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HALL-HIGMAN TYPE THEOREMS V

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This paper sets out the inductive machinery which makes the computations of other numbers of this sequence useful. Representation theory is cast in the framework of wreath products which are then used to study the behavior of regular orbits and modules under various types of induction. Tensor induction is defined and studied along with the related concept of form primitivity.

Let AG be a solvable group with normal subgroup G and nilpotent complement A where $(|A|, |G|) = 1$. Assume that k is a field and that V is a faithful irreducible k $[AG]$-module. This series considers the following two questions.

(1) If $k = GF(r)$ for a prime r, then when does the permutation representation of A on $V^\#$ contain a regular orbit?

(2) If $R \leqq G$ is a normal extraspecial r-subgroup of AG where $Z(R) \leqq Z(AG)$, $C_A(R) = 1$, and $R/Z(R)$ is an AG-chief factor, then when does $V|_A$ contain a regular k $[A]$-module?

This paper sets up the inductive machinery needed to study these two questions. The actual method whose tools are described in this paper is given in [12]. The introductions to [1, 3, 4, 5, 8, 10] describe various aspects of this method. In its purest form, the method is applied in [9]. Various other applications occur in [2, 7, 11, 13, 19].

In § 2, the relevant inductive schemes are defined in the suitably computational framework of wreath products. In § 3, these inductive schemes are studied for wreath products to determine when regular orbits (modules) induce to regular orbits (modules). Section 4 translates the results on wreath products into the terms of general representation theory.

Two methods of induction play important roles; namely, usual group theoretic induction, and tensor induction (Definition (2.6)). The method of tensor induction is applicable, but not directly, to the study of primitive linear groups which contain normal extraspecial subgroups. Section 5 is devoted to the method by which tensor induction is applied to a general primitive linear group having a normal extraspecial subgroup (Theorem (5.18)). Solvability is assumed but is not entirely necessary for this analysis.

If R is a normal extraspecial subgroup of a group G where

$Z(R) \leqq Z(G)$ and $R/Z(R)$ is a chief factor of G then obtaining the main hypothesis of § 5 reduces to the "form invariant" induction structure of the G-module $\bar{R} = R/Z(R)$ where $\bar{R}$ is endowed with a nonsingular symplectic form g fixed by the action of G [20, Satz 13.7]. The module $\bar{R}$ will be "form invariantly" induced from a "form primitive" submodule. Without any solvability hypotheses, § 7 defines and derives the major properties about form induction. Similarly, § 6 derives the effect of ground field extensions on forms (Theorem (6.7). There are two obvious situations which can occur for a "form primitive" module. Theorem (7.9) shows that a third situation can also occur. Section 8 is devoted to a translation of this third case into the first two cases, by altering the group, the form, and the module in a fixed reversible way. (Theorems (6.7), (8.10), (8.13), and Proposition (8.18).)

ACKNOWLEDGMENTS. Version 1 of this paper contained an incorrect definition of minimal module. This fact was discovered and pointed out to me by participants in a seminar run by T. O. Hawkes at the University of Warwick in 1972. I even had unwittingly provided the counterexample [4, (2.3)] of a quaternion group of order 8. Version 2 gave a correct definition with some additional analysis needed to correct the original arguments. These corrections took sufficient time that this paper was delaying the publishing of other numbers of this sequence. At this point R. Beaumont, an editor of the Pacific Journal of Mathematics, generously assisted me. Refereeing of version 2 was completed by Spring 1975. The referee offered many significant comments for the improvement of exposition and correction of errors. During 1975 I gave a series of lectures at Australian National University on this material. That seminar, and especially the aid of C. Praeger, M. Herzog, and L. Kovács, led to further improvements. The present definition of minimal module was given by L. Kovács and is equivalent to others given in this series. I am grateful to the many who have contributed to the final product. Of course, any errors in this version are my fault.

1. **Preliminary remarks.** All groups considered in this paper are assumed to be finite and solvable. The notation is standard and conforms to that used in other papers of this sequence [3-8].

We state below two useful results with proofs sketched.

(1.1) PROPOSITION. *Assume that k is a field, G is a group, and V is a faithful $k[G]$-module. Suppose that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ where the V_i are $k[G]$-submodules, and set $G_i = \ker V_i$. If G/G_i permutes the elements of V_i with at least s regular orbits for*

$1 \leqq i \leqq t$ then G permutes the elements of V with at least s^t regular orbits.

Let $v_{il}, \cdots, v_{is}$ generate s distinct G/G_i-regular orbits on V_i. Then for all choices of indices $i_1, \cdots, i_t \leqq s$, the vectors $v_{1i_1} + v_{2i_2} + \cdots + v_{ti_t}$ all generate distinct regular G-orbits on V.

(1.2) PROPOSITION. *Assume that k is a field, G is a group, and V is a faithful $k[G]$-module. Suppose that $V = V_1 \otimes \cdots \otimes V_t$ where the V_i are $k[G]$-modules and set $G_i = \ker V_i$. If a direct sum decomposition of V_i contains at least s copies of the regular $k[G/G_i]$-module for $1 \leqq i \leqq t$ then a direct sum decomposition of V contains at least s^t copies of the regular $k[G]$-module.*

We may choose vectors $v_{i1}, \cdots, v_{is} \in V_i$ so that the set $\{hv_{ij} \mid 1 \leqq j \leqq s, h \in G/G_i\}$ is linearly independent, and therefore, the v_{ij} generate distinct independent regular $k[G/G_i]$-modules. For all choices of indices $i_1, \cdots, i_t \leqq s$, the vectors $v_{1i_1} \otimes v_{2i_2} \otimes \cdots \otimes v_{ti_t}$ all generate distinct independent regular $k[G]$-modules. Since the regular $k[G]$-module is injective [14, (62.1) (58.6)], the proposition follows.

The following result on unitary forms will be useful.

(1.3) PROPOSITION. *Assume that $\boldsymbol{K}$ is a field, ν is an automorphism of order two on $\boldsymbol{K}$, and $\boldsymbol{F}$ is the fixed field of ν in $\boldsymbol{K}$. Suppose that V is a vector space over $\boldsymbol{K}$, f is a nonsingular unitary form on V with automorphism ν, $g = \tau\mu f$ where $0 \neq \mu \in \boldsymbol{K}$ and $\tau: \boldsymbol{K} \to \boldsymbol{F}$ is the trace. If* char $\boldsymbol{K} \neq 2$, *$0 \neq \omega \in \boldsymbol{K}$ satisfies $\omega^\nu = -\omega$, and $V = W + \omega W$ where W is a totally isotropic $\boldsymbol{F}$-subspace of V for g then for $u, v \in V$*

$$f(u, v) = (2\mu)^{-1}g(u, v) + (2\mu\omega)^{-1}g(\omega u, v) .$$

We may write $u = u_1 + \omega u_2$ and $v = v_1 + \omega v_2$ where $u_i, v_j \in W$. Now $g(\omega u_2, \omega v_2) = \mu f(\omega u_2, \omega v_2) + \mu^\nu f(\omega u_2, \omega v_2)^\nu = -\omega^2[\mu f(u_2, v_2) + \mu^\nu f(u_2, v_2)^\nu] = -\omega^2 g(u_2, v_2) = 0$ so that ωW is totally isotropic. Continuing,

$$\begin{aligned} g(\omega u_2, v_1) &= \mu f(\omega u_2, v_1) + \mu^\nu f(\omega u_2, v_1)^\nu \\ &= -[\mu f(u_2, \omega v_1) + \mu^\nu f(u_2, \omega v_1)^\nu] \\ &= -g(u_2, \omega v_1) . \end{aligned}$$

Further,

$$0 = g(u_1, v_1) = \mu f(u_1, v_1) + \mu^\nu f(u_1, v_1)^\nu$$

so that

$$\mu f(u_1, v_1) = -\mu^\nu f(u_1, v_1)^\nu .$$

From this we obtain

$$\begin{aligned} g(\omega u_2, v_1) &= \mu f(\omega u_2, v_1) + \mu^\nu f(\omega u_2, v_1)^\nu \\ &= \omega\mu f(u_2, v_1) + \omega(-\mu^\nu f(u_2, v_1)^\nu) \\ &= 2\omega\mu f(u_2, v_1) . \end{aligned}$$

Computing

$$\begin{aligned} &(2\mu)^{-1}g(u_1 + \omega u_2, v_1 + \omega v_2) + (2\mu\omega)^{-1}g(\omega u_1 + \omega^2 u_2, v_1 + \omega v_2) \\ &\quad = (2\mu)^{-1}[g(u_1, \omega v_2) + g(\omega u_2, v_1)] + (2\mu\omega)^{-1}[g(\omega u_1, v_1) + \omega^2 g(u_2, \omega v_2)] \\ &\quad = (2\mu)^{-1}[-g(\omega u_1, v_2) + g(\omega u_2, v_1)] + (2\mu\omega)^{-1}[g(\omega u_1, v_1) - \omega^2 g(\omega u_2, v_2)] \\ &\quad = \omega[-f(u_1, v_2) + f(u_2, v_1)] + [f(u_1, v_1) - \omega^2 f(u_2, v_2)] \\ &\quad = [f(u_1, \omega v_2) + f(\omega u_2, v_1)] + [f(u_1, v_1) + f(\omega u_2, \omega v_2)] \\ &\quad = f(u_1 + \omega u_2, v_1 + \omega v_2) \end{aligned}$$

completing the proof.

The following form of Frobenius Reciprocity seems to be well known.

(1.4) PROPOSITION. (Frobenius Reciprocity). *Suppose that k is a field, G is a group with a subgroup H, V is a $k[G]$-module, and U is a $k[H]$-module. Then as k-vector spaces*:

$$(1) \qquad \mathrm{Hom}_{k[H]}(U, V|_H) \cong \mathrm{Hom}_{k[G]}(U|^G, V) ,$$

and

$$(2) \qquad \mathrm{Hom}_{k[H]}(V|_H, U) \cong \mathrm{Hom}_{k[G]}(V, U|_G) .$$

The proofs of (1) and (2) are similar. We sketch only the proof of (1). Let $x_1 = 1, x_2, \cdots, x_t$ be a transversal of H in G, so that $U|^G = \Sigma^\oplus x_i \otimes U$. Let Φ be the mapping of $\mathrm{Hom}_{k[G]}(U|^G, V)$ to $\mathrm{Hom}_k(U, V|_H)$ determined by

$$\Phi(\phi)(u) = \phi(x_1 \otimes u)$$

where $\phi \in \mathrm{Hom}_{k[G]}(U|^G, V)$ and $u \in U$. Let Ψ be the mapping of $\mathrm{Hom}_{k[H]}(U, V|_H)$ to $\mathrm{Hom}_k(U|^G, V)$ determined by

$$\Psi(\psi)(\Sigma x_i \otimes u_i) = \Sigma x_i \psi(u_i)$$

where $\psi \in \mathrm{Hom}_{k[H]}(U, V|_H)$ and $u_i \in U$. It is straightforward to verify that Φ and Ψ are the inverse k-linear isomorphisms needed to prove (1).

(1.5) COROLLARY. *Suppose that k is a field, G is a group with normal subgroup N, and V is an irreducible $k[G]$-module for which*

$V|_N = V_1 + \cdots + V_t$ where the V_i are homogeneous components. If H is the stabilizer in G of V_1 then restriction to V_1 is an isomorphism of $\mathrm{Hom}_{k[G]}(V, V)$ *onto* $\mathrm{Hom}_{k[H]}(V_1, V_1)$.

By Schur's lemma $\hat{k} = \mathrm{Hom}_{k[G]}(V, V)$ is a division ring. Note also that any nonzero element of $\hat{k}$ induces a $k[N]$-isomorphism of V_1 into V. Since V_1 is a homogeneous component, $\hat{k}$ stabilizes V_1. Restriction obviously induces an isomorphism of $\hat{k}$ into $\tilde{k} = \mathrm{Hom}_{k[H]}(V_1, V_1)$ since $\hat{k}$ is a division ring. Since $\mathrm{Hom}_{k[N]}(V_1, V) \cong \mathrm{Hom}_{k[N]}(V_1, V_1)$ as k-vector spaces, $\hat{k} \cong \mathrm{Hom}_{k[H]}(V_1, V)$ as k-vector spaces. By Clifford's theorems $V_1|^G(\text{from } H) \cong V$ so by Frobenius Reciprocity $\tilde{k}$ and $\hat{k}$ have the same dimension over k. We conclude that the restriction map is onto $\tilde{k}$ proving the corollary.

(1.6) PROPOSITION. *Assume that k is a field, G is a group with normal subgroup H, and V is a $k[G]$-module. If $C = \mathrm{Hom}_{k[H]}(V, V)$ then G acts naturally as automorphisms of C by*

$$\alpha^z v = z^{-1}\alpha z v$$

where $z \in G$, $\alpha \in C$, and $v \in V$.

It suffices to show that if $\alpha \in C$ and $z \in G$ then $\alpha^z \in C$. Accordingly, let $x \in H$ and $v \in V$ so that $\alpha^z x v = z^{-1}\alpha z x v = z^{-1}\alpha(zxz^{-1})zx = z^{-1}(zxz^{-1})\alpha z v = x\alpha^z v$ completing the proof.

(1.7) PROPOSITION. *Assume that k is a finite field of odd characteristic, G is a group with subgroup H of index 2, and V is a $k[G]$-module for which $V|_H$ is irreducible. Let J be the 1-dimensional faithful $k[G/H]$-module, $\hat{k} = \mathrm{Hom}_{k[G]}(V, V)$, and $\tilde{k} = \mathrm{Hom}_{k[H]}(V, V)$. Then $V \otimes_k J \cong V$ if and only if $\tilde{k} \neq \hat{k}$. If $\tilde{k} \neq \hat{k}$ then $[\tilde{k}: \hat{k}] = 2$ and $x \in G\backslash H$ acts upon $\tilde{k}$ as an automorphism of order 2.*

Let $v \in V$ and define $z \cdot v = zv$ if $z \in H$ or $-zv$ if $z \in G\backslash H$. This makes V into a $k[G]$-module isomorphic to $V \otimes_k J$. Assume that $V \otimes_k J \cong V$. Thus there is a k-isomorphism ϕ of V to V such that if $z \in G$ then $z \cdot v = \phi^{-1} z \phi v$ for $v \in V$. In particular, $zv = z \cdot v = \phi^{-1} z \phi v$ for $z \in H$ so that $\phi \in \tilde{k}$. If $x \in G\backslash H$ then $x^2 \in H$ so that x operates on $\tilde{k}$ as an automorphism of order 1 or 2. Now $\phi^{-1} x \phi v = x \cdot v = -xv$ for $v \in V$ so that $\phi^x = x^{-1}\phi x = -\phi$ proving that x acts with order 2 and $\hat{k} \neq \tilde{k}$ (since clearly $\hat{k} \leqq \tilde{k}$).

Assume now that $\hat{k} \neq \tilde{k}$. By [14, (29.13)] V is an absolutely irreducible $\hat{k}[G]$-module and an absolutely irreducible $\tilde{k}[H]$-module.

Let $\tilde{V} = \tilde{k} \otimes_{\hat{k}} V$, so that $\tilde{V}$ is an absolutely irreducible $\tilde{k}[G]$-module. The mapping given by $\alpha \otimes v \mapsto \alpha v$ defines a $\tilde{k}[H]$-homomorphism of $\tilde{V}$ onto V. But $\dim_{\tilde{k}} \tilde{V} = \dim_{\hat{k}} V = [\tilde{k}: \hat{k}] \dim_{\tilde{k}} V$ so that the kernel of this homomorphism is a proper $\tilde{k}[H]$-submodule of $\tilde{V}$. Consequently, $\tilde{V}|_H$ is reducible. Since $[G: H] = 2$ and $\tilde{V}|_H$ is a sum of absolutely irreducible constituents, $\tilde{V}|_H$ is the sum of two such irreducibles of equal dimension, one being V as a $\tilde{k}[H]$-module. Thus $[\tilde{k}: \hat{k}] = 2$. By (1.5) G acts as automorphisms of $\tilde{k}$. Since $\tilde{k} \geqq \hat{k}$, if G acts trivially, $\tilde{k} = \hat{k}$. Therefore G acts as G/H, a group of order 2.

Choose $\omega \in \tilde{k}$ and $x \in G \backslash H$ so that $\omega^x \neq \omega$. Set $\mu = \omega^x - \omega$ so that $\mu^x = -\mu \neq 0$. If $z \in G$ then $z \cdot v = \mu^{-1} z \mu$ for $v \in V$ so that μ induces an isomorphism of V with usual action to V with $\cdot$- action proving that $V \cong V \otimes_k J$. The proof is complete.

