} (-1)^j \lambda^{j-1} \frac{p_{n-j}}{P_{n-j}} \prod_{i=0}^{j} \frac{P_{n-i}}{p_{n-i} - \lambda P_{n-i}} = (-1)^k \lambda^k \prod_{j=0}^{k} \frac{P_{n-j}}{P_{n-j} - \lambda P_{n-j}}.$$

With (2), one verifies by induction that (1) is true.

THEOREM 1. *Let A be a weighted mean method. Then $\sigma(A) \subseteqq \{z \mid |z - 1/2| \leqq 1/2\}$.*

Proof. Let $\lambda = x + iy$ satisfy $|\lambda - 1/2| > 1/2$. This inequality is equivalent to $\alpha > -1$, where $-1/\lambda = \alpha + i\beta$. Since $\alpha > -1$ and $0 \leqq p_j/P_j \leqq 1$ for all j, $|1 - p_j/\lambda P_j| \geqq |1 + \alpha p_j/P_j = 1 + \alpha p_j/P_j$. For $k < n$, $|d_{nk}| \leqq p_k/|\lambda|^2 P_n \prod_{j=k}^{n} (1 + \alpha p_j/P_j) = f_{nk}$, say.

Using finite induction we can show, for each $0 < r < n$,

$$\sum_{k=0}^{r} f_{nk} = \frac{P_r}{|\lambda|^2 P_n(1+\alpha)\prod_{j=r+1}^{n}(1+\alpha p_j/P_j)} .$$

Therefore $\sum_{k=0}^{n} |d_{nn}| \leqq |d_{nn}| + \sum_{k=0}^{n-1} f_{nk} = |d_{nn}| + P_{n-1}/|\lambda|^2 P_n$.

$$\begin{aligned}(1+\alpha)(1+\alpha p_n/P_n) &\leqq |p_n/P_n - \lambda|^{-1} + \beta|\lambda|^{-2}(1+\alpha)^{-1} \\ &\leqq \beta|\lambda|^{-1}(1 + 1/|\lambda|(1+\alpha)) ,\end{aligned}$$

where $\beta = 1$ if $\alpha \geqq 0$ and $\beta = (1+\alpha)^{-1}$ if $-1 < \alpha < 0$. Since $d_{nn} \neq 0$ for each n, from Problem 32 [16, p. 232], the convergence domain of D, (D), is equal to c, and $\lambda \in \rho(A)$, the resolvent of A.

Theorem 1 is a special case of [2, Theorem 1]. Since 0 is not an interior point of $\sigma(A)$, Theorem 1 provides another proof of the fact that every weighted mean method lies in the closure of the maximal group of invertible elements in Δ, the subalgebra of $B(c)$ consisting of triangular matrices. (See [11, p. 287].)

Let $\delta = \overline{\lim}_n\, p_n/P_n$, $\gamma = \underline{\lim}_n\, p_n/P_n$.

THEOREM 2. *Let A be a regular weighted mean method. Then $\sigma(A) \supseteqq \{\lambda \mid |\lambda - (2-\delta)^{-1}| \leqq (1-\delta)/(2-\delta)\} \cup S$, where $S = \overline{\{p_n/P_n \mid n \geqq 0\}}$.*

Proof. Fix λ satisfying $|\lambda - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)$ and $\lambda \neq p_n/P_n$ for any n. From (1) we obtain

$$(3) \qquad |d_{nk}| = \frac{p_k}{|\lambda|^2 P_{k-1}\prod_{j=k}^{n}\left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{p_j}{P_{j-1}}\right|} .$$

Note that $|1 + (1 - (1/\lambda)p_{n+1}/P_n| \leqq 1$ if and only if

$$(1 + (1+\alpha)p_{n+1}/P_n)^2 + (\beta p_{n+1}/P_n)^2 \leqq 1 ,$$

where $-1/\lambda = \alpha + i\beta$; i.e.,

$$(4) \qquad 2(1+\alpha)p_{n+1}/P_n + ((1+\alpha)^2 + \beta^2)(p_{n+1}/P_n)^2 < 0 .$$

For each n such that $p_{n+1} = 0$, (4) is automatically satisfied. For each n such that $p_{n+1} > 0$, (4) is equivalent to

$$(5) \qquad 2(1+\alpha) + ((1+\alpha)^2 + \beta^2)p_{n+1}/P_n \leqq 0 .$$

For (5) to be true for all n sufficiently large, it is sufficient to have δ satisfy

$$(6) \qquad 2(1+\alpha) + ((1+\alpha)^2 + \beta^2)\delta/(1-\delta) < 0 ,$$

since $p_{n+1}/P_n = p_{n+1}/P_{n+1}(1 - p_{n+1}/P_{n+1})$, which is monotone increasing in p_n/P_n. Inequality (6) is equivalent to $|\lambda - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)$.

Therefore, for all $n \geqq N$, using (3),

$$\sum_{k=N}^{n-1} |d_{nk}| \geqq \frac{1}{|\lambda|^2} \sum_{k=N}^{n-1} \frac{p_k}{P_{k-1}} \geqq \frac{1}{|\lambda|^2} \sum_{k=N}^{n-1} \frac{p_k}{P_k},$$

which diverges by the Abel-Dini theorem [7, p. 290].

If $\lambda = p_n/P_n$ then λ belongs to the spectrum of A. Theorem 2 follows since the spectrum is always closed.

COROLLARY 1. *Let A be a regular weighted mean method with $\delta = 0$. Then $\sigma(A) = \{\lambda \mid |\lambda - 1/2| \leqq 1/2\}$.*

Proof. Combine Theorems 1 and 2, observing that S is already contained in the disc.

Special cases of Corollary 1 for λ real appear in [1], [6], and [10].

THEOREM 3. *Let A be a regular weighted mean method with $\gamma > 0$. Then $\sigma(A) \subseteq \{\lambda \mid |\lambda - (2-\gamma)^{-1}| < (1-\lambda)/(2-\gamma)\} \cup S$.*

Proof. Let λ be fixed and satisfy $|\lambda - (2-\gamma)^{-1}| > (1-\gamma)/(2-\gamma)$ and $\lambda \neq p_n/P_n$ for any n. We shall show that $\lambda \in \rho(A)$, the resolvent of A. From Theorem 1 we need consider only those values of λ satisfying $|\lambda - 1/2| \leqq 1/2$; i.e., $\alpha < -1$. The value $\alpha = -1$ corresponds to $\lambda = 1$, which we know lies in the spectrum, since $p_0/P_0 = 1$. Therefore we shall assume $\alpha < -1$.

Under the assumption on λ we wish to verify that

$$|1 + (1 - 1/\lambda)p_j/P_{j-1}|$$

is strictly larger than one for all j sufficiently large. To this end, define $f(t) = 1 + 2(1+\alpha)t + ((1+\alpha)^2 + \beta^2)t^2$. f has a minimum at $t_0 = -(1+\alpha)/((1+\alpha)^2 + \beta^2)$.

The assumption on λ is equivalent to

$$\gamma(\alpha^2 + \beta^2) + 2\alpha > \gamma - 2\,. \tag{7}$$

Therefore

$$\frac{\gamma}{2(1-\gamma)} > \frac{-(1+\alpha)}{(1+\alpha)^2 + \beta^2} = t_0$$

and f is monotone increasing for all $t > \gamma/2(1-\gamma)$.

Let $\varepsilon > 0$ and small. $f((\gamma/(1-\gamma)) - \varepsilon) = f(\gamma/(1-\gamma)) - 2\epsilon g(\varepsilon)$, where $g(\varepsilon) = 1 + \alpha + ((1+\alpha)^2 + \beta^2)(\gamma/(1-\gamma) - \varepsilon/2)$. $g(\varepsilon) > 0$ for small ε, since f is monotone increasing for $t > \gamma/2(1-\gamma)$.

We shall now show that $f(\gamma/(1-\gamma)) > 1$. From the hypothesis on λ and (6),

$$\alpha^2 + \beta^2 + \frac{2\alpha}{\gamma} > \frac{\gamma - 2}{\gamma},$$

which is equivalent to

$$\left|\frac{1}{1-\gamma} - \frac{\gamma}{\lambda(1-\lambda)}\right| > 1 .$$

But $1/(1-\gamma) = 1 + \gamma/(1-\gamma)$, so we have

$$(f(\gamma/(1-\gamma)) = |1 + (1 - 1/\lambda)\gamma/(1-\gamma)|^2 > 1 .$$

Now choose $\varepsilon > 0$ and so small that $f(\gamma/(1-\gamma) - \varepsilon) = f(\gamma/(1-\gamma)) - 2\varepsilon g(\varepsilon) = m^2 > 1$. Then, by the definition of γ there exists an N such that $n > N$ implies $p_{n+1}/P_n > \gamma/(1-\gamma) - \varepsilon$, so that $f(p_n/P_{n-1}) > f(\gamma/(1-\gamma) - \varepsilon) = m^2$.