(1.8) PROPOSITION. *Suppose that k is a field; G is a group; and V_1 and V_2 are completely reducible nonisomorphic $k[G]$-modules. Assume that H is normal in G; G/H is a four group; S_1, S_2, S_3 are the maximal subgroups of G containing H; and $V_1|_{S_i} \cong V_2|_{S_i}$ for $i = 1, 2$. Then $V_1|_{S_3} \ncong V_2|_{S_3}$.*

Let χ_1, χ_2 be the Brauer (or complex) characters of V_1, V_2 respectively. If char $k = p > 0$ we consider only p-regular elements z of G. The modules we are considering are completely reducible in any finite extension field of k [14, 69.9] so that the modules are determined up to isomorphism by their Brauer characters [14, (82.7), (29.11)]. By hypothesis, $\chi_1 \neq \chi_2$ so that for some $z \in G$, $\chi_1(z) \neq \chi_2(z)$. Since $\chi_1|_{S_i} = \chi_2|_{S_i}$ for $i = 1, 2$, $z \notin S_1 \cup S_2$. Since $S_i \geqq H$ and G/H is a four group, $z \in S_3$. Thus $\chi_1|_{S_3} \neq \chi_2|_{S_3}$ proving the proposition.

(1.9) PROPOSITION. *Assume that $\hat{k}$ is a field, ν is an automorphism of $\hat{k}$ of finite order, and k is the fixed field of ν. Assume that G is a group, V is a $\hat{k}[G]$-module with $\hat{k}$-basis $\{v_1, \cdots, v_t\}$. For $\sum \alpha_i v_i \in V$, $\alpha_i \in \hat{k}$ set $\nu \cdot \sum \alpha_i v_i = \sum \alpha_i^{\nu} v_i$. Make V into a $\hat{k}[G]$-module νV by letting $x \in \hat{k}[G]$ act upon $v \in V$ as $\nu^{-1} \cdot (x\nu) \cdot v$. Then $V \cong \nu V$ as $k[G]$-modules.*

Choose $\omega_1, \cdots, \omega_s$ as a k-basis for $\hat{k}$. Then $\mathscr{B} = \{\omega_i v_j \mid 1 \leqq i \leqq s, 1 \leqq j \leqq t\}$ is a k-basis for V. Now ν induces a k-linear transformation $\hat{\nu}$ of the k-space V. If $T(x)$ is the representation of $x \in G$ in the basis $\mathscr{B}$ then $\nu^{-1} \cdot (x\nu) \cdot v = \hat{\nu}^{-1} T(x) \hat{\nu} v$ for $v \in V$. In particular, the representation $\hat{T}(x) = \hat{\nu}^{-1} T(x) \hat{\nu}$ of $k[G]$ on νV is similar to $T(x)$

on V proving the proposition.

2. The wreath product and representation theory.

(2.1) HYPOTHESIS. *Let k be a field, G a group, H a subgroup, U a $k[H]$-module, and H^* a normal subgroup of H contained in* $\ker U$.

We map G homomorphically into a wreath product $\tilde{G}$. We then prove that $k[G]$-modules induced from H extend naturally to $k[\tilde{G}]$-modules induced from an appropriate subgroup of $\tilde{G}$.

A. The Frobenius embedding. We fix the following notation: a factor group $H_0 = H/H^*$; a transversal $\mathscr{T} = \{1 = x_1, \cdots, x_n\}$ for H in G; and a set of integers $\Omega = \{1, 2, \cdots, n\}$. Consider the homomorphism of G into S^n given by the action of G on the cosets of H. That is, if $x \in G$ and $x \to \bar{x}$ where $\bar{x}$ is the image of x in S^n then $\bar{x}$ is defined by the equalities

$$xx_iH = x_jH$$

where $\bar{x}(i) = j$. We let $\bar{G}$ be the image of G in S^n. The *wreath product* of H_0 by $\bar{G}$ ($\tilde{G} = H_0 \sim \bar{G}$) is defined to be the semidirect product of H_0^Ω by $\bar{G}$ where $H_0^\Omega = \{f: \Omega \to H_0 \mid f \text{ a function}\}$ is the group given by pointwise multiplication (i.e., H_0^Ω is the direct product of n copies of H_0), and where the action of $\bar{x} \in \bar{G}$ upon $f \in H_0^\Omega$ is given by $f^{\bar{x}}(i) = f(\bar{x}(i))$.

For $x \in G$ set

$$f_x(i) = x_j^{-1}xx_iH^* \in H_0 \quad \text{where} \quad \bar{x}(i) = j \tag{2.2}$$

and set

$$\Phi(x) = \bar{x}f_x \in \tilde{G} = H_0 \sim \bar{G}\,. \tag{2.3}$$

(2.4) PROPOSITION. (Frobenius Homomorphism). *$\Phi: G \to \tilde{G}$ is a homomorphism with kernel $\bigcap H^{*x}$* (over $x \in G$).

The proof is a straightforward computation using (2.2) and (2.3). The proof is given in both [20, Satz 15.9] and [23, III.5.k] for the case where $H^* = 1$ and H is normal in G. Dropping these additional hypotheses on H does not alter the proof.

The homomorphism Φ has the following useful conjugacy property.

(2.5) PROPOSITION. *Suppose that $\mathscr{T}'$ is a transversal for H in G with elements $x_i' = x_ih_i$ where $h_i \in H$. If $f_x'(x \in G)$ and Φ' are*

defined by (2.2) *and* (2.3) *respectively for the transversal* $\mathscr{T}'$ *then*

$$\Phi'(x) = f^{-1}\Phi(x)f(x \in G)$$

where $f \in H_0^Q$ *is defined by* $f(i) = h_i$.

Let $x \in G$ so that

$$f^{-1}\Phi(x)f = \bar{x}f^{-\bar{x}}f_x f\,.$$

Evaluating at i and letting $j = \bar{x}(i)$,

$$\begin{aligned} f^{-\bar{x}}f_x f(i) &= h_j^{-1}x_j^{-1}xx_ih_iH^* \\ &= x_j'^{-1}xx_i'H^* \\ &= f_x'(i) \end{aligned}$$

proving the proposition.

B. Induced modules. For each integer $i = 1, 2, \cdots, n$ consider the $k[x_iHx_i^{-1}]$-submodule $x_i \otimes U$ of $k[G] \otimes U$ where tensoring is over $k[H]$. The direct sum of the modules $x_i \otimes U$ is just the induced module $U|^G = k[G] \otimes U$. We may also obtain a module by tensoring (over k) the modules $x_i \otimes U$. This latter process gives a $k[G]$-module under more general hypotheses.

Instead of assuming that U is a $k[H]$-module of the usual kind, we assume that U is a *projective* $k[H]$*-module*, that is, there is a factor set $\alpha: H \times H \to k$ such that if $h, h' \in H$ and $u \in U$ then $h(h'u) = \alpha(h, h')(hh')u$.

REMARKS. (1) If the factor set α of the projective module U is 1, then U is a module in the usual sense. To emphasize this fact, we will call such a module *nonprojective.*

(2) Since the projective and injective modules of general module theory play no role here, there will be no confusion in this usage. Strictly speaking, U is a module affording a projective representation of H with factor set α [14, § 51].

(2.6) DEFINITION. Set $U|^{\otimes G} = (x_1 \otimes U) \otimes (x_2 \otimes U) \otimes \cdots \otimes (x_n \otimes U)$ where tensoring between $x_i \otimes U$'s is over k. We call $U|^{\otimes G}$ the *tensor induced module.*

There are two tensor signs used here, one over $k[H]$ and one over k. The positions of these signs make clear which tensor symbol is meant, and therefore, we omit future reference to distinctions between the two types.

REMARK. In early versions of this paper and in some of the author's papers, the cumbersome notation $\tilde{U}^G$ was used in place of $U|^{\otimes G}$.

(2.7) PROPOSITION. *If U is a projective $k[H]$-module with $H^* \leqq \ker U$ and factor set α then $U|^{\otimes G}$ is a projective $k[G]$-module with multiplication given by*

$$x \cdot [(x_1 \otimes u_1) \otimes \cdots \otimes (x_n \otimes u_n)] = (x_1 \otimes w_1) \otimes \cdots \otimes (x_n \otimes w_n)$$

where $x \in G$, $u_i \in U$, and $w_j = f_x(i)u_i$ when $j = \bar{x}(i)$, and factor set β given by

$$\beta(x, y) = \prod_{i=1}^{n} \alpha(f_x(\bar{y}(i)), f_y(i))$$

where $x, y \in G$.

Since H^* is in the kernel of U, U is naturally a projective $k[H_0]$-module. Let $U^{\Omega} = U \times \cdots \times U$ (n copies) and define the mapping $m_x: U^{\Omega} \to U|^{\otimes G}$ for $x \in G$ by

$$m_x(u_1, \cdots, u_n) = (x_1 \otimes w_1) \otimes \cdots \otimes (x_n \otimes w_n)$$

where u_i and w_j are as in the proposition. It is easy to verify that m_x is a balanced mapping linear in each variable [14, (12.3)]. If $c: U^{\Omega} \to U|^{\otimes G}$ is the mapping sending $(u_1, \cdots, u_n)$ to $(x_1 \otimes u_1) \otimes \cdots \otimes (x_n \otimes u_n)$, then there is a unique linear transformation $x\cdot$ of $U|^{\otimes G}$ (whose restriction to tensors is given in the proposition) making the diagram commutative.

$$\begin{array}{ccc} U^{\Omega} & \xrightarrow{m_x} & U|^{\otimes G} \\ {\scriptstyle c}\downarrow & \nearrow_{x\cdot} & \\ U|^{\otimes G} & & \end{array}$$

To show that $U|^{\otimes G}$ is a projective $k[G]$-module with factor set β we compute the composition of $x, y \in G$ on $U|^{\otimes G}$.

$$\begin{aligned} w &= x \cdot (y \cdot [(x_1 \otimes u_1) \otimes \cdots \otimes (x_n \otimes u_n)]) \\ &= x \cdot \prod{}^{\otimes} (x_i \otimes f_y(j)u_j) \\ &= \prod{}^{\otimes} (x_i \otimes f_x(k)f_y(l)u_l) \end{aligned}$$

where $j = \bar{y}^{-1}(i)$, $k = \bar{x}^{-1}(i)$, and $l = \overline{xy}^{-1}(i)$. But $f_x(k)f_y(l)u_l = \alpha(f_x(k), f_y(l))f_{xy}(l)$ so that

$$w = [\prod_{i=1}^{n} \alpha(f_x(k), f_y(l))][\prod^{\otimes} (x_i \otimes f_{xy}(l)u_l]$$
$$= [\prod_{i=1}^{n} \alpha(f_x(k), f_y(l))][(xy) \cdot \prod^{\otimes} (x_i \otimes u_i)] ,$$

since k and l run through $1, 2, \cdots, n$ as i does, and since $k = \bar{y}(l)$, we have

$$\prod_{i=1}^{n} \alpha(f_x(k), f_y(l)) = \prod_{l=1}^{n} \alpha(f_x(\bar{y}(l)(, f_y(l)) = \beta(x, y) .$$

We have proven that if $z \in G$ induces a linear transformation T_z on $U|^{\otimes G}$ then $T_x T_y = T_{xy}\beta(x, y)$ proving the proposition.

Two properties of tensor induction are mentioned here without proof. They are analogous to properties of ordinary induction. First, recall the definition of equivalent projective representations. Stated for modules: two projective $k[G]$-modules V and W are *equivalent* if there is a crossed homomorphism $\gamma: G \to k$ and a vector space isomorphism $\phi: V \to W$ such that if $x \in G$ and $v \in V$ then

$$\phi(xv) = \gamma(x)x\phi(v) .$$

That is, V and W are essentially "isomorphic", the deviation from isomorphism being that the cocycle of the one module is obtained by altering the cocycle of the other module by a coboundary.

(2.8) PROPOSITION. *Consider a second projective $k[H]$-module W and a subgroup $H \leqq K \leqq G$. Then*

(a) $(U \otimes_k W)|^{\otimes G}$ *is equivalent to* $(U|^{\otimes G}) \otimes_k (W|^{\otimes G})$;

and

(b) $(U|^{\otimes K})|^{\otimes G}$ *is equivalent to* $U|^{\otimes G}$.

Next we prove that a Mackey decomposition holds for tensor induction. Note that $x \otimes U$ is a projective $k[xHx^{-1}]$-module with factor set

$$\alpha_x(xhx^{-1}, xh'x^{-1}) = \alpha(h, h') \quad \text{for} \quad h, h' \in H .$$

(2.9) PROPOSITION. (Mackey Decomposition). *Let K be a subgroup of G. Let $W = \prod^{\otimes} (x \otimes U|_{xHx^{-1} \cap K})|^{\otimes K}$ where the tensor product is over a set of (K, H)-double coset representatives x in G. Then W is equivalent to $U|^{\otimes G}|_K$.*

The proof is carried out by computing with tensors. If in the proof of the Mackey decomposition [14, (44.2)] one replaces sums of vectors by their tensors over k, then one obtains the proof for tensor induction.

REMARK. Induction and tensor induction are special cases of more general "induction" procedures which we sketch now. Fix an integer m between 1 and n and let $\mathscr{M}$ be the set of all m-tuples $(i_1, i_2, \cdots, i_m)$ such that $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$. There is an action $\bar{G}$ upon $\mathscr{M}$ given as follows: if $\bar{x} \in \bar{G}$ and $g = (i_1, \cdots, i_m) \in \mathscr{M}$ then $\{\bar{x}(i_1), \cdots, \bar{x}(i_m)\}$ is a set of m distinct integers, which when ordered, gives an m-tuple $g^{\bar{x}} = (j_1, \cdots, j_m) \in \mathscr{M}$. Form the tensor products

$$U_g = (x_{i_1} \otimes U) \otimes \cdots \oplus (x_{i_m} \otimes U)$$

where $g \in \mathscr{M}$. Finally, let V_m be the direct sum of all modules U_g for $g \in \mathscr{M}$. Using this notation, if $u_1, \cdots, u_m \in U$ we set

$$\begin{aligned} x \cdot [(x_{i_1} \otimes u_1) \otimes \cdots \otimes (x_{i_m} \otimes u_m)] \\ = (x_{j_1} \otimes w_1) \otimes \cdots \otimes (x_{j_m} \otimes w_m) \end{aligned} \tag{2.10}$$

where $x \in G$, $g = (i_1, \cdots, i_m)$, $g^{\bar{x}} = (j_1, \cdots, j_m) \in \mathscr{M}$, and $w_s = f_x(i_t)u_t$ when $j_s = \bar{x}(i_t)$. This action extends linearly to V_m and makes it into a $k[G]$-module. When $m = 1$, V_m is the induced module, and when $m = n$, V_m is the tensor induced module. If $\mathscr{O}_1, \cdots, \mathscr{O}_t$ are the $\bar{G}$-orbits on $\mathscr{M}$ and $W_i = \sum^{\oplus} U_g$ $(g \in \mathscr{O}_i)$ then $V_m = \sum^{\oplus} W_i$ is a $k[G]$-decomposition of V_m. Thus there is an "induced" module belonging to U for each $m = 1, 2, \cdots, n$ and each orbit $\mathscr{O}$ of $\bar{G}$ on $\mathscr{M}$.

In what follows, we extend induced modules to a wreath product and prove that certain mappings are compatible. This wreath product may be used to show that induction is independent of the transversal $\mathscr{T}$. We discuss only induction and tensor induction, but the arguments apply equally to all the "induced" modules described in this remark.

C. **Extension to wreath products.** Recall the wreath product $\tilde{G} = H_0 \sim \bar{G}$. Let $\tilde{G}_1 = \{\bar{x}f \mid \bar{x}(1) = 1,\ f \in H_0^{\bar{x}}\}$ so that $\Phi(H) \leqq \tilde{G}_1$. We make U into a $k[\tilde{G}_1]$-module (projective or otherwise) by setting

$$\bar{x}f \cdot u = f(1)u \quad \text{for} \quad \bar{x}f \in \tilde{G}_1 \quad \text{and} \quad u \in U\,. \tag{2.11}$$

For the moment we call this module U_0 to distinguish it from U. Using the identification mapping $\tau\colon U \to U_0$ it is easy to see, since U is naturally a $k[H_0]$-module, that

$$\tau(hu) = \Phi(h)\tau(u)$$

for $h \in H$ and $u \in U$. We may "induce" the module U_0 to $\tilde{G}$ to obtain a module V_0 which is either $U_0|^{\tilde{G}}$ or $U_0|^{\otimes\tilde{G}}$. We also "induce" U to G to obtain a module V which is $U|^G$ or $U|^{\otimes G}$. Define mappings:

(2.12) (a) *If* $V = U|^G$ *and* $V_0 = U_0|^{\tilde{G}}$ *then define* I: $V \to V_0$ *by linear extension of*

$$I(x_i \otimes u) = \bar{x}_i \otimes \tau(u) .$$

(b) *If* $V = U|^{\otimes G}$ *and* $V_0 = U_0|^{\otimes \tilde{G}}$ *then define* I: $V \to V_0$ *by linear extension of*

$$I(\prod^{\otimes}(x_i \otimes u_i)) = \prod^{\otimes}(\bar{x}_i \otimes \tau(u_i)) .$$

These vector space isomorphisms have the following property.