Using (3), $|d_{nk}|/|d_{n+1,k}| = (f(p_{n+1}/P_n)) > m^2 > 1$ for all $n \geqq N$. Therefore $|d_{nk}|$ is monotone decreasing in n for each $k, n \geqq N$, so that D has bounded columns. Thus, to show that D has finite norm it is sufficient to show that $|d_{nn}|$ is bounded, and that $\sum_{k=N}^{n-1} |d_{nk}|$ is bounded.

Recall that p_n/P_{n-1} is monotone increasing in p_n/P_n. For the ε we are using, we can enlarge N, if necessary, to ensure that $p_n/P_{n-1} < \delta/(1-\delta) + 1$ for $n \geqq N$.

From (3),

$$\begin{aligned}\sum_{k=N}^{n-1} \left| |d_{nk}| \right. &\leqq \frac{1}{|\lambda|^2}\left(\frac{\delta}{1-\delta} + 1\right)\sum_{k=N}^{n-1}\left(\prod_{j=k}^{n}\left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{p_j}{P_{j-1}}\right|^{-1}\right. \\ &\leqq \frac{1}{|\lambda|^2}\left(\frac{\delta}{1-\delta} + 1\right)\sum_{k=N}^{n-1} m^{-n+k-1} < H ,\end{aligned}$$

where H is independent of n.

$$\begin{aligned}|d_{nn}| &= \frac{P_n}{|p_n - \lambda P_n|} = \frac{P_n}{|\lambda|\,|P_n - p_n/\lambda|} = \frac{P_n}{|\lambda|\,|P_{n-1} + (1 - 1/\lambda)p_n|} \\ &= \frac{P_n/P_{n-1}}{|\lambda|\,|1 + (1 - 1/\lambda)p_n/P_{n-1}|} = \frac{(1 + p_n/P_{n-1})}{|\lambda|\,|1 + (1 - 1/\lambda)p_n/P_{n-1}|} \\ &< \frac{1 + \delta/(1-\delta) + 1}{|\lambda| m} .\end{aligned}$$

Therefore D has finite norm. From [16, loc. cit.], $(D) = c$ and $\lambda \in \rho(A)$.

COROLLARY 2. *Let A be a regular weighted mean method with $\lim_n p_n/P_n = \gamma > 0$. Then $\sigma(A) = \{\lambda \mid |\lambda - (2-\gamma)^{-1}| \leqq (1-\gamma)/(2-\gamma)\} \cup E$, where $E = \{p_n/P_n \mid p_n/P_n < \gamma/(2-\gamma)\}$.*

Proof. Combine Theorems 2 and 3 and note that $S\backslash E$ is already contained in the disc, and E is a finite set.

We now obtain a necessary and sufficient condition for a weighted mean method to be equivalent to convergence.

THEOREM 4. *Let A be a regular weighted mean method. Then $(A) = c$ if and only if $\theta = \underline{\lim}_n p_{n+1}/P_n > 0$.*

Proof. $\theta > 0$ implies $p_{n+1}/P_n \geqq \theta/2$ for all n sufficiently large. For each $np_{n+1}/P_{n+1} = (p_{n+1}/P_n)/(1 + p_{n+1}/P_n)$. Note that $f(y) = y/(1+y)$ is monotone increasing in y, so that, for all $n \geqq N$, $p_{n+1}/P_{n+1} \geqq \theta/(2+\theta)$, and the diagonal entries of A are nonzero for $n \geqq N$. If $a_{nn} = 0$ for any $n < N$, replace the zero by 1. The new matrix B has the same convergence domain as A. For $n \geqq N$, the nonzero terms of B^{-1} are $b_{nn}^{-1} = P_n/p_n$, $b_{n,n-1}^{-1} = -P_{n-1}/p_n$.

Suppose $a_{kk} = 0$ for some $k < N$. Then $p_k = 0$, $b_{kk} = 1$ and $b_{nk} = 0$ for $n > k$. Thus $b_{kk}^{-1} = 1$, $b_{k+1,k}^{-1} = 0$ and, by induction, $b_{nk}^{-1} = 0$ for $n > k$.

Therefore $\|B^{-1}\| = \sup_n [P_{n-1}/p_n + P_n/p_n] \leqq \sup_n 2P_n/p_n \leqq 2(2+\theta)/\theta < \infty$. By [16], $(B) = c$. Thus $(A) = c$.

Suppose $\theta = 0$. Then there exists a subsequence $\{n_k\}$ of n such that $\lim_k p_{n_k+1}/P_{n_k} = 0$.