(2.13) THEOREM. *In* (2.12), *if* $x \in G$ *and* $v \in V$ *then*

$$I(xv) = \Phi(x)I(v) .$$

The proof is by direct computation, and for tensor and ordinary induction, the proofs are of the same general shape. Therefore, we only sketch the computation for tensor induction. Observe that $\mathscr{T} = \{\bar{x}_i | 1 \leqq i \leqq n\}$ is a transversal for $\tilde{G}_1$ in $\tilde{G}$ so that I defines a vector space isomorphism of V onto V_0. If we set $\tilde{x} = \bar{x}f_x$ then we first compute $\tilde{f}_{\tilde{x}}(i)u$ where $u \in U_0$ and $\tilde{f}_{\tilde{x}}$ is defined for $\tilde{G}$ with respect to the transversal $\mathscr{T}$ by (2.2). If $\tilde{H}^*$ is in the kernel of $\tilde{G}_1$ on U_0 then with $j = \bar{x}(i)$

$$\begin{aligned} \tilde{f}_{\tilde{x}}(i) &= \bar{x}_j^{-1}(\bar{x}f_x)\bar{x}_i\tilde{H}^* \\ &= \bar{x}_j^{-1}\bar{x}\bar{x}_i f_x^{\bar{x}_i}\tilde{H}^* . \end{aligned}$$

Since $\bar{x}_j^{-1}\bar{x}\bar{x}_i(1) = 1$, we have

$$\begin{aligned} \tilde{f}_{\tilde{x}}(i) \cdot u &= f_x^{\bar{x}_i}(1)u \\ &= f_x(i)u . \end{aligned}$$

Consequently, using the fact that $\tau(u) = u$ we have

$$\begin{aligned} &\Phi(x)I(\prod^{\otimes}(x_i \otimes u_i)) \\ &\quad = \bar{x}f_x \cdot \prod^{\otimes}(\bar{x}_i \otimes u_i) \\ &\quad = \prod^{\otimes}(\bar{x}_j \otimes w_j) \\ &\quad = I(\prod^{\otimes}(x_j \otimes w_j)) \\ &\quad = I(x \cdot \prod^{\otimes}(x_j \otimes u_j)) \end{aligned}$$

where $j = \bar{x}(i)$ and $w_j = \tilde{f}_{\tilde{x}}(i)u_i = f_x(i)u_i$.

Using this embedding we may prove:

(2.14) PROPOSITION. *Induction and tensor induction are independent of transversal.*

The transversal does not affect the group $\bar{G}$. In fact, trans-

versals $\mathscr{T}$ and $\mathscr{T}'$ for H in G only alter the homomorphism Φ of (2.4). When applying (2.13), by (2.5) these two transversals give rise to conjugate subgroups of $\tilde{G}$. The element f of (2.5) then gives the necessary equivalence of modules by its action on V_0.

The embedding (2.13) allows us to enlarge the group G to $\tilde{G}$ acting upon $V_0 \cong V$ (via Φ, I). Since $\tilde{G}$ is the split extension of a permutation group by a direct product of copies of H_0, it is often easier to compute the action of $\tilde{G}$ on V_0 than that of G on V. It is this computational advantage which we exploit in later sections.

REMARK. The Clifford theorems for tensor decomposed modules only hold in a very narrow setting where the Fitting subgroup $F(G)$ has class 2. This setting reduces to ordinary Clifford theory on $F(G)/Z(G)$ viewed as a G-module. We shall discuss the appropriate concepts in § 5.

3. The wreath product and permutation representations. Let G be a group, and $C = \langle(12\cdots n)\rangle \leqq S^n$ where $\Omega = \{1, 2, \cdots, n\}$.

Assume that G is given as a permutation group on a set Γ. Form the set $\Gamma^\Omega = \{g: \Omega \to \Gamma \mid g \text{ a function}\}$. We shall write elements of $G \sim C$ as yf where $y \in C$ and $f \in G^\Omega$. If $g \in \Gamma^\Omega$ then yf acts upon g by

$$g^{yf}(i) = (g(y(i)))f(i) .$$

Thus $G \sim C$ acts naturally upon Γ^Ω.

We shall study the following type of configuration. We have a certain subgroup H of $G \sim C$ for which we know that $HG^\Omega = G \sim C$. Further, we assume certain facts about the orbit structure of G upon Γ. For example, G may have regular orbits upon Γ. Our question then is: when will H have regular orbits upon Γ^Ω? Certainly, if $G \sim C$ has regular orbits upon Γ^Ω, then H will also. We study this case first.

(3.1) PROPOSITION. *If G permutes the elements of Γ so that:*

(a) *there are at least two regular G-orbits on Γ and $n > 2$ then $G \sim C$ has at least two regular orbits on Γ^Ω;*

(b) *there are at least three regular G-orbits on Γ then $G \sim C$ has at least three regular orbits on Γ^Ω.*

Similar results are proved in [22]. In proving results like these it is only necessary to obtain the right number of distinct regular orbit representatives of G on Γ then to use these to construct regular orbit representatives for $G \sim C$ on Γ^Ω. Consequently, proofs proceed by writing down the answer, then verifying that it is cor-

rect. Most verifications follow a fixed pattern. Therefore, only a few verifications are included, and they vary in completeness.

Let $s = 2, 3$ and $\mathcal{O}_i$, $1 \leqq i \leqq s$, be regular orbits of G on Γ. Let $\omega_i \in \mathcal{O}_i$. Let

$$\begin{aligned} g_1(i) &= \omega_1 \quad \text{if} \quad i = 1 \\ &\quad\ \omega_2 \quad \text{if} \quad i > 1 \\ g_2(i) &= \omega_2 \quad \text{if} \quad i = 1 \\ &\quad\ \omega_1 \quad \text{if} \quad i > 1 \\ g_2'(i) &= \omega_1 \quad \text{if} \quad i = 1 \\ &\quad\ \omega_3 \quad \text{if} \quad i > 1 \\ g_3(i) &= \omega_2 \quad \text{if} \quad i = 1 \\ &\quad\ \omega_3 \quad \text{if} \quad i > 1 . \end{aligned}$$

If we are in case (a) and $s = 2$ we consider the orbits generated by $\{g_1, g_2\}$. In case (b) where $s = 3$ we consider the orbits generated by $\{g_1, g_2', g_3\}$.

Assume $\omega, \omega' \in \Gamma$ generate distinct regular G-orbits. Let

$$\begin{aligned} g(i) &= \omega' \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 . \end{aligned}$$

First we show that g generates a regular $G \sim C$-orbit on Γ^Q. Suppose $yf \in G \sim C$ fixes g. Then

$$g(i) = g^{yf}(i) = (g(y(i)))f(i)$$

for all i. Assume $y \neq 1$ and choose i so that $y(i) = 1$. Then $g(i) = \omega'$ and $g(y(i)) = g(1) = \omega$. Thus

$$(\omega)f(i) = \omega' .$$

But $(\omega)z \neq \omega'$ for any $z \in G$. Thus $y = 1$. Here we have

$$(g(i))f(i) = g(i)$$

for every i. But ω, ω' generate regular G-orbits so

$$f(i) = 1$$

for each i. Thus $yf = 1$. This proves g generates a regular $G \sim C$-orbit. In particular, g_1, g_2, g_2', g_3 all generate regular $G \sim C$-orbits.

Second, as an example of the computations, we show that for $n > 2$, g_1, g_2 generate distinct regular $G \sim C$-orbits. Assume $yf \in G \sim C$ and

$$g_1^{yf} = g_2 .$$

Two possibilities occur.

(1) $y = 1$.

Now $g_1^f(i) = g_2(i)$ for all i. In particular, for $i = 1$ we get $(g_1(1))f(1) = g_2(1)$ or

$$(\omega_1)f(1) = \omega_2$$

which is not possible since ω_1, ω_2 belong to distinct G-orbits.

(2) $y \neq 1$.

Choose i so that $i \neq 1$ and $y(i) \neq 1$. This is possible since $n > 2$. Then

$$(\omega_2)f(i) = (g_1(y(i)))f(i) = g_2(i) = \omega_1$$

which is again not possible.

Therefore, g_1, g_2 generate distinct regular $G \sim C$-orbits.

This completes the example. The rest of the proof is similar.

Recall the groups $H \leqq G \sim C$ and G. In applications, we will consider the case where both H and G are nilpotent. As is evident from (3.1) we are looking for a situation where H has several regular orbits upon Γ^Ω. The wreath product collects such orbits quite rapidly for most subgroups H. For nilpotent groups which involve more than one prime in their order this is especially true. The next result makes this idea more precise.

(3.2) PROPOSITION. *Let* $K = L \times M \leqq G \sim C$ *with* $L \geqq C$, $1 \neq M \leqq G^\Omega$, *and* $(|L|, |M|) = 1$. *Assume that* G *has at least* $s \geqq 1$ *regular orbits upon* Γ.

(a) *If* $n > 4$ *then* K *has at least three regular orbits upon* Γ^Ω.

(b) *If* $n = 3$ *and*

(i) $|M| > 2$, *or*

(ii) $|M| = 2$ *and* $s \geqq 2$, *or*

(iii) $|M| = 2$, $L = C$, *and* Γ *is not the regular* G*-orbit,*

then K *has at least three regular orbits upon* Γ^Ω.

(c) *If* $n = 2$ *and*

(i) $|M| > 5$, *or*

(ii) $|M| = 3, 5$ *and* $s \geqq 2$, *or*

(iii) $|M| = 5$, $L = C$, *and* Γ *is not the regular* G*-orbit,*

then K *has at least three regular orbits upon* Γ^Ω.

(d) *If* $n = 3$, $|M| = 2$, *then* K *has at least one regular orbit upon* Γ^Ω.

(e) *If* $n = 2$ *and* $|M| = 3, 5$ *then* K *has at least* t *regular orbits upon* Γ^Ω *where* $t = 1$ *if* $|M| = 3$ *or* $t = 2$ *if* $|M| = 5$.

For the proof we list, in each case, elements which generate distinct regular K-orbits on Γ^Ω. Suppose $\omega \in \Gamma$ generates a regular

G-orbit. If there is another regular G-orbit $\mathcal{O}$ on Γ let $\mu \in \mathcal{O}$ be a generator.

Note that $C \leqq L$ and $[L, M] = 1$. For $y \in C$ and $f \in M$ this means $f^y(i) = f(i)$ for all i. Thus $f(1) = f(i)$ for all i. In particular, if $f(1) = z$ then $f(i) = z$ for all i. With this in mind, we choose $f \in M^\#$ where $f(1) = z$. We now list the orbit generators in Γ^Ω.

(a) If $n > 4$ set

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_2(i) &= \omega z \quad \text{if} \quad i = 1, 2 \\ &\quad\ \omega \quad \text{if} \quad i > 2 \\ g_3(i) &= \omega z \quad \text{if} \quad i = 1, 3 \\ &\quad\ \omega \quad \text{if} \quad i = 2 \quad \text{or} \quad i > 3 . \end{aligned}$$

(b) (i) Choose $f' \in M^\#$ so that $f'(i) = u \neq z$. Set

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_2(i) &= \omega u \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_3(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega u \quad \text{if} \quad i = 2 \\ &\quad\ \omega \quad \text{if} \quad i = 3 \end{aligned}$$

(ii) Let $\mu \in \Gamma - \omega^G$ generate a regular G-orbit distinct from ω^G. Set

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_2(i) &= \omega \quad \text{if} \quad i = 1 \\ &\quad\ \mu \quad \text{if} \quad i > 1 \\ g_3(i) &= \mu \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \end{aligned}$$

(iii) Let $\mu \in \Gamma - \omega^G$ generate a G-orbit distinct from ω^G. Set

$$\begin{aligned} g_1(i) &= \omega \quad \text{if} \quad i = 1 \\ &\quad\ \mu \quad \text{if} \quad i > 1 \\ g_2(i) &= \mu \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_3(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 . \end{aligned}$$

(c) (i) Choose $h, k \in M^\sharp$ so that f, h, k are unequal and not inverses of one another. Suppose $h(1) = u$ and $k(1) = w$. Set

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad = 2 \\ g_2(i) &= \omega u \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad = 2 \\ g_3(i) &= \omega w \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad = 2 \end{aligned}$$

(ii) Suppose $\mu \in \Gamma - \omega^G$ generates a second regular G-orbit. Set

$$\begin{aligned} g_1(i) &= \mu \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i = 2 \\ g_2(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i = 2 \\ g_3(i) &= \mu z \quad \text{if} \quad i = 1 \\ &\quad\ \mu \quad \text{if} \quad i = 2 \end{aligned}$$

(iii) Let $\mu \in \Gamma - \omega^G$ generate an orbit distinct from ω^G. Let $h \in M^\sharp$, $h(i) = u \neq z,\ z^{-1}$.

$$\begin{aligned} g_1(i) &= \omega \quad \text{if} \quad i = 1 \\ &\quad\ \mu \quad \text{if} \quad i = 2 \\ g_2(i) &= \omega u \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i = 2 \\ g_3(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i = 2\,. \end{aligned}$$

(d) Set

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1\,. \end{aligned}$$

(e) Assume $|M| = 5$. Choose $h \in M^\sharp$, $h(i) = u \neq z,\ z^{-1}$.

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1 \\ g_2(i) &= \omega u \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1\,. \end{aligned}$$

(e′) Assume $|M| = 3$

$$\begin{aligned} g_1(i) &= \omega z \quad \text{if} \quad i = 1 \\ &\quad\ \omega \quad \text{if} \quad i > 1\,. \end{aligned}$$

Let us look at a few examples in the proof. Suppose $g(1) = \omega$ and $g(i) = \omega z$ for $i > 1$. We show that g generates a regular K-orbit. Suppose $yf^* \in K$ fixes g. Consider first $y \neq 1$. Then

$$g(i) = g^{yf^*}(i) = g(y(i))f^*(i) .$$

Now taking $i = 1$ we have, since $y(i) > 1$, that

$$\omega z = g(y(1))f^*(1) = \omega f^*(1) .$$

Since ω^G is regular, $f^*(1) = z$. For $y(i_0) = 1$, $i_0 > 1$ we have $\omega = g(1)f^*(i_0) = \omega z f^*(i_0)$. Now $f^*(i_0) = z^{-1}$. Finally, for $i \neq i, i_0$ we have

$$\omega = g(y(i))f^*(i) = \omega f^*(i)$$

so that $f^*(i) = 1$. This tells us the value of f^*. Namely,

$$f^*(i) = \begin{cases} z & i = 1 \\ z^{-1} & i = i_0 \\ 1 & i \neq 1, i_0 . \end{cases}$$

Let K_g be the stabilizer in K of g. Since $[L, M] = 1$ and $(|L|, |M|) = 1$ we have $K_g = L_g \times M_g$. For $yf^* \in K_g$, $y \neq 1$, we may assume that $yf^* \in L_g$ by taking an appropriate power of yf^*. But then $yf^* \in L$ so that $y^{-1}yf^* = f^* \in L$ since $y \in C \leqq L$. Since the order of z divides $|M|$ ($f \in M$), f^* lies in $L \cap M = 1$. This contradicts $z \neq 1$. We conclude that $y = 1$.

Now $g(i) = g^{f^*}(i) = g(i)f^*(i)$. For $i = 1$ we obtain $\omega z = \omega z f^*(1)$. For $i > 1$ we obtain $\omega = \omega f^*(i)$. In any case, $f^*(i) = 1$ for all i. So $yf^* = 1$.

We have proved that the various g_i's all generate regular K-orbits. We prove the case (e) to illustrate the method used to prove the various orbits are distinct. Suppose $g_1^{yf^*} = g_2$. Assume that $y \neq 1$. So $y = (12)$. Now

$$\omega u = g_2(1) = g_1^{yf^*}(1) = g_1(2)f^*(1) = \omega f^*(1) ,$$

and

$$\omega = g_2(2) = g_1^{yf^*}(2) = g_1(1)f^*(2) = \omega z f^*(2) .$$

We have shown that $f^*(1) = u$ and $f^*(2) = z^{-1}$. Since $y \in K$, $y^{-1}yf^* \in K$. By order we have $f^* \in M$. But then $f^*(1) = f^*(2)$. Since $u \neq z^{-1}$ we conclude that $y = 1$. Again computing we have

$$\omega u = g_2(1) = g_1^{f^*}(1) = \omega z f^*(1) ,$$

and

$$\omega = g_2(2) = g_1^{f^*}(1) = \omega f^*(1) .$$

Thus $f^*(1) = z^{-1}u$ and $f^*(2) = 1$. By the order of f^* we must have

$f^* \in M$. So f^* is constant. We conclude that $z^{-1}u = 1$. But $u \neq z$. This contradiction completes the proof. We know that g_1, g_2 generate distinct regular K-orbits.

The remaining computations are much the same.

The next result enables us to treat wreath-free groups.

(3.3) PROPOSITION. *Suppose $K \leqq G \sim C$ is nilpotent and $KG^{\Omega} = G \sim C$. Assume n is a prime and $\boldsymbol{Z}_n \sim \boldsymbol{Z}_n$ is not involved in K. If G has at least one regular orbit $\mathcal{O}$ on Γ and $\Gamma \neq \mathcal{O}$ then K has at least one regular orbit $\tilde{\mathcal{O}}$ on Γ^{Ω} and $\Gamma^{\Omega} \neq \tilde{\mathcal{O}}$.*

Choose $\omega \in \Gamma$ so that ω^G is a regular G-orbit. Then choose $\mu \in \Gamma - \omega^G$.

Set

$$g_1(i) = \begin{cases} \mu & \text{if } i = 1 \\ \omega & \text{if } i > 1 \end{cases}$$

$$g_2(i) = \omega \quad \text{all } i\,.$$

Certainly g_1, g_2 generate distinct $G \sim C$-orbits so Γ^{Ω} consists of at least two K-orbits.

It is not difficult to see that $C_{G\sim C}(g_1) \cap K \leqq \{f \in G^{\Omega} \mid f(1) \in C_G(\mu);\ f(i) = 1,\ i > 1\}$. If $C_{G\sim C}(g_1) \cap K = 1$ then g_1 generates a regular K-orbit on Γ^{Ω}. So we may assume that $C_{G\sim C}(g_1) \cap K = K_0 > 1$. Now if $K \cap C = C$ then $\langle C, K_0 \rangle \leqq K$ is either not nilpotent or involves $\boldsymbol{Z}_n \sim \boldsymbol{Z}_n$. So we may assume $K \cap C = 1$.

Recall $g_2(i) = \omega$ all i. Now $C_{G\sim C}(g_2) = C$ and $K \cap C = 1$. Therefore g_2 generates a regular K-orbit. So the proof is complete.

This improvement of an earlier lemma of the author is due to E. C. Dade. Actually, we will need this lemma in a slightly different form also. Let $(\Gamma^{\Omega})^*$ be the collection

$$\bigcup_{\alpha \in \Omega} \Gamma^{\Omega - \{\alpha\}}\,.$$

Thus if $g \in (\Gamma^{\Omega})^*$, g will have domain $\Omega - \{\alpha\}$ for some choice of $\alpha \in \Omega$. For $yf \in G \sim C$ we still have the action

$$g^{yf}(i) = (g(y(i)))f(i)$$

where, if $g(i)$ has domain $\Omega - \{\alpha\}$ then $g(y(i))$ has domain $\Omega - \{y^{-1}(\alpha)\}$. Thus $G \sim C$ has a natural action upon $(\Gamma^{\Omega})^*$.

(3.4) PROPOSITION. *Suppose $K \leqq G \sim C$ is nilpotent and $KG^{\Omega} = G \sim C$. Assume n is a prime and $\boldsymbol{Z}_n \sim \boldsymbol{Z}_n$ is not involved in K. If G has at least one regular orbit on Γ then K has at least one regular orbit on $\Gamma^{\Omega} \cup (\Gamma^{\Omega})^*$.*

Choose $\omega \in \Gamma$ to generate a regular G-orbit. Set

$$g_1(i) = \omega \quad \text{if} \quad i > 1$$
$$g_2(i) = \omega \quad \text{all} \quad i .$$

Now $g_1 \in (\Gamma^\Omega)^*$ and $g_2 \in \Gamma^\Omega$. Using the same argument as in (3.3) we see that either g_1 or g_2 generates a regular K-orbit.

4. **Applications.** In this section we combine the results of §§ 2 and 3 to obtain information about induced and tensor induced representions. We are interested in the regular structure of induced modules.

(4.1) HYPOTHESIS. *Assume the following*:

(a) *G is a nilpotent group with subgroup H;*

(b) *k is a field and V is a $k[H]$-module faithful on $\bar{H} = H/H_0$, $H_0 = \ker [H \to \mathrm{Aut}\, V]$.*

(c) *G is faithful on $V|^G$ (or $V|^{\otimes G}$, as the case may be).*

A. **Induction.**

(4.2) PROPOSITION. *If $\bar{H}$ permutes the elements of $V^\sharp$ so that*

(a) *there are at least two regular H-orbits and $[G: H]$ is odd, then G has at least two regular orbits on $(V|^G)^\sharp$;*

(b) *there are at least three regular H-orbits, then G has at least three regular orbits on $(V|^G)^\sharp$;*

(c) *there is at least one regular $\bar{H}$-orbit and G does not involve $Z_p \sim Z_p$ for any prime $p | [G: H]$, then G has at least one regular orbit on $(V|^G)^\sharp$.*

In each case, the proof is the same. For (a) we use (3.1)(a). For (b) we use (3.1)(b) and finally for (c) we use (3.4). We will only prove (c). The proof is by induction upon $[G: H]$. If $[G: H] = 1$, then (c) is obvious, so that we assume $[G: H] > 1$ and (c) holds for all indices smaller than $[G: H]$. Suppose $[G: H]$ is not a prime. Since G is nilpotent, we may choose H_1 so that $H < H_1 < G$. Induction applies with H_1 in place of G. Thus there is a regular H_1-orbit on $W^\sharp = (V|^{H_1})^\sharp$. Now induction applies with H_1 in place of H and W in place of V. Thus we may assume that $[G: H] = p$ is a prime.

By (2.4) there is a natural embedding of G into $\bar{H} \sim C$ where $C = \langle (12 \cdots p) \rangle$. By (2.13) this embedding is compatible with induction. If we take $V^\sharp = \Gamma$, then $g \in \Gamma^\Omega$ may be identified with the vector

$$x_1 \otimes g(1) + \cdots + x_p \otimes g(p) \in V|^G$$

($x_1, \cdots, x_p$ a transversal of H in G)

where all $g(i) \neq 0$. Further, $g \in (\Gamma^Q)^*$ may be identified with the vector

$$x_1 \otimes g(1) + \cdots + x_p \otimes g(p) \in V|^G$$

where we set $g(i) = 0$ if i is not in the domain of g. Now (c) is immediate by (3.4).

(4.3) PROPOSITION. *Let* $[G: H] = p$ *a prime,* $\bar{H} = Q \times B$ *where* Q *is a* p*-group and* B *is a* p'*-group,* $|B| \neq 1$. *Assume* $\bar{H}$ *has at least one regular orbit on* $V^\sharp$.

(a) *If* $p > 3$ *then* G *has at least three regular orbits on* $(V|^G)^\sharp$.

(b) *If* $p = 3$ *and*

(i) $|B| > 2$, *or*

(ii) $|B| = 2$ *and* $\bar{H}$ *has at least two regular orbits on* $V^\sharp$, *or*

(iii) $|B| = 2$, $|Q| = 1$ *and* $V^\sharp$ *is not the regular* B *orbit,*

then G *has at least three regular orbits on* $(V|^G)^\sharp$.

(c) *If* $p = 2$ *and*

(i) $|B| > 5$, *or*

(ii) $|B| = 3, 5$ *and* $\bar{H}$ *has at least two regular orbits on* $V^\sharp$, *or*

(iii) $|B| = 5$, $|Q| = 1$, *and* $V^\sharp$ *is not the regular* $\bar{H}$*-orbit,*

then G *has at least three regular orbits on* $(V|^G)^\sharp$.

(d) *If* $p = 3$, $|B| = 2$ *then* G *has at least one regular orbit on* $(V|^G)^\sharp$.

(e) *If* $p = 2$ *and* $|B| = 3, 5$ *then* G *has at least* s *regular orbits on* $(V|^G)^\sharp$ *where* $s = 1$ *if* $|B| = 3$ *and* $s = 2$ *if* $|B| = 5$.

To prove this we apply (2.4), (2.13), and (3.2). In looking at G embedded in $\bar{H} \sim C$ we may have to enlarge G to $G_0 = \langle G, Q^Q \rangle$ to make certain that $C \leqq G_0$. Observe that G_0 is nilpotent. Then (3.2) applies to G_0. The result for G is obtained by restriction. We let $\Gamma = V^\sharp$ and identify Γ^Q with the set of vectors $x_1 \otimes v_1 + \cdots + x_p \otimes v_p \in V|^G$ with all $v_i \neq 0$ ($x_1, \cdots, x_p$ a transversal for H in G) via $g \mapsto x_1 \otimes g(1) + \cdots + x_p \otimes g(p)$.

B. Tensor induction. The results here go exactly as in the case for ordinary induction. The pattern is as follows. Suppose $V = V_1 + \cdots + V_s + W$ where each V_i is a regular $k[\bar{H}]$-module. Then we may choose $S_i = \{hv_i \mid h \in \bar{H}\}$ for some fixed $v_i \in V_i$ so that S_i is a k-basis for V_i. If $W \neq (0)$ we choose $w \in W^\sharp$ and let $T = \{hw \mid h \in \bar{H}\}$. We take $\Gamma = T \cup (\bigcup S_i)$. Then $\bar{H}$ permutes the elements of Γ with at least s regular orbits. Let $\{x_1, \cdots, x_p\}$ be a

transversal of H in G if $[G: H] = p$. We identify Γ^{Ω} with

$$\{(x_1 \otimes v_1) \otimes \cdots \otimes (x_p \otimes v_p) \mid v_i \in \Gamma\};$$

via $g \mapsto (x_1 \otimes g(1)) \otimes \cdots \otimes (x_p \otimes g(p))$ where $g: \Omega \to \Gamma$ is a function. Let $\mathscr{S}$ be the set of all $g \in \Gamma^{\Omega}$ with $g(i) \in \bigcup S_j$ for all i. It is clear that $\mathscr{S}$ is a linearly independent set of vectors in $V|^{\otimes G}$ and is a basis for the $k[G]$-module $\langle \mathscr{S} \rangle$. Let $\mathscr{T}$ be the set of all $g \in \Gamma^{\Omega}$ such that $g(i) \in T$ for some i. Then $\mathscr{T}$ contains a basis for the $k[G]$-module $\langle \mathscr{T} \rangle$. Further $\langle \mathscr{S} \rangle + \langle \mathscr{T} \rangle$ is a direct sum of $k[G]$-modules. Note that $\langle \mathscr{S} \rangle$ has the G-permutation basis $\mathscr{S}$. If $W \neq (0)$ then $\langle \mathscr{T} \rangle \neq (0)$. Embedding G in $\bar{H} \sim C$ we see that Γ^{Ω} corresponds with its counterpart of § 3.

We now have the following results for tensor induction.

(4.4) PROPOSITION. *If V contains at least s copies of the regular $\bar{H}$-module where*

(a) *$s = 2$ and $[G: H]$ is odd then $V|^{\otimes G}$ contains at least two copies of the regular G-module;*

(b) *$s = 3$ then $V|^{\otimes G}$ contains at least three copies of the regular G-module;*

(c) *$s = 1$, V is not the regular $\bar{H}$-module, and G does not involve $\boldsymbol{Z}_p \sim \boldsymbol{Z}_p$ for any prime $p \mid [G: H]$ then $V|^{\otimes G}$ contains at least one copy of the regular G-module and $V|^{\otimes G}$ is not the regular G-module.*

Actually we use (2.4), (2.13) (i.e., the compatibility of the embedding with tensor induction) and (3.1), (3.3) in the proof. Other than that everything proceeds as in the induction case.

In a similar way we obtain the following variation of (4.3).

(4.5) PROPOSITION. *Let $[G: H] = p$, a prime, $\bar{H} = Q \times B$ where Q is a p-group and B is a p'-group, $|B| \neq 1$. Assume V contains at least one copy of the regular $\bar{H}$-module.*

(a) *If $p > 3$ then $V|^{\otimes G}$ contains at least three copies of the regular G-module.*

(b) *If $p = 3$ and*

(i) *$|B| > 2$, or*

(ii) *$|B| = 2$ and V contains at least two regular $\bar{H}$-modules, or*

(iii) *$|B| = 2$, $|Q| = 1$, and V is not the regular $\bar{H}$-module,*

then $V|^{\otimes G}$ contains at least three copies of the regular G-module.

(c) *If $p = 2$ and*

(i) *$|B| > 5$, or*

(ii) *$|B| = 3, 5$ and V contains at least two regular $\bar{H}$-modules:*

or

(iii) $|B| = 5$, $|Q| = 1$, *and* V *is not the regular* $\bar{H}$*-module, then* $V|^{\otimes G}$ *contains at least three copies of the regular* G*-module.*

(d) *If* $p = 3$, $|B| = 2$ *then* $V|^{\otimes G}$ *contains at least one regular* G*-module.*

(e) *If* $p = 2$ *and* $|B| = 3, 5$ *then* $V|^{\otimes G}$ *contains at least* s *regular* G*-modules where* $s = 1$ *if* $|B| = 3$ *and* $s = 2$ *if* $|B| = 5$.

REMARK. A few observations are in order on these results. Recall the choice of Γ^2 in the proofs of (4.2) and (4.3). Excepting possibly the case where $[G: H] = 2$ in (4.2) (c) there are more orbits of G upon $(V|^G)^\sharp$ than those in Γ^2.

For tensor induction, a similar conclusion holds. The module $\langle \mathscr{S} \rangle$ cannot be the given number of regular modules. For example, look at (4.5) (e) when $|B| = 5$. Then $\langle \mathscr{S} \rangle$ cannot be a sum of just two regular modules. Note that $|G| \,|\, p|Q|^p|B| = 10|Q|^2$. On the other hand $|\mathscr{S}| = |QB|^p = 25|Q|^2$. This latter number is much larger than $2|G|$. Thus $V|^{\otimes G}$ is more than just a sum of two regular G-modules.

The results (4.2) and (4.4) apply in a general setting. The results (4.3) and (4.5) apply when $|\bar{H}|$ has composite order. These latter two results indicate that it is much harder for a group of composite order to avoid some kind of regular structure.

5. A transference theorem.

A. The main construction. In this section we prove a technical theorem which makes the method of tensor induction applicable in the study of primitive linear groups. The setting is rather complex so that we fix the following hypotheses.

(5.1) HYPOTHESIS. *Assume* G *is a solvable group with normal extraspecial* r*-subgroup* R *where* $Z(R) \leqq Z(G)$ *and* $R/Z(R)$ *is a chief factor of* G. *Suppose* $H \leqq G$, $R_1 \leqq R$ *so that*

(1) R_1 *is extraspecial,*

(2) $xR_1x^{-1} = R_1$, *or* $[xR_1x^{-1}, R_1] = 1$ *for all* $x \in G$,

(3) H *normalizes* R_1, *and*

(4) *as a* G*-module* $R/Z(R) \cong (R_1/Z(R))|^G$ *where* $R_1/Z(R)$ *is viewed as an* H*-module.*

REMARK. For $T \leqq R$ we set $\bar{T} = TZ(R)/Z(R)$. Clearly condition (4) implies that $H = N_G(R_1)$. Now since $\bar{R}$ is a chief factor for G, G acts irreducibly on it. This together with the fact that $\bar{R}_1|^G \cong \bar{R}$ tells us that $R \leqq H$.

(5.2) PROPOSITION. *There is a subgroup N of G so that*
(1) $NR = G$,
(2) $N \cap R = Z(R)$, *and*
(3) $N \geqq C(R)$.
If $M \leqq G$ satisfies (1), (2), *and* (3) *then $M^x = N$ for some $x \in G$.*

Set $C = C_G(R/Z(R))$. Let T/C be minimal normal in G/C. Since G is solvable, T/C is an elementary abelian t-group for a prime t. But $R/Z(R)$ represents G/C faithfully and irreducibly so that T/C is an r'-group acting fixed point freely upon $R/Z(R)$. Let S be a t-Sylow of T so that $T = SC$. The normality of T in G means that $G = N(S)T = N(S)C$ by the Frattini argument. (Also S is fixed point free on $\bar{R}$ since $T = SC$.)

By [18, (5.4.6)] we have $C = C(R)R$ so that we set $N = N(S)C(R)$. Since S is fixed point free upon $\bar{R}$ $(=R/Z(R))$, so is $SC(R)/C(R)$, and $S^xC(R)/C(R) \neq SC(R)/C(R)$ for any $x \in R \backslash Z(R)$. If $x \in N \cap R$ then $x = yz$ where $y \in N(S)$, $z \in C(R)$. We now have $S^xC(R) = S^{yz}C(R) = S^zC(R) = SC(R)$ so that $x \notin R \backslash Z(R)$. This shows that $N \cap R \leqq Z(R)$, and since $Z(R) \leqq N \cap R$ we conclude that $Z(R) = N \cap R$. Therefore, (1), (2), and (3) hold for N.

Suppose that $M \leqq G$ satisfies (1), (2), and (3). Now $G/C(R)$ $(=MR/C(R))$ has $RC(R)/C(R)$ $(\cong \bar{R})$ as a normal subgroup which is minimal (since G acts irreducibly on $\bar{R}$), and clearly is unique (as any other minimal normal subgroup would centralize it). Also $M/C(R)$ and $N/C(R)$ are complements for $\bar{R}$ in $G/C(R)$, and so there is $y \in G$ such that $M^y = N$. This completes the proof.

We set up some notation now. Choose N as in (5.2) and form the semidirect product

$$G^* = N \cdot R ,$$

and then set

$$N_0 = N \cap H ,$$
$$G_0^* = N_0 \cdot R_1 ,$$

and

$$H^* = N_0 \cdot R$$

where the starred groups are subgroups of G^*. The mapping ϕ: $G^* \longrightarrow G$ given by

$$\phi(x, y) = xy$$

is a homomorphism of G^* onto G. In addition, ϕ maps G_0^* onto $G_0 = N_0R_1$ and H^* onto $H = N_0R$. It is the group G^* which will concern us for the time being.

Let $R_1^{\perp} = C_R(R_1) = \Pi(xR_1x^{-1})$ where the product is over all $x \in G$ such that $xR_1x^{-1} \neq R_1$. It is easy to see that R is the central product $R_1R_1^{\perp}$.