Case I. $p_n = 0$ for at most a finite number of values of n. Let B be the matrix A with each zero diagonal entry replaced by 1. Then $(B) = (A)$. Since $p_{n+1}/P_{n+1} = (p_{n+1}/P_n)/(1 + p_{n+1}/P_n)$, $\lim_k P_{n_k}/p_{n_k} = 0$. Therefore $\|B^{-1}\| \geqq \sup_k |b_{n_k,n_k}^{-1}| = +\infty$, and $(B) \neq c$.

Case II. $p_n = 0$ for an infinite number of values of n. Let $\{n_k\}$ denote this set. Define a sequence $\{x_n\}$ by $x_{n_k} = 1$, $x_k = 0$ otherwise. Then $Ax = 0$, and $(A) \neq c$.

The special case of this theorem for $0 < p_n \leqq 1$ appears in [4]. A special case of the sufficiency of this theorem appears in [5, p. 59].

We now consider the pathology which may arise when $\gamma < \delta$.

With $p_0 = 1$, $p_n \geqq 0$ for $n > 0$, $c_n = p_n/P_n$, then, as in [12, pp. 163-4], one can show that $p_n = c_n \prod_{j=1}^n (1 - c_j)^{-1}$, $c_0 = 1$, $0 \leqq c_n < 1$ for $n > 0$, and $P_n \to \infty$ is equivalent to $\sum_{n=0}^\infty c_n = \infty$.

For any sequence $s = \{s_n\}$ define $u_n = \sum_{k=0}^n p_k s_k/P_n$. Then $u_n - (1 - c_n)u_{n-1} = c_n s_n$. Let

$$t_n = u_n - \lambda s_n \,. \tag{8}$$

For each $c_n \neq 0$,

$$t_n = \lambda(1 - c_n)u_{n-1}/c_n + (1 - \lambda/c_n)u_n \, . \tag{9}$$

Now for the examples. Let p, q be real numbers satisfying $1 < p < q$. Define $\{c_n\}$ by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, $n > 0$. Using (8) and (9), $t_0 = (1 - \lambda)u_0$, $t_{2n} = (p - 1)\lambda u_{2n-1} + (1 - p\lambda)u_{2n}$, and $t_{2n+1} = (q - 1)\lambda u_{2n} + (1 - q\lambda)u_{2n+1}$. Therefore $t = Bu$, where $b_{00} = 1$, $b_{2n,2n} = 1 - p\lambda$, $b_{2n-1,2n-1} = 1 - q\lambda$, $b_{2n,2n-1} = (q - 1)\lambda$, $b_{2n-1,2n-2} = (p - 1)\lambda$, $n > 0$, $b_{nk} = 0$ otherwise. From Theorem 4, $(A) = c$.

Suppose $\lambda \neq \{1/p, 1/q, 1\}$, and let $E = B^{-1}$. If $\|E\| < \infty$, then from [16, loc. cit.] E is conservative and $(B) = c$. Therefore $t \in c \Rightarrow u \in c \Rightarrow s \in c$ and $(A - \lambda I) = c$, which implies $\lambda \notin \sigma(A)$. Conversely, if $\lambda \notin \sigma(A)$, then $(A - \lambda I) = c$, so that $t \in c \Rightarrow s \in c \Rightarrow u \in c \Rightarrow E$ is conservative $\Rightarrow \|E\| < \infty$. We have shown that, if $\lambda \neq \{1/p, 1/q, 1\}$ then $\lambda \notin \sigma(A)$ if and only if $\|E\| < \infty$.

To compute the norm of E, observe that $b_{nn}e_{nk} + b_{n,n-1}e_{n-1,k} = 0$ for $k < n$, so that $e_{nk} = -b_{n,n-1}e_{n-1,k}/b_{nn}$.

Thus $e_{2n,k} = -(p - 1)\lambda e_{2n-1,k}/(1 - p\lambda)$, $k < 2n$, $n = 1, 2, \cdots$, and $e_{2n+1,k} = -(q - 1)\lambda e_{2n,k}/(1 - q\lambda)$. Let $R_n = \sum_{k=0}^{n} |e_{nk}|$. For $n \geqq 1$,

$$\begin{aligned} R_{2n} &= \sum_{k=0}^{2n-1} |e_{2n,k}| + |e_{2n,2n}| \\ &= \frac{(p - 1)|\lambda|}{|1 - p\lambda|} \sum_{k=0}^{2n-1} |e_{2n-1,k}| + \frac{1}{|1 - p\lambda|} \\ &= \frac{1}{|1 - p\lambda|}[(p - 1)|\lambda| R_{2n-1} + 1] \, , \end{aligned} \tag{10}$$

and, for $n \geqq 0$,

$$R_{2n+1} = \frac{1}{|1 - q\lambda|}[(q - 1)|\lambda| R_{2n} + 1] \, . \tag{11}$$