The construction we now undertake shows that if U is a projective $k[G_0]$-module such that $U|_{R_1}$ is nonprojective, faithful, and irreducible then U has a projective extension to U^*, a $k[H]$-module, such that $U^*|^{\otimes G} = V$ is a projective $k[G]$-module for which $V|_R$ is nonprojective, faithful, and irreducible. Since U^* is constructed in a canonical fashion, we will alter the definition of tensor induction so that $U|^{\otimes G}$ is defined to be $U^*|^{\otimes G}$. This abuse of language will not cause any confusion since the situation surrounding this construction of $U^*|^{\otimes G}$ is so complex that it will clearly indicate that when $U|^{\otimes G}$ is written, really $U^*|^{\otimes G}$ is meant.

If $T \leqq N$ is being viewed as a subgroup of N then we set $T^* = \{(t, 1) | t \in T\}$, and if $T \leqq R$ is being viewed as a subgroup of R then we set $T^* = \{(1, t) | t \in T\}$. With these conventions, we view $Z(R)$ as a subgroup of R unless explicitly stated otherwise.

We have pre-empted the bar notation for $\bar{R} = R/Z(R)$ so that we use $\hat{}$-notation where we have used a bar in previous sections. Let $\mathscr{T} = \{x_1 = 1, x_2, \cdots, x_s\}$ be a transversal for N_0 in N. Since $NH = G$ and $N \cap H = N_0$, $\mathscr{T}$ is also a transversal for H in G. Let $x \longrightarrow \hat{x}$ be the permutation representation of G upon $\Omega = \{1, 2, \cdots, s\}$ given by $\hat{x}(i) = j$ if and only if $xx_iH = x_jH$. Form the wreath product

$$\tilde{G} = G_0^* \sim \hat{G} = \hat{G} \cdot G_0^{*\Omega} .$$

In the following, we embed G^* into a factor group of $\tilde{G}$. If $L \leqq G^*$ let $L^{\Omega} = \{f \in G_0^{*\Omega} | f(i) \in L \text{ for all } i\}$.

Next define

$$M = \{f \in Z(R)^{*\Omega} | \Pi f(i) = 1\} ,$$

that is, if $f(i) = (1, z_i)$ where $z_i \in Z(R)$ and $f \in M$ then $\Pi z_i = 1$. It is straightforward to show that $M \lhd \tilde{G}$. We wish to embed G^* in $\tilde{G}/M$.

Let $\mathscr{R}$ be a transversal for $Z(R)$ in R_1 containing 1. By the bar-convention, if $K \leqq R$ then $\bar{K} = KZ(R)/Z(R)$. Set

$$R_i = x_iR_1x_i^{-1} .$$

From (5.1) we have

$$\bar{R} = \bar{R}_1 + \bar{R}_2 + \cdots + \bar{R}_s .$$

(Clearly $s = [N: N_0] = [G: H]$ = number of distinct conjugates of R_1 in R.) Let $x \in R$. Then there are unique elements $r_1, r_2, \cdots, r_s \in \mathscr{R}$ so that

$$\bar{x} = x_1 \cdot \bar{r}_1 + \cdots + x_s \cdot \bar{r}_s .$$

Consequently

$$x = z\Pi(x_i r_i x_i^{-1})$$

for some unique $z \in Z(R)$. We set $r_1(x) = zr_1$ and $r_i(x) = r_i$ for $i > 1$ so that

$$x = \Pi(x_i r_i(x) x_i^{-1})$$

where the r_i are functions upon R.

For $x \in N$, $y \in R$ set

$$\Xi(x, y) = \hat{x} f_x h_y M$$

where

(a) $f_x(i) = (x_{\hat{x}(i)}^{-1} x x_i, 1)$,

(b) $h_y(i) = (1, r_i(y))$.

(5.3) PROPOSITION. *The mapping $\Xi: G^* \to \tilde{G}/M$ is a monomorphism.*

First, we prove that Ξ is a homomorphism. Let $(x, y), (u, v) \in G^*$. Then

$$(x, y)(u, v) = (xu, y^u v) .$$

Starting with $y^u v$ we have

$$y^u v = (u^{-1} y u) v = [\Pi(u^{-1} x_i r_i(y) x_i^{-1} u)] v .$$

Let j be an index dependent upon i by the relation $\hat{u}^{-1}(i) = j$. Then $u^{-1} x_i = x_j u_i$ for some $u_i \in N_0$, so that

$$y^u v = [\prod_i (x_j u_i r_i(y) u_i^{-1} x_j^{-1})] v .$$

We may assume that j is independent and that $i = \hat{u}(j)$ so that

$$\begin{aligned} y^u v &= [\prod_j (x_j u_i r_i(y) u_i^{-1} x_j^{-1})][\prod_j x_j r_j(v) x_j^{-1}] \\ &= \prod_j (x_j [u_i r_i(y) u_i^{-1} r_j(v)] x_j^{-1}) \end{aligned}$$

since R is the *central* product of the R_i's. But we also know that

$$y^u v = \Pi(x_j r_j(y^u v) x_j^{-1}) .$$

Comparing components in $\bar{R}$ we now must have

$$z_j r_j(y^u v) = u_i r_i(y) u_i^{-1} r_j(v) \tag{5.4}$$

where $i = \hat{u}(j)$ and $z_j \in Z(R)$.

But now

$$\begin{aligned} y^u v &= \prod_j (x_j z_j r_j (y^u v) x_j^{-1}) \\ &= [\prod_j x_j r_j (y^u v) x_j^{-1})][\prod_j z_j] \\ &= (y^u v)(\prod_j z_j) \ . \end{aligned}$$

Therefore, $\Pi z_i = 1$.

Let $\tilde{f}(i) = (1, z_i)$ for all i so that $\tilde{f} \in M$. Then

$$\begin{aligned} \Xi(x, y)\Xi(u, v) &= \hat{x} f_x h_y \hat{u} f_u h_v M \\ &= \hat{x}\hat{u}(f_x^{\hat{u}} f_u)(f_u^{-1} h_y^{\hat{u}} f_u h_v) M \ . \end{aligned}$$

In a fashion analogous to [23, III.5.k, 20, Satz 15.9] we have $f_x^{\hat{u}} f_u = f_{xu}$. Computing the value at each i, by setting $\hat{u}(i) = j$, we have

$$\begin{aligned} w(i) = f_u^{-1} h_y^{\hat{u}} f_u h_v(i) &= f_u^{-1}(i) h_y(j) f_u(i) h_v(i) \\ &= (x_j^{-1} u x_i, 1)^{-1}(1, r_j(y))(x_j^{-1} u x_i, 1)(1, r_i(v)) \\ &= (1, x_i^{-1} u^{-1} x_j r_j(y) x_j^{-1} u x_i r_i(v)) \\ &= (1, u_j r_j(y) u_j^{-1} r_i(v)) \end{aligned}$$

where $x_i^{-1} u^{-1} x_j = u_j \in N_0$. By equation (5.4)

$$\begin{aligned} w(i) &= (1, z_i r_i(y^u v)) \\ &= h_{y^u v} \tilde{f}(i) \ . \end{aligned}$$

We conclude that $w = h_{y^u v} \tilde{f}$. Since $\tilde{f} \in M$ we have

$$\begin{aligned} \Xi(x, y)\Xi(u, v) &= \hat{x}\hat{u} f_{xu} h_{y^u v} M \\ &= \Xi(xu, y^u v) \\ &= \Xi((x, y)(u, v)) \ . \end{aligned}$$

Consequently $\Xi: G^* \to \tilde{G}/M$ is a homomorphism.

Finally we show that Ξ is a monomorphism. If $\Xi(x, y) = M$ then $\hat{x} f_x h_y M = M$. Using the proof cited in [23, III.5.k] we may show that $\hat{x} f_x = 1$, so that $x = 1$. Therefore, $h_y \in M$ and $h_y(i) \in Z(R)^*$ for each i. But $\mathscr{R} \cap Z(R) = 1$ so that $h_y(i) = 1$ for all $i > 1$. Since $\Pi r_i(y) = 1$ (i.e., $h_y \in M$) we conclude that $r_1(y) = 1$ and $h_y(i) = 1$ for all i. Therefore, $h_y = 1$ so that $(x, y) = (1, 1)$. The proof that Ξ is a monomorphism is now complete.

(5.5) PROPOSITION. *Let k be a field and U a projective $k[G_0^*]$-module. Assume that $T \leqq G_0^*$ is a subgroup containing R_1^* and that $U|_T$ is nonprojective. Further, suppose that $U|_{R_1^*}$ is faithful and absolutely irreducible. The subgroup of $\hat{G}$ fixing 1 in $\Omega = \{1, 2, \cdots, s\}$ is $\hat{H}$. If $\hat{x} \in \hat{H}$ and $f \in G_0^{*\Omega}$ then set*

$$\hat{x} f \cdot u = f(1)u$$

for $u \in U$. This defines a projective $k[\hat{H}G_0^{*\Omega}]$-module $\tilde{U}(=U)$ which is nonprojective upon restriction to $\hat{H}T^\Omega$. Set $\tilde{V} = \tilde{U}|^{\otimes G}$.

(1) *The kernal of $R_1^{*\Omega}$ upon $\tilde{V}$ is M.*

(2) *The module $\tilde{V}|_{R_1^{*\Omega}}$ is absolutely irreducible.*

(3) *The module $\tilde{V}|_{\hat{G}T^\Omega}$ is nonprojective.*

Since $\mathscr{T} = \{\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_s\}$ is a transversal for $\hat{H}G_0^{*\Omega}$ in $\tilde{G}$, $\tilde{V}$ is the tensor product of the modules $\hat{x}_i \otimes \tilde{U}$. If $u \in \tilde{U}$, $f \in G_0^{*\Omega}$, $\hat{x} \in \hat{G}$ then

$$\hat{x}f \cdot \hat{x}_i \otimes u = \hat{x}_j \otimes f(i)u$$

where $j = \hat{x}(i)$. First, assume that $f \in R_1^{*\Omega}$. If $f \notin Z(R)^{*\Omega}$ then we may choose $u \in U$ and i so that $f(i)u$ and u are linearly independent since $f(i)$ does act not via scalar multiplication on U [18, (5.5.4)]. Therefore, $v = (\hat{x}_1 \otimes u) \otimes \cdots \otimes (\hat{x}_s \otimes u)$ and

$$f \cdot v = (\hat{x}_1 \otimes f(1)u) \otimes \cdots \otimes (\hat{x}_i \otimes f(i)u) \otimes \cdots \otimes (x_s \otimes f(s)u)$$

are linearly independent. If $z \in Z(R)^*$ then $zu = \lambda(z)u$ for $u \in U$ and λ a faithful linear character of $Z(R)^*$ since z acts via scalar multiplication on U. Let $f' \in Z(R)^{*\Omega}$ so that if $u_1, \cdots, u_s \in \tilde{U}$ and if $v' = (\hat{x}_1 \otimes u_1) \otimes \cdots \otimes (\hat{x}_s \otimes u_s)$ then

$$\begin{aligned} f' \cdot v' &= (\hat{x}_1 \otimes \lambda(f'(1))u_1) \otimes \cdots \otimes (\hat{x}_s \otimes \lambda(f'(s))u_s) \\ &= \lambda(\Pi f'(i))v' \ . \end{aligned}$$

These calculations show that an element $f \in R_1^{*\Omega}$ is in the kernel of $\tilde{V}$ if and only if $f \in M$ which proves that (1) holds.

It is easy to see that $R_1^{*\Omega}/M$ is isomorphic to the central product of s copies of R_1. If R_1 has order r^{2e+1} then $\dim U = r^e$ so that $R_1^{*\Omega}/M$ is extraspecial of order r^{2se+1} and $\dim \tilde{V} = r^{se}$. If $\tilde{V}|_{R_1^{*\Omega}}$ is nonprojective then (2) follows from (1) and knowledge of the dimension of faithful absolutely irreducible modules of extraspecial groups [18, (5.5.5)]. Therefore, part (2) follows from (3).

If $\hat{x}, \hat{y} \in \hat{G}$; $f, g \in T^\Omega$, $u \in \tilde{U}$, and α is the factor set of G_0^* upon U then $\alpha|_{T \times T} = 1$ since $U|_T$ is nonprojective, so that with $j = \hat{x}(i)$ and $k = \hat{y}(j)$,

$$\begin{aligned} \hat{y}g \cdot (\hat{x}f \cdot \hat{x}_i \otimes u) &= \hat{x}_k \otimes g(j)f(i)u \\ &= (\hat{y}g\hat{x}f) \cdot \hat{x}_i \otimes u \end{aligned}$$

since $\alpha(g(j), f(i)) = 1$. Therefore, $\hat{G}T^\Omega$ has trivial factor set upon $\tilde{V}$. Part (3) holds completing the proof of (5.5).

If we start with the projective $k[G_0^*]$-module U then we finish with a projective $k[G^*]$-module $V(=\tilde{V})$ defined by

$$xv = \Xi(x) \cdot v$$

for $x \in G^*$ and $v \in V(=\tilde{V})$. Since Ξ maps R^* onto R_1^{*Q}/M, $V|_{R^*}$ is a nonprojective faithful absolutely irreducible $k[R^*]$-module by (5.5) since $R^* \cong R_1^{*Q}/M$. Before describing properties of this module, let us show that the construction of V is essentially choice-free.

The choices involved in the construction for V were those of $\mathscr{T}$ and $\mathscr{R}$. Our choices are restricted by the conditions that $1 \in \mathscr{T} \subseteq N$ and $1 \in \mathscr{R}$. Suppose that we alter our choices to $\mathscr{T}' = \{x_i' = x_i a_i \mid 1 \leqq i \leqq s\}$ where $a_i \in N_0$ and to $\mathscr{R}'$ where for $y \in R$ $r_i'(y)$ is the r_i-function defined for the $\mathscr{T}'$, $\mathscr{R}'$-system. Let $f(i) = (a_i, 1)$, and conjugate $\Xi(x, y) = \hat{x} f_x h_y M$, by f. We have, as in (2.5),

$$f^{-1}\hat{x} f_x h_y f = \hat{x}(f^{-\hat{x}} f_x h_y f) .$$

By a simple computation we obtain

$$g(i) = f^{-\hat{x}} f_x h_y f(i) = f_x'(i)(1, r_i(y)^{a_i})$$

where f_x' is the appropriate function defined for the $\mathscr{T}'$, $\mathscr{R}'$-system.

Computing the value of $y \in R$ in the $\mathscr{T}$, $\mathscr{R}$- and $\mathscr{T}'$, $\mathscr{R}'$-systems gives

$$\begin{aligned} y &= \Pi(x_i r_i(y) x_i^{-1}) \\ &= \Pi(x_i' r_i'(y) x_i'^{-1}) \\ &= \Pi(x_i a_i r_i'(y) a_i^{-1} x_i^{-1}) . \end{aligned}$$

Since r_i, r_i' are constructed from transversals for $Z(R)$ in R_1, and since $a_i \in N_0$ normalizes R_1 we must have

$$r_i(y) = a_i r_i'(y) a_i^{-1} \zeta_i(y)$$

for $\zeta_i(y) \in Z(R)$ and $\Pi\zeta_i(y) = 1$. Consequently

$$\begin{aligned} g(i) &= f_x'(i)(1, r_i'(y)\zeta_i(y)) \\ &= f_x' h_y'(i)(1, \zeta_i(y)) \end{aligned}$$

where h_y' is the appropriate function defined in the $\mathscr{T}'$, $\mathscr{R}'$-system. Set $f'(i) = \zeta_i(y)$ so that $f' \in M$. Then

$$f^{-1}\hat{x} f_x h_y f = \hat{x} f' h_y' f'$$

proving that the $\mathscr{T}'$, $\mathscr{R}'$-system gives Ξ', an embedding of G^* conjugate in $\tilde{G}/M$ to that given by Ξ. The transformation induced by fM upon $\tilde{V}$ is therefore an equivalence from the $\mathscr{T}$, $\mathscr{R}$-system construction of V to the $\mathscr{T}'$, $\mathscr{R}'$-system construction of V.

(5.6) PROPOSITION. *The $k[G^*]$-module defined upon $\tilde{U}|^{\otimes \tilde{a}}$ by $xv = \Xi(x) \cdot v$ for $x \in G^*$ and $v \in \tilde{U}|^{\otimes \tilde{a}}$ is independent of the choices for $\mathscr{T}$ and $\mathscr{R}$.*

We may "factor" $\tilde{G}$ as a semidirect product

$$\tilde{N} \cdot R_1^{*\Omega}$$

where $\tilde{N} = N_0^* \sim \hat{G}$. Using the Mackey Decomposition (2.9), we have that $\tilde{U}|^{\otimes \tilde{G}}|_{\tilde{N}}$ is equivalent to $\tilde{U}|_{\hat{G}_1 N_0^{*\Omega}}|^{\otimes \tilde{N}}$. The embedding Ξ restricted to N^* maps it into $\tilde{N}$ via

$$\Xi(x, 1) = \hat{x} f_x .$$

Therefore, by (2.13) $\tilde{U}|^{\otimes \tilde{G}}$, viewed as an N^*-module via Ξ is equivalent to $U|_{N_0^*}|^{\otimes N^*}$.

(5.7) PROPOSITION. *The $k[G^*]$-module $\tilde{U}|^{\otimes \tilde{G}}$ defined in* (5.6) *is equivalent as a $k[N^*]$-module to $U|_{N_0^*}|^{\otimes N^*}$.*

We may now transfer this all back to G.

(5.8) DEFINITION. Let U be a projective $k[G_0]$-module such that $U|_{R_1}$ is faithful, absolutely irreducible, and nonprojective. If $(x, y) \in G_0^*$ then set $(x, y)u = xyu$ for $u \in U$. Let $\mathscr{S}$ be a transversal for $Z(R)$ in N. Let $\tilde{V}$ be the $k[G^*]$-module $\tilde{U}|^{\otimes \tilde{G}}$ defined in (5.6). Define a projective $k[G]$-module $V(= \tilde{V})$ by setting

$$xyv = (x, y)v$$

for $x \in \mathscr{S}$, $y \in R$, $v \in \tilde{V}$. We call V the *tensor induced module* of the $k[G_0]$-module U and write $V = U|^{\otimes G}$. This definition requires the hypothesis (5.1), and therefore, should not be confused with ordinary tensor induction.

The transversal $\mathscr{S}$ belongs to the central extension

$$1 \longrightarrow \ker \phi \longrightarrow G^* \xrightarrow{\phi} G \longrightarrow 1 , \tag{5.9}$$

and therefore, the cocycles introduced by $\mathscr{S}$ and another choice $\mathscr{S}'$ belong to the extension (5.9) and are equivalent. In particular, different choices for $\mathscr{S}$ in Definition (5.8) give equivalent $k[G]$-modules V.

The construction of V requires the choice of the group N. By (5.2) all choices for N are conjugate in G. This conjugation gives an equivalence of modules constructed for two distinct choices of N.

Summarizing our results thus far we have

(5.10) THEOREM. *Assume* (5.1). *Let k be a field and assume that U is a projective $k[G_0^*]$-module such that $U|_{R_1}$ is nonprojective, faithful, and absolutely irreducible. Let $V = U|^{\otimes G}$ be as in Defini-*

tion (5.8). *Then the following hold*:

(1) *V is a projective $k[G]$-module*;

(2) *$V|_R$ is nonprojective, faithful, and absolutely irreducible*;

(3) *up to equivalence of projective modules, V is independent of the choices of N, $\mathscr{T}$, $\mathscr{R}$, and $\mathscr{S}$*;

(4) *if B is a subgroup of N and $B \cap Z(R) = 1$ then $V|_B$ is equivalent to $U|_{N_0}|^{\otimes N}|_B$ where ordinary tensor induction is meant here; and*

(5) *in* (4), *if $U|_{N_0 \cap B^x}$ is equivalent to a nonprojective module for x running over a set of B, N_0-double coset representatives in N then $V|_{BR}$ is equivalent to a nonprojective module.*

Part (1) follows from (5.6), (5.9), and Definition (5.8). Since Ξ maps R^* onto $R_1^{*\Omega}/M$, (2) follows from parts (1)-(3) of (5.5). Part (3) follows from (5.6) and the discussion following Definition (5.8). We may choose $\mathscr{S}$ so that for (4), $B \subseteqq \mathscr{S}$. Then (4) follows from (5.7). To prove (5) we need the Mackey Decomposition (2.9)

$$U|_{N_0}|^{\otimes N}|_B \text{ is equivalent to } \Pi^{\otimes}(x \otimes U)|_{xN_0x^{-1} \cap B}|^{\otimes B}.$$

By (4) $V|_B$ will be equivalent to a nonprojective module if each module $(x \otimes U)|_{xN_0x^{-1} \cap B}$ is equivalent to a nonprojective module. Mapping via x^{-1} we ask only that $U|_{N_0 \cap B^x}$ be equivalent to a nonprojective module as x runs over B, N_0-double coset representatives in N. Thus $V|_B$ is equivalent to a nonprojective module. Since $B \subseteqq \mathscr{S}$ it follows that $V|_{BR}$ is equivalent to a nonprojective module. The proof of (5) and the theorem are now complete.

REMARK. Can one give a wreath product free construction of $U|^{\otimes G}$ in (5.10)? The sequence (5.9) gives an extension G^* of G by a group isomorphic to $Z(R)$. In turn, the embedding Ξ gives an extension

$$1 \longrightarrow M \longrightarrow G^+ \longrightarrow G^* \longrightarrow 1$$

by M (where G^+ is the inverse image in $\tilde{G}$ of $\Xi(G^*) \leqq \tilde{G}/M$). We may identify G_0^* as a subgroup of $G^+ \subseteqq \tilde{G}$ such that $G_0^* \cap M = 1$. The embedding Ξ allows us to extend the module U projectively to $G_0^* R_1^{*\Omega}$ in G^+. The action of $G_0^* R_1^{*\Omega x^{-1}}$ upon $x \otimes U$ involves a cocycle α_x whose action "comes from" an element of $Z(R)^{*\Omega} \supseteq M$. Since G^+ acts nontrivially upon M, it acts nontrivially upon these cocycles α_x. Thus α_x is not a factor set (central cocycle) of G. To eliminate the wreath products from this construction, one must find the cocycles α_x explicitly, giving the G-action upon them. One must then show that, at least for R, tensor induction of U (as a projective $k[H]$-module where $H = G_0^* R_1^{*\Omega}/K$ for an appropriate $K \geqq M$) to G ($=G^+/L$ for an

appropriate $L \geqq M$) reduces the product of the cocycles upon the various $x \otimes U$ to a trivial factor set. Such a construction seems quite formidable, more so than the present wreath product construction. Clues may appear in [16, 21].

We turn next to applications of Theorem (5.10). First we need a lemma allowing us to compute the sign of certain determinants.

B. A lemma on permutation groups.

(5.11) NOTATION. Let $\mathscr{D} = \{1, 2, \cdots, d\}$ and $\Omega = \{1, 2, \cdots, n\}$ for integers $n, d > 1$. Let $\mathscr{D}^\Omega$ be the set of functions from Ω to $\mathscr{D}$. Let S^n be the symmetric group upon Ω. If $f \in \mathscr{D}^\Omega$, $x \in S^n$, and $i \in \Omega$, then set $f^x(i) = f(x(i))$. Let $\bar{x}$ denote the permutation induced by x upon $\mathscr{D}^\Omega$, and $\bar{G}$ denote the image of a subgroup $G \leqq S^n$ under the mapping $x \to \bar{x}$.

Clearly $x \to \bar{x}$ defines a homomorphism of S^n into the symmetric group S^* upon $\mathscr{D}^\Omega$. Let A^* be the alternating group upon $\mathscr{D}^\Omega$. We wish to determine when $\overline{S^n} \leqq A^*$.

(5.12) PROPOSITION. $\overline{S^n} \leqq A^*$ *unless*
(1) $d \equiv -1 \pmod 4$, *or*
(2) $d \equiv 2 \pmod 4$ *and* $n = 2$.

Let $x = (12) \in S^n$ be a transposition. It is sufficient to determine the parity of $\bar{x}$. Let $\Omega_0 = \{1, 2\}$ and $\Omega_1 = \{3, \cdots, n\}$. There is a canonical one-one correspondence between $\mathscr{D}^{\Omega_0} \times \mathscr{D}^{\Omega_1}$ and $\mathscr{D}^\Omega$. The pair (g, h) corresponds to f if and only if $g(i) = f(i)$ for $i = 1, 2$; and $h(i) = f(i)$ for $i = 3, \cdots, n$. Note that (g^x, h) corresponds to f^x. So x acts upon $\mathscr{D}^{\Omega_0} \times \mathscr{D}^{\Omega_1}$ exactly as it acts upon $\mathscr{D}^\Omega$, so for each two cycle of x on $\mathscr{D}^{\Omega_0}$ we get $|\mathscr{D}^{\Omega_1}| = d^{n-2}$ two-cycles on $\mathscr{D}^{\Omega_0} \times \mathscr{D}^{\Omega_1}$. Let $\hat{x}$ be the restriction of $\bar{x}$ to $\mathscr{D}^{\Omega_0}$. Then the parity of $\bar{x}$, $\pi(\bar{x})$, is equal to $(\pi(\hat{x}))^{d^{n-2}}$ where $\pi(\hat{x})$ is the parity of $\hat{x}$. Now $\hat{x}$ fixes the d constant functions of $\mathscr{D}^{\Omega_0}$ and permutes the remaining $d^2 - d$ functions in orbits of length two. Thus

$$\pi(\hat{x}) = (-1)^{(d^2-d)/2}.$$

Finally

$$\begin{aligned}\pi(\bar{x}) &= (-1)^{d^{n-2}(d^2-d)/2} \\ &= (-1)^{d^{n-1}(d-1)/2}.\end{aligned}$$

Now $\pi(\bar{x}) = 1$ unless $d^{n-1}(d-1)/2$ is odd or

$$d^{n-1}(d-1) \neq 0 \pmod 2.$$

But

$$d^{n-1}(d-1) \equiv 0 \pmod 4$$

unless conditions (1) and (2) occur. In those cases, $\pi(\bar{x}) = -1$.

(5.13) COROLLARY. *Assume* $G \leqq S^n$. *Then the group* $\bar{G} \leqq \overline{S^n}$ *is in* A^* *unless*

(1) $d \equiv -1 \pmod 4$, $G \not\leqq A^n$, *or*

(2) $d \equiv 2 \pmod 4$, $G = S^2$, $n = 2$.

This is an immediate consequence of (5.12).

C. Unique extensions.

(5.14) THEOREM. *Let* R *be an extra special* r*-group. Suppose that* A *is an* r'*-group of operators on* R *centralizing* $Z(R)$. *Form the semidirect product* AR *of* A *and* R. *Let* k *be a field of characteristic* c, *unequal to* r, *containing a primitive* $|R|$th *root of unity. Let* U_λ *be an irreducible* $k[Z(R)]$*-module with character* $\lambda \neq 1$. *Then there is a* $k[AR]$*-module* V_λ *determined uniquely up to isomorphism such that*:

(1) $V_\lambda|_{Z(R)}$ *is isomorphic to a sum of copies of* U_λ;

(2) $V_\lambda|_R$ *is irreducible; and*

(3) *if* $x \in A$ *induces a transformation* $\mathfrak{X}(x)$ *on* V_λ *then* $\det \mathfrak{X}(x) = 1$ *for all choices of* $x \in A$.

There is a proof of this for k of characteristic 0 in [17]. The uniqueness of $V_\lambda|_R$ satisfying (1) and (2) is a general fact about extraspecial groups. Because it will be of some value later, we shall indicate how a proof is carried out for finite fields. Actually, we prove a slightly more general result from which this theorem follows as a corollary.

(5.15) THEOREM. *Assume that* G *is a group with normal subgroup* N, *and that* $\mathfrak{X}$ *is a* G*-stable absolutely irreducible representation of* N *in a finite field* k *having degree* m. *There is a projective extension* $\hat{\mathfrak{X}}$ *of* $\mathfrak{X}$ *to* G. *If* $y \in G$ *we say that* $\hat{\mathfrak{X}}(y)$ *has* (*) *if its order is finite and relatively prime to* m, *and if its determinant is* 1.

(1) *If* $y \in G \backslash N$ *and* $\hat{\mathfrak{X}}(y)$ *has* (*) *then* $\hat{\mathfrak{X}}(y)$ *is uniquely determined.*

(2) *If* $\mathscr{S} \subseteq G \backslash N$ *is a set of elements of* G *of order prime to* m *then* $\hat{\mathfrak{X}}$ *may be chosen so that* $\hat{\mathfrak{X}}(y)$ *has* (*) *for every* $y \in \mathscr{S}$.

(3) *If* K *is a subgroup of* G *of order prime to* m *for which* $K \cap N = 1$, *then for each* $x, y \in K$, $\hat{\mathfrak{X}}(y)$ *may be chosen so that it has* (*) $\hat{\mathfrak{X}}(xy) = \hat{\mathfrak{X}}(x)\hat{\mathfrak{X}}(y)$, *and, therefore,* $\hat{\mathfrak{X}}|_{KN}$ *is nonprojective.*

Since $\mathfrak{X}$ is absolutely irreducible, by extending $\mathfrak{X}$ linearly to $k[N]$ we obtain a central simple algebra $\mathfrak{X}(k[N])$. If $z \in G$ then z induces an automorphism of $\mathfrak{X}(k[N])$, centralizing its center, given by $\mathfrak{X}(u) \to \mathfrak{X}(z^{-1}uz)$ for $u \in k[N]$. By the Skolem-Noether theorem, this automorphism induced by z is inner. In fact, by Schur's lemma, there must be an invertible element Z of $\mathfrak{X}(k[N])$, uniquely determined up to a multiple by any nonzero element of k, such that

$$\mathfrak{X}(z^{-1}uz) = Z^{-1}\mathfrak{X}(u)Z, \, u \in k[N] \, . \tag{5.16}$$

We set $\hat{\mathfrak{X}}(z) = \mathfrak{X}(z)$ if $z \in N$, and leaving the k-multiple to be determined, for each $z \in \mathscr{T}$ where $\mathscr{T}$ is a fixed transversal for N in G we set $\hat{\mathfrak{X}}(z) = Z$ where Z is determined by (5.16). Finally, we set $\hat{\mathfrak{X}}(xy) = \hat{\mathfrak{X}}(x)\hat{\mathfrak{X}}(y)$ where $x \in \mathscr{T}$ and $y \in N$, so that $\hat{\mathfrak{X}}$ is defined on all of G. The important point to note here is that if $y \in G \backslash N$ then $\hat{\mathfrak{X}}(y)$ is uniquely determined by (5.16) up to any nonzero k-multiple.

Since both $\hat{\mathfrak{X}}(x)\hat{\mathfrak{X}}(y)$ and $\hat{\mathfrak{X}}(xy)$ taken for Z (with xy taken for z) satisfy (5.16) where $x, y \in G$, and since Z is unique up to a multiple from k,

$$\hat{\mathfrak{X}}(x)\hat{\mathfrak{X}}(y) = \hat{\mathfrak{X}}(xy)\alpha(x, y) \tag{5.17}$$

where $\alpha(x, y) \in k$. Therefore, $\hat{\mathfrak{X}}$ projectively extends $\mathfrak{X}$ to G.

Suppose that $y \in G \backslash N$ and $\hat{\mathfrak{X}}(y)$ has (*). Assume also that $\mathfrak{X}^*$ is a projective extension of $\mathfrak{X}$ to G such that $\mathfrak{X}^*(y)$ also has (*). By (5.16) we know that $\mathfrak{X}^*(y) = \hat{\mathfrak{X}}(y)b$ for some scalar b. Since both $\mathfrak{X}^*(y)$ and $\hat{\mathfrak{X}}(y)$ have order prime to m, there is an integer t, prime to m, such that $\mathfrak{X}^*(y)^t = \hat{\mathfrak{X}}(y)^t = 1$ so that $b^t = 1$. We conclude that b is an m' root of unity. Taking determinants,

$$1 = \det \mathfrak{X}^*(y) = b^m \det \hat{\mathfrak{X}}(y) = b^m \, ,$$

proves that b is also an mth root of unity. We conclude that $b = 1$ and $\mathfrak{X}^*(y) = \hat{\mathfrak{X}}(y)$ proving (1).

Suppose next that $y \in G \backslash N$ has order t, prime to m. From (5.17) we conclude that

$$\hat{\mathfrak{X}}(y)^t = b\hat{\mathfrak{X}}(y^t) = bI$$

where b is a scalar and I is the identity. Since $k^\times$ is finite and $(t, m) = 1$, there is a scalar $c \in k^\times$ such bc^t has finite order prime to m. Thus $\hat{\mathfrak{X}}(y)c$ has finite order s, prime to m. If $\det \mathfrak{X}(y)c = d$ then $1 = [\det \mathfrak{X}(y)c]^s = d^s$ so that d has order prime to m. We may now choose a power a of d such that $a^{-m} = d$. We conclude that $\hat{\mathfrak{X}}(y)ca$ has order prime to m and determinant 1. For each $y \in \mathscr{S}$ we may find c and a and replace $\hat{\mathfrak{X}}(y)$ by $\hat{\mathfrak{X}}(y)ca$ proving (2).

Taking $K \subseteqq \mathscr{T}$ and $K\backslash 1 = \mathscr{S}$ we find that only the final assertion of (3) requires proof. To complete the proof, it is sufficient to show that $\hat{\mathfrak{x}}|_K$ is nonprojective. Accordingly, let $x, y \in K$ so that by (5.17) $\hat{\mathfrak{x}}(x)\hat{\mathfrak{x}}(y) = \hat{\mathfrak{x}}(xy)a$ for some $a \in k$. Taking determinants on both sides we find that $a^m = 1$. Let $\hat{K}$ be the linear group $\{\hat{\mathfrak{x}}(y)w \mid y \in K, w \in k^\times\}$. This group is an extension of $k^\times$ by some factor group K/K_0 of K. Since K/K_0 has order prime to m and since $k^\times$ is central, there are unique subgroups $\hat{K}_1 \leqq \hat{K}$ and $\hat{K}_2 \leqq k^\times$ such that $\hat{K} = \hat{K}_1 \times \hat{K}_2$, $\hat{K}_1$ has order prime to m, and every prime divisor of $|\hat{K}_2|$ divides m. Note that $\hat{\mathfrak{x}}(x)$, $\hat{\mathfrak{x}}(y)$ and $\hat{\mathfrak{x}}(xy)$ must all lie in $\hat{K}_1$ by order considerations. We conclude that a must be an m'-root of unity. Since $a^m = 1$ also, $a = 1$ proving that $\hat{\mathfrak{x}}|_K$ is nonprojective. The proof of the theorem is finished.