Substituting (11) into (10) we have

$$R_{2n+2} = \frac{(p - 1)(q - 1)|\lambda|^2}{|1 - p\lambda|\,|1 - q\lambda|} R_{2n} + \frac{(p - 1)|\lambda|}{|1 - p\lambda|\,|1 - q\lambda|} + \frac{1}{|1 - p\lambda|} \, ,$$

and

$$R_{2n+1} = \frac{(p - 1)(q - 1)|\lambda|^2}{|1 - p\lambda|\,|1 - q\lambda|} R_{2n-1} + \frac{(q - 1)|\lambda|}{|1 - p\lambda|\,|1 - q\lambda|} + \frac{1}{|1 - q\lambda|} \, .$$

Let $\{\sigma_n\}$ be defined by $\sigma_{n+1} = a\sigma_n + b$, where a and b are fixed positive constants. Then

$$\frac{\sigma_{n+1}}{a^{n+1}} - \frac{\sigma_n}{a^n} = \frac{b}{a^{n+1}} ,$$

so that

$$\frac{\sigma_{n+1}}{a^{n+1}} - \frac{\sigma_0}{a^0} = \frac{b}{a} \frac{(1 - a^{-n-1})}{(1 - a^{-1})} ,$$

or $\sigma_{n+1} - \sigma_0 a^{n+1} = b(a^{n+1} - 1)/(a - 1)$. For $0 < a < 1$, $\{\sigma_n\}$ is bounded, and, for $a \geqq 1$, $\{\sigma_n\}$ is unbounded. Therefore

$$\begin{aligned} \sigma(A) &= \{\lambda \mid \|E\| = \infty\} \cup \{1/p, 1/q, 1\} \\ &= \{\lambda \mid (p-1)(q-1)|\lambda|^2 \geqq |1 - p\lambda|\,|1 - q\lambda|\} , \end{aligned}$$

since $1/p, 1/q$ and 1 already belong to those values of λ for which $\|E\| = \infty$.

For $p = 2, q = 3$, $\partial\sigma(A)$ is an oval with x-intercepts of $1/4, 1$. For $p = 2, q = 8$, the boundary consists of a pair of ovals which are tangent at $x = (10 - \sqrt{8})/23$. For $p = 3, q = 13$, $\sigma(A)$ is contained in two disjoint ovals. The left oval has x-intercepts at $1/15, 1/9$, and the right oval has x-intercepts at $1/7, 1$.

References

1. S. Aljančić, *Sur le Theorème Mercerien de Čakalov*, Publ. L'Inst. Math. (Beograd), **19**, (1975) 9-15.
2. W. Beekman, *Mercer-Sätze für abschnittsbeschrankte Matrix transformationen*, Math. Z., **97**, (1967), 154-157.
3. L. N. Čakalov, *Generalization of a theorem of Mercer on convergence*, Izv. Mat. Inst. Acad. Bulgar. Sci., **1**., (1954), 85-88.
4. C. W. Groetsch, *Summation methods associated with an iteration*, Nanta Math., **7** (1974) 13-16.
5. G. H. Hardy, *Divergent Series*, Oxford University Press, 1949.
6. K. Knopp, *Zur Theorie der C-und-H-Summierbarkeit*, Math. Z., **19** (1923) 97-113.
7. ———, *Theory and Application of Infinite Series*, Blackie and Son Ltd., London, 1947.
8. M. E. Landau, *Darstellung und Begründing einiger neurer Ergebnisse der Funktiontheorie*, Berlin, (1916), 30.
9. J. Mercer, *On the limit of real variants*, Proc. London Math. Soc., (2) **5** (1907), 206-224.
10. Y. Okada, *A theorem on limits*, Tôhoku Math. J., **15** (1919), 280-283.
11. B. E. Rhoades, *Triangular summability methods and the boundary of the maximal group*, Math. Z., **105** (1968), 284-290.
12. ———, *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc., **196** (1974), 161-176.
13. N. K. Sharma, *Spectra of conservative matrices*, Proc. Amer. Math. Soc., **35** (1972), 515-518.
14. W. Sierpinski, *Sur la dependance entre l'existence de limites des suites* $x_n + q(x_1 + x_2 + \cdot + x_n)/n$ *et* x_n, Tôhoku Math. J., **11** (1914), 1-14.
15. G. Sunouchi, *Notes on Fourier analysis* (xviii), Absolute summability of series with constant terms, Tôhoku Math. J., (2) **1** (1949-50), 57-65.

16. A. Wilansky, *Functional Analysis*, Blaisdell, 1964.

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