REMARK. This proof works as soon as we know that the factor set $\alpha(x, y)$ of (5.17) maps $G \times G$ into roots of unity in k. Details on the order of $\alpha(x, y)$ in fields of characteristic 0 are given in [15].

D. **Applications of transference.** The unique module whose existence is given by (5.14) will be called $V_\lambda(AR)$ in order to keep track of the essential ingredients: λ; A; and R.

(5.18) THEOREM. *Let k be an algebraically closed field of characteristic c, prime to r. Assume that*

(a) *G satisfies* (5.1), *and*

(b) *A is an r'-subgroup of G.*

Then there is a conjugate of H (in place of H) *and a subgroup N of G such that*

(i) $G = NR$, $N \cap R = Z(R)$, $A \leqq N$;

(ii) $H = (N \cap H)R$, *$A \cap N \cap H$ is an r'-subgroup of $N \cap H$;*

(iii) $G_0 = (N \cap H)R_1$;

(iv) *there is a projective extension U of $V_\lambda((A \cap H)R_1)$ to G_0 such that U is nonprojective for $(A \cap H)^x R_1$ for all $x \in N \cap H$;*

(v) $U|^{\otimes G} = V$ (as in Definition (5.8)) *is a projective $k[G]$-module which is equivalent to a nonprojective module when restricted to AR; and*

(vi) $V|_{AR}$ *is equivalent to $V_\lambda(AR)$ unless both* $\dim U \equiv -1 \pmod 4$ *and the permutation representation ρ of A upon the cosets of H in G is not in the alternating group of degree $[G: H]$. In this exceptional case, $V|_{AR}$ is equivalent to $V_\lambda(AR) \otimes_k W$ where W is a 1-dimensional module affording the alternating character of ρ.*

(vii) $V|_A$ *is equivalent to $\Pi^{\otimes}[x^{-1} \otimes V_\lambda(A \cap (H \cap N)^x)]^{\otimes A}$ where x^{-1} runs over a complete set of $A, H \cap N$-double cosets in N and*

$V_\lambda(A \cap (H \cap N)^x)$ is taken to be $V_\lambda([A \cap (H \cap N)^x]R_1)$ restricted to $A \cap (H \cap N)^x$.

Conditions (i)-(iii) are easily demonstrated by (5.2), and by using an appropriate conjugate of H and a correct choice for N. By (5.15) we may extend U to $(N \cap H)R_1 = G_0$ in such a way that if $\mathfrak{X}$ is the projective representation of G_0 afforded by U then for all r'-elements $y \in G_0$ both $\det \mathfrak{X}(y) = 1$, and $\mathfrak{X}$ has factor set 1 for $\langle y \rangle R_1$. The uniqueness given in (5.14) and (5.15) together imply that $U_{TR_1} \cong V_\lambda(TR_1)$ for every r'-subgroup T of $(N \cap H)R_1$. Consequently, U satisfies (iv).

Fix $z \in Z(R)$ and $u \in U$. Then $zu = \lambda(z)u$ for the character λ of $Z(R)$. The element $z \in Z(R)$ acts upon V as $\Xi(1, z) = h_z M$ does. But $h_z(1) = z$ and $h_z(i) = 1$ for $i > 1$. Therefore if $u_1 \otimes \cdots \otimes u_s \in V$ where $u_i \in \hat{x}_i \otimes \tilde{U}$ then $z \cdot (u_1 \otimes \cdots \otimes u_s) = (zu_1) \otimes u_2 \otimes \cdots \otimes u^s = \lambda(z)(u_1 \otimes \cdots \otimes u_s)$. In particular, $Z(R)$ acts upon V via the character λ. By (5.10) and (iv), V is a projective $k[G]$-module which is equivalent to a nonprojective module for AR, proving (v). In addition, $V|_R$ is nonprojective, faithful, and absolutely irreducible. If V affords the representation $\mathscr{V}$ and $\det \mathscr{V}(x) = 1$ for all $x \in A$ then $V|_A$ is equivalent to $V_\lambda(AR)$ by (5.14).

Every element of the set $\mathscr{A} = \bigcup (A \cap H)^x$, $x \in H$, acts with determinant 1 upon U. Thus, every element of the set $\mathscr{A}^{*\Omega} \subseteq \tilde{G}$ acts upon $\tilde{U}|^{\otimes \tilde{G}}$ with determinant 1. The elements of $\Xi(\mathscr{A}^*)$ all lie in $\hat{A}\mathscr{A}^{*\Omega}$ and have the form $\hat{x}f$ for $\hat{x} \in \hat{A}$ and $f \in \mathscr{A}^{*\Omega}$. Therefore, the determinant of $\mathscr{V}(x)$ for $x \in A$ is equal to the determinant of $\hat{x}$ upon $\tilde{U}|^{\otimes \tilde{G}}$.

Choose a basis $u_1, \cdots, u_d$ for $\tilde{U}$ and let $\mathscr{D} = \{1, 2, \cdots, d\}$. We set $u_f = (\hat{x}_1 \otimes u_{f(1)}) \otimes \cdots \otimes (\hat{x}_s \otimes u_{f(s)})$ where $f: \Omega \to \mathscr{D}$. The set $\{u_f \mid f \in \mathscr{D}^\Omega\}$ is a basis for $\tilde{U}|^{\otimes \tilde{G}}$. The action of $\hat{x}$ upon this basis is given by $\hat{x} \cdot u_f = u_{f^{\hat{x}-1}}$. Thus $\hat{A}$ acts upon the u_f's contragrediently to the action $f \to f^{\hat{x}}$. In particular, the determinant of $\hat{x}$ upon $\tilde{U}|^{\otimes \tilde{G}}$ is just the parity of $\hat{x}$ upon the f's. We are in a position to apply (5.13). That is, $\det \mathscr{V}(x) = 1$ for $x \in A$ unless (1) or (2) of (5.13) holds.

Assume that (5.13) (2) holds. If $d \equiv 2 \pmod 4$ then $\dim U = d = r^t \equiv 2 \pmod 4$ so that $d = r^t = 2$ where $|R_1| = r^{2t+1}$. Further, $s = 2$ and $\hat{A} \not\leqq A^s$, the alternating group of degree s so that $2 \| A|$. But then 2 divides $(r, |A|) = 1$. Therefore, (5.13) (2) cannot hold.

Assume that (5.13) (1) holds. Then $d \equiv -1 \pmod 4$ and $\hat{A} \not\leqq A^s$. If $\hat{x} \notin A^s$, in this case, $\det \mathscr{V}(x) = -1$. Part (vi) follows from these considerations.

By (5.10) (4) and Mackey's Decomposition (2.9) $V|_A$ is equivalent

to $\Pi^{\otimes}(x^{-1} \otimes U)|_{(H\cap N)^x\cap A}|^{\otimes A}$ where x^{-1} runs over A, $H \cap N$-double coset representatives in G. But $x^{-1} \otimes U|_{(H\cap N)^x\cap A}$ is equivalent to $x^{-1} \otimes V_\lambda(A \cap (H \cap N)^x)$ so that (vii) follows. The proof of (5.18) is now complete.

Using Clifford's theorems again [14, (51.7)] we may prove the following extension of this theorem.

(5.19) COROLLARY. *Assume the situation of* (5.18) *holds. If Y is an irreducible $k[G]$-module such that $Z(R)$ acts upon Y via the character λ and $Y|_R$ is nonprojective, then there is a projective $k[G]$-module X with R in its kernel such that Y is equivalent to $V \otimes_k X$.*

REMARK. If $Y|_R$ is irreducible then X must be one dimensional.

6. Field extensions and forms. We shall say that a bilinear form g on a vector space V is *classical* if g is (i) symmetric, (ii) symplectic (alternating), or (iii) unitary. If g is unitary, we shall use ν to denote the associated field automorphism of order two.

(6.1) HYPOTHESIS. *Assume that $\boldsymbol{K}$ is a finite field; G is a group; and V is an irreducible $\boldsymbol{K}[G]$-module. Suppose that $g: V \times V \to \boldsymbol{K}$ is a nonsingular classical bilinear form on V which is fixed by G* (i.e., $g(xu, xv) = g(u, v)$ *for all $u, v \in V$ and $x \in G$*).

The object of this section is to study the form g in extension fields of $\boldsymbol{K}$.

(6.2) PROPOSITION. *If $\hat{\boldsymbol{K}} = \mathrm{Hom}_{K[G]}(V, V)$ then $\hat{\boldsymbol{K}}$ is a finite extension field of $\boldsymbol{K}$. If I is the anihilator of V in $\boldsymbol{K}[G]$ and $A = \mathrm{Hom}_{\hat{K}}(V, V)$, then*

$$\boldsymbol{K}[G]/I \cong A .$$

There is an antiautomorphism α of A such that if

(i) *$\bar{x}$ is the image of $x \in G$ in A then $\bar{x}^\alpha = \bar{x}^{-1}$,*

(ii) *$\alpha|_{\hat{K}}$ is an automorphism of order one or two of $\hat{\boldsymbol{K}}$ (if g is unitary then $\alpha|_K = \nu$), and*

(iii) *$g(au, v) = g(u, a^\alpha v)$ for all $u, v \in V$ and $a \in A$.*

Define $\alpha_0: G \to G$ by setting $x^{\alpha_0} = x^{-1}$ for $x \in G$. Thus defined, α_0 is an antiautomorphism of $\boldsymbol{K}[G]$. If g is unitary, we extend ν linearly to an automorphism of $\boldsymbol{K}[G]$ by making it act trivially on all elements of G. We take $\alpha_1 = \alpha_0$ except when g is unitary when we let $\alpha_1 = \nu\alpha_0$. Since $g(xu, v) = g(u, x^{-1}v)$ for all $u, v \in V$ and $x \in G$,

we must have

$$g(au, v) = g(u, a^{\alpha_1}v)$$

for all $u, v \in V$ and $a \in \boldsymbol{K}[G]$. Since g is nonsingular, this latter identity implies that α_1 must stabilize the anihilator I of V in $\boldsymbol{K}[G]$.

Since V is irreducible, Schur's lemma implies that $\hat{\boldsymbol{K}}$ is a division algebra, and by Wedderburn's theorem on finite division algebras, $\hat{\boldsymbol{K}}$ is a finite extension field of $\boldsymbol{K}$. By the Wedderburn structure theorems for simple algebras, $\boldsymbol{K}[G]/I \cong A$ where $\hat{\boldsymbol{K}}$ is the center of A. Now we see that α_1 induces an antiautomorphism α of A such that $\bar{x}^\alpha = \bar{x}^{-1}$ where $\bar{x}$ is the image in A of $x \in G$. Since α_1 has order two on $\boldsymbol{K}[G]$, α will certainly have order one or two on A. Further, since α is induced by α_1, and since V is naturally an A-module, we have

$$g(au, v) = g(u, a^\alpha v)$$

for all $u, v \in V$ and $a \in A$ completing the proof of (i) and (iii).

In order to prove (ii), assume that $a \in \hat{\boldsymbol{K}}$, the center of A, and that $\bar{x}$ is the image in A of any $x \in G$. Then

$$\bar{x}a^\alpha = (\bar{x}^{-1})^\alpha a^\alpha = (a\bar{x}^{-1})^\alpha = (\bar{x}^{-1}a)^\alpha = a^\alpha(\bar{x}^{-1})^\alpha = a^\alpha\bar{x}$$

for all $x \in G$. We conclude that $a^\alpha \in \hat{\boldsymbol{K}}$ so that α fixes $\hat{\boldsymbol{K}}$, and therefore, must be an automorphism of $\hat{\boldsymbol{K}}$ of order one or two. If $t \in \boldsymbol{K}$ then $(t + I)^\alpha = (t^{\alpha_1} + I) = t^\nu + I$ when g is unitary. In this latter case, $\alpha|_{\boldsymbol{K}} = \nu$ completing the proof of the proposition.

NOTATION. We shall let $\hat{\boldsymbol{K}} = \mathrm{Hom}_{\boldsymbol{K}[G]}(V, V)$ and $A = \mathrm{Hom}_{\hat{\boldsymbol{K}}}(V, V)$. We denote trace mappings as follows:

(a) $\Delta: A \to \hat{\boldsymbol{K}}$ of A (linear algebra trace); and

(b) $\tau: \hat{\boldsymbol{K}} \to \boldsymbol{K}$ of $\hat{\boldsymbol{K}}$.

(6.3) PROPOSITION. *If $b \in A$ and α is as in* (6.2) *then*

$$\Delta(b^\alpha) = \Delta(b)^\alpha .$$

Fix a $\hat{\boldsymbol{K}}$-basis for V and identify an element $a \in A$ with its matrix $\tilde{a}$ written in this basis. Let $\tilde{\alpha}$ be the automorphism of A given by applying $\alpha|_{\hat{\boldsymbol{K}}}$ to the entries of a matrix $\tilde{a}$. Let t be the antiautomorphism of A given by transposing a matrix $\tilde{a}$. It is straightforward to show that the composition $\alpha\tilde{\alpha}t$ is an automorphism of the central simple algebra A which centralizes $\hat{\boldsymbol{K}}$. By the Skolem-Noether theorem, $\alpha\tilde{\alpha}t$ is inner. That is, written as matrices, $\tilde{a}^{\alpha\tilde{\alpha}t} = \tilde{b}^{-1}\tilde{a}\tilde{b}$ for some invertible matrix $\tilde{b}$. Written differently with

$\tilde{c}^{-1} = \tilde{b}^{\tilde{t}\alpha}$ we have

$$\tilde{a}^{\alpha} = \tilde{c}^{-1}\tilde{a}^{\tilde{t}\alpha}\tilde{c} .$$

Taking traces we have

$$\begin{aligned} \Delta(a^{\alpha}) &= \text{Trace}\,(\tilde{a}^{\alpha}) \\ &= \text{Trace}\,(\tilde{a}^{\tilde{t}\alpha}) \\ &= [\text{Trace}\,(\tilde{a})]^{\alpha} \\ &= \Delta(a)^{\alpha} \end{aligned}$$

completing the proof.

Fix a nonzero vector $v \in V$, and a primitive idempotent $e \in A$ such that $ev = v$. The mapping $\phi: Ae \longrightarrow V$ defined by $\phi(ae) = av$, $a \in A$, is an A-isomorphism, so that

$$\bar{g}(ae, be) = g(\phi(ae), \phi(be)) \tag{6.4}$$

defines a nonsingular classical form $\bar{g}$ on Ae equivalent to g on V. The representation $x \mapsto \bar{x}$ of x on V gives a homomorphism of G into A and thus defines an action of G on Ae isomorphic to that of G on V. For the time being, we will treat the module Ae and the form $\bar{g}$ in place of V and g respectively.

(6.5) PROPOSITION. *There is an element $d \in A$ such that*

(i) *$d^{\alpha} = d$ if g is symmetric or unitary, and $d^{\alpha} = -d$ if g is symplectic, and*

(ii) *$\bar{g}(u, w) = \tau\Delta(dw^{\alpha}u)$ for all $u, w \in Ae$.*

It is straightforward to verify that if $a, b \in A$ then $a, b \mapsto \Delta(ab)$ defines a nonsingular symmetric $\hat{\boldsymbol{K}}$-bilinear form on the $\hat{\boldsymbol{K}}$-space A. Likewise, $r, s \mapsto \tau(rs)$ defines a nonsingular symmetric $\boldsymbol{K}$-bilinear form on the $\boldsymbol{K}$-space $\hat{\boldsymbol{K}}$. Since Δ is $\hat{\boldsymbol{K}}$-bilinear, we conclude that for $a, b \in A$, $a, b \mapsto \tau\Delta(ab)$ defines a nonsingular symmetric $\boldsymbol{K}$-bilinear form on the $\boldsymbol{K}$-space A.

The mapping

$$a \longmapsto \bar{g}(ae, e)$$

defines a $\boldsymbol{K}$-linear functional on the $\boldsymbol{K}$-space A. By nonsingularity, we conclude that there is a $d \in A$ such that

$$\bar{g}(ae, e) = \tau\Delta(da)$$

for all $a \in A$. If $u, w \in Ae$ then

$$\begin{aligned} \bar{g}(u, w) &= g(ue, we) \\ &= g(w^{\alpha}ue, e) \\ &= \tau\Delta(dw^{\alpha}u) . \end{aligned}$$

To complete the proof we need only show that $d^\alpha = \varepsilon d$ where $\varepsilon = 1$ if g is symmetric or unitary and $\varepsilon = -1$ if g is symplectic. If g is not unitary, we let $\nu = 1$. If $a \in A$ then by the symmetry of $\Delta(ab)$ we have

$$\begin{aligned} 0 &= \bar{g}(ae, e) - \bar{g}(ae, e) \\ &= \bar{g}(ae, e) - \varepsilon\bar{g}(e, ae)^\nu \\ &= \bar{g}(ae, e) - \varepsilon g(a^\alpha e, e)^\nu \\ &= \tau\Delta(da) - \varepsilon[\tau\Delta(da^\alpha)]^\nu \\ &= \tau\Delta(da) - \varepsilon[\tau\{\Delta(d^\alpha a)^\alpha\}]^\nu . \end{aligned}$$

Because $\tau: \hat{K} \to K$ is the trace, $\tau(u^\alpha) = \tau(u)^\nu$ for $u \in \hat{K}$ so that

$$\begin{aligned} 0 &= \tau\Delta(da) - \varepsilon\tau\Delta(d^\alpha a) \\ &= \tau\Delta([d - \varepsilon d^\alpha]a) \end{aligned}$$

for all $a \in A$. Since $\tau\Delta(ab)$ is nonsingular, $d - \varepsilon d^\alpha = 0$ or $d^\alpha = \varepsilon d$ completing the proof of the proposition.

(6.6) PROPOSITION. *Define*

$$\bar{h}(u, w) = \Delta(dw^\alpha u)$$

for $u, w \in Ae$. *Then*

(i) $\bar{h}$ *is fixed by* G,

(ii) *if* $u, w_1, w_2 \in Ae$ *and* $c \in \hat{K}$ *then*

$$\bar{h}(cw_1 + w_2, u) = c\bar{h}(w_1, u) + \bar{h}(w_2, u) ,$$

(iii) *if* $u, w \in Ae$ *then*

$$\bar{h}(u, w) = \varepsilon\bar{h}(w, u)^\alpha$$

where $\varepsilon = 1$ *if* g *is symmetric or unitary and* $\varepsilon = -1$ *if* g *is symplectic,*

(iv) $\bar{h}$ *is nonsingular, and*

(v) $\bar{g} = \tau\bar{h}$.

Let $x \in G$ and $\bar{x}$ be the image of x in A. Then

$$\begin{aligned} \bar{h}(xu, xw) &= \bar{h}(\bar{x}u, \bar{x}w) \\ &= \Delta(d(\bar{x}w)^\alpha(\bar{x}u)) \\ &= \Delta(dw^\alpha(\bar{x}^\alpha\bar{x})u) \\ &= \Delta(dw^\alpha u) \\ &= \bar{h}(u, w) \end{aligned}$$

for all $u, w \in Ae$ so that G fixes $\bar{h}$.

Part (ii) is an obvious calculation using the $\hat{K}$-linearity of Δ. To prove part (iii), use the symmetry of $\Delta(ab)$ to note that

$$\begin{aligned}\bar{h}(u, w) &= \Delta(dw^{\alpha}u) \\ &= \Delta(u^{\alpha}wd^{\alpha})^{\alpha} \\ &= \Delta(d^{\alpha}u^{\alpha}w)^{\alpha} \\ &= \varepsilon\Delta(du^{\alpha}w) \\ &= \varepsilon\bar{h}(w, u)^{\alpha} .\end{aligned}$$

Now (v) follows from Proposition (6.5) (ii), so that the nonsingularity of $\bar{g}$ implies the nonsingularity of $\bar{h}$ completing the proof of (iv) and the proposition.

(6.7) THEOREM. *Suppose that Hypothesis* (6.1) *holds. Set* $\hat{K} = \mathrm{Hom}_{k[G]}(V, V)$ *so that* $\hat{K}$ *is a finite extension field of* K. *Then one of the following occurs.*

(i) *There is a nonsingular classical form* $\tilde{g}$ *on the* $\hat{K}$*-space* V *of the same type as* g *which is fixed by* G *and for which*

$$g = \tau\tilde{g}$$

where $\tau: \hat{K} \to K$ *is the trace mapping.*

(ii) *The form* g *is symmetric or symplectic; there is an automorphism* α *of order two of* $\hat{K}$ *which fixes* K; *there is an element* μ *such that* $\mu = 1$ *if* g *is symmetric and* $\mu^{\alpha} = -\mu$ *if* g *is not symmetric; and there is a nonsingular unitary form* h *on the* $\hat{K}$*-space* V *fixed by* G *such that*

$$g = \tau(\mu h)$$

where $\tau: \hat{K} \to K$ *is the trace mapping.*

Since $\phi: Ae \to V$ is an A-isomorphism, we may define

$$\tilde{g}(u, w) = \bar{h}(ae, be)$$

where $u = av$, $w = bv$ for $a, b \in Ae$ and $\bar{h}$ is as in Proposition (6.6). If g is unitary or α is trivial on $\hat{K}$ then part (i) follows directly from Proposition (6.6).

Assume now that α is nontrivial on $\hat{K}$ and that g is symmetric or symplectic. If g is not symmetric then $\hat{K}$ has odd characteristic and we may find $\mu \in \hat{K}$, $\mu \neq 0$, such that $\mu^{\alpha} = -\mu$. In all other cases let $\mu = 1$. Then $\mu^{\alpha} = \varepsilon\mu$ where $\varepsilon = 1$ if g is symmetric and $\varepsilon = -1$ otherwise. Set

$$h = \mu^{-1}\tilde{g} .$$

By part (iii) of Proposition (6.6),

$$h(u, w) = h(w, u)^{\alpha}$$

for $u, w \in V$. By (i)-(iv) of that proposition, h is a nonsingular unitary form on the $\hat{K}$-space V fixed by the action of G. Finally,

$$g = \tau\tilde{g} = \tau(\mu h)$$

by part (v) of Proposition (6.6) completing the proof of the theorem.

Nonsingular and totally isotropic subspaces of V may sometimes be followed through this extension process.

(6.8) PROPOSITION. *Consider the situation of Theorem* (6.7). *If N is normal in G, U is a homogeneous component of the $\hat{K}[N]$-module U, and f is the extended form of the theorem ($f = \tilde{g}$ in* (i) *or $f = h$ in* (ii)) *then U is totally isotropic* (nonsingular) *for f if and only if it is also for g.*

If $\tau: \hat{K} \to K$ then there is a μ such that $\tau\mu f = g$. From this it follows that if U is totally isotropic for f then it is also for g, and if U is nonsingular for g it is also for f. Suppose that f is nonsingular. If $0 \neq v \in U$, then there is a $u \in U$ such that $f(v, u) \neq 0$. By taking a scalar multiple of u in place of u we may assume $f(v, u)$ has some preassigned nonzero value. In particular, this preassigned value ω may be taken so that $\tau(\mu\omega) \neq 0$. Thus $g(v, u) = \tau\mu f(v, u) \neq 0$ proving that g is nonsingular. Finally assume that U is totally isotropic for g. By our argument above, U must be singular for f. Let $S = \mathrm{stab}(G, U)$ be the stabilizer in G of U. Since V is an irreducible $\hat{K}[G]$-module, U is an irreducible $\hat{K}[S]$-module. But the radical of f in U is not (0) and is S-invariant implying that U is totally isotropic, proving the proposition.

REMARK. The results of this section may be used to simplify some parts of § 3 of [4].

7. Minimal $K[G]$-modules.

(7.1) HYPOTHESIS. *Assume that K is a finite field, G is a group, and V is an irreducible $K[G]$-module. Suppose that $g: V \times V \to K$ is a nonsingular classical bilinear form on V fixed by G.*

We shall say that V is *form induced* by U if there is a subgroup S of G and a nonsingular $K[S]$-submodule U of V such that both $U|^G \cong V$ and the distinct subspaces among xU, $x \in G$, of V are

pairwise orthogonal. An irreducible module which is not form induced from a *proper* submodule will be called *form primitive.*

The following obvious result shows that form induction is just "form invariant" induction.

(7.2) PROPOSITION. *Assume* (7.1). *Let S be a subgroup of G and U a $\boldsymbol{K}[S]$-submodule of V which form induces V. Let $x_1, x_2, \cdots, x_t$ be a transversal for S in G. Define $\tilde{g}$ on $U|^G \times U|^G$ by setting*

$$\tilde{g}(\sum_i x_i \otimes u_i, \sum_j x_j \otimes v_j) = \sum_i g(u_i, v_i)$$

for $u_i, v_j \in U$. Then $\tilde{g}$ is a nonsingular form on $U|^G$. Further, the isomorphism $\phi: U|^G \to V$ defined by $\phi(x \otimes u) = xu$, $x \in G$, $u \in U$, sends $\tilde{g}$ to $g = \phi\tilde{g}$.

Suppose that $\sum x_i \otimes u_i$, $\sum x_j \otimes v_j \in U|^G$ where $u_i, v_i \in U$. Then

$$\begin{aligned}\tilde{g}(\sum x_i \otimes u_i, \sum x_j \otimes v_j) &= \sum g(u_i, v_i)\\ &= \textstyle\sum_i g(x_i u_i, x_i v_i)\\ &= \textstyle\sum_{i,j} g(x_i u_i, x_j v_j)\\ &= g(\sum x_i u_i, \sum x_j v_j)\\ &= \phi\tilde{g}(\phi \sum x_i \otimes u_i, \phi \sum x_j \otimes v_j)\end{aligned}$$

since $g(x_i u_i, x_j v_j) = 0$ for all $i \neq j$.

Equally obvious is the following fact.

(7.3) PROPOSITION. *Assume* (7.1). *There is a subgroup S of G and a form primitive $\boldsymbol{K}[S]$-submodule U of V so that U form induces V.*

If V is form primitive we simply take $G = S$ and $V = U$. Induction upon $|G|$ and the transitivity of module induction complete the proof.

Like the primitive modules, the form primitive modules are difficult to know. In classifying primitive modules, one usually studies quasiprimitive ones. We follow an analogous course here. Since "form quasiprimitive module" is too cumbersome, we opt for simpler vocabulary.

(7.4) DEFINITION. Assume (7.1). We say that V is a *minimal $\boldsymbol{K}[G]$-module* if for any normal subgroup N of G either $V|_N$ is homogeneous or $V|_N = V_1 + V_2$ where the V_i are the homogeneous components and are totally isotropic subspaces.

REMARK. In other numbers of this sequence, a different, apparently more restrictive, definition of minimal module is given. By Corollary (7.11) this other definition is equivalent to the present one. This fact was first noted by L. Kovács.

The object of this section is to show that form primitive modules are minimal and to derive a few simple properties of minimal modules. We first study the effect of restriction to normal subgroups.

(7.5) PROPOSITION. *Assume* (7.1). *If* $H \lhd G$ *and* $V|_H = V_1 + \cdots + V_t$ *where the* V_i *are homogeneous components then either all* V_i *are nonsingular or all are totally isotropic.*

Let $S = \text{stab}(G, V_1)$ be the stabilizer in G of V_1. Since S fixes g, $V_1 \cap V_1^\perp$ is a $\boldsymbol{K}[S]$-module. But V_1, as an S-module, is irreducible since $V_1|^G$ (from S) $\cong V$ and V is irreducible. Therefore, $V_1 \cap V_1^\perp = V_1$ or (0). So V_1 is totally isotropic or nonsingular. Since $V_i = xV_1$ for some $x \in G$ and since G fixes g, we must have V_i nonsingular if and only if V_1 is also. A similar situation holds if V_1 is totally isotropic. The proof is complete.

(7.6) PROPOSITION. *In* (7.5) *if* V_1 *is nonsingular then all the* V_i *are pairwise orthogonal so that* V_1 *form induces* V *from* $S = \text{stab}(G, V_1)$.

Since $V_i = xV_1$ for some $x \in G$ and since G fixes g, it is sufficient to prove that V_1 and V_j are orthogonal for $j > 1$. The subspace $\sum_{i>1} V_i$ is the unique $\boldsymbol{K}[H]$-complement to V_1 in V because the V_i are homogeneous components. But H fixes g, and V_1 is nonsingular, so that $V_1^\perp$ is a $\boldsymbol{K}[H]$-complement to V_1 in V. Therefore, $V_1^\perp = \sum_{i>1} V_i$ completing the proof.

(7.7) PROPOSITION. *In* (7.5) *if* V_1 *is totally isotropic then there is an* $x \in G$ *so that*

(a) $x^2 \in S = \text{stab}(G, V_1)$ *and* $x \in N_G(S)$,

(b) $U = V_1 + xV_1$ *is a nonsingular* $\boldsymbol{K}[K]$*-module where* $K = \langle S, x \rangle$,

(c) U *form induces* V, *and*

(d) $C_S(V_1) = C_S(xV_1)$.

Since V_1 is totally isotropic $V_1 \subseteqq V_1^\perp$. By complete reducibility of $V|_H$ we may find a $\boldsymbol{K}[H]$-complement V^* to the $\boldsymbol{K}[H]$-module $V_1^\perp$ in V so that $V = V^* + V_1^\perp$. The nonsingularity of V guarantees that $V^* + V_1$ is a nonsingular space. The form g is fixed by H so that nonsingularity of g on $V^* + V_1$ assures us that the module V^*

is contragredient to V_1. In particular, V^* is a homogeneous $\boldsymbol{K}[H]$-module. Since $\dim V^* = \dim V_1$ we know that $V^* = V_j$ is a homogeneous component of $V|_H$. There is some $x \in G$ so that $V_j = xV_1$. There is a unique $\boldsymbol{K}[H]$-complement in V to $V_1 + V_j$, and it is $\sum_{i \neq 1,j} V_i$. But $(V_1 + V_j)^\perp$ is a $\boldsymbol{K}[H]$-complement to $V_1 + V_j$ so that $(V_1 + V_j) = \sum_{i \neq 1,j} V_i$.

Let $y \in G$ be such that yV_1 or yxV_1 is V_1. Then $y(V_1 + xV_1) = V_1 + V_i$ for some i. Since $V_1^\perp \supseteq \sum_{s \neq 1,j} V_s$ and since $V_1 + V_i$ is nonsingular, we must have $i = j$ so that $\{yV_1, yxV_1\}$ is $\{V_1, xV_1\}$. With $y = x$ we have $\{xV_1, x^2V_1\} = \{V_1, xV_1\}$ so that $x^2V_1 = V_1$ and $x^2 \in S$. If $y \in S$ then $yV_1 = V_1$ so that $\{V_1, yxV_1\} = \{V_1, xV_1\}$. From this we conclude that $yxV_1 = xV_1$ or $x^{-1}yx \in S$. This completes the proof of (a). Part (b) follows since $K = \langle S, x\rangle$ has S as a subgroup of index two, since $V_1|^G$ (from S) $\cong V$ is irreducible, and since $V_1|^G$ (from S) $\cong (V_1|^K)|^G \cong U|^G$. Now $yU \subseteq \sum_{i \neq 1,j} V_i$ for $y \notin K$, and $U^\perp = \sum_{i \neq 1,j} V_i$ so that $yU \subseteq U^\perp$ for $y \notin K$. Therefore, the distinct modules among yU, $y \in G$, are pairwise orthogonal proving (c).

The group S fixes g and acts upon the complementary totally isotropic subspaces V_1 and xV_1 of U. Thus the action of S on V_1 is contragredient to that on xV_1. In particular, $y \in C_S(V_1)$ if and only if $y^{-1} \in C_S(xV_1)$ proving (d).

We may now prove:

(7.8) THEOREM. *In* (7.1) *if* V *is form primitive then* V *is a minimal module.*

If $V|_N$ is homogeneous for all $N \lhd G$, then V is a minimal module. So assume that $N \lhd G$ and $V|_N = V_1 + \cdots + V_t$ where the V_i are homogeneous components and $t > 1$. By (7.5) either all the V_i are totally isotropic or all are nonsingular. Let $S = \text{stab}\,(G, V_1)$. If the V_i are nonsingular, then by (7.6) $V_1|^G$ (from S) $\cong V$ and V is form induced. Therefore, all the V_i must be totally isotropic. By (7.7) there is a group $K > S$ such that $[K: S] = 2$ and $V_1|^K$ (from S) $\cong V_1 + V_j$ (for some $j > 1$) is nonsingular. In addition, with $U = V_1 + V_j$, U form induces V. We conclude that $U = V$ and $t = 2$ completing the proof of (7.8).

We turn now to the structure of minimal modules.

(7.9) THEOREM. *Assume* (7.1) *holds. Suppose that* S *and* T *are distinct subgroups of* G *of index* 2 *such that* $V|_S = V_1 + V_2$ *and* $V|_T = U_1 + U_2$ *where* V_i *and* U_j, $i, j = 1, 2$, *are homogeneous totally isotropic components of* V. *Fix* $x \in S \backslash T$ *and* $y \in T \backslash S$, *and set* $H = S \cap T$ *and* $K = \langle H, xy\rangle$. *Then either*

(1) *all the modules* V_i *and* U_j, $i, j = 1, 2$, *are irreducible iso-*

morphic $K[H]$-modules such that $V_i \cap U_j = (0)$ for $i, j = 1, 2$; or

(2) *$V_1 \cap U_1 + V_2 \cap U_2$ and $V_1 \cap U_2 + V_2 \cap U_1$ are orthogonal nonisomorphic irreducible $K[K]$-modules either of which form induces V.*

We set $V_{ij} = V_i \cap U_j$ for $i, j = 1, 2$. Note that V_{ij} is a $K[H]$-module. Since S and T are of index 2, G/H is a four group. Since $[S: H] = 2$, and since V_1 is an irreducible $K[S]$-module, $V_1|_H$ is the sum of at most two $K[H]$-modules.

Since $V_{1j} \neq V_1$, if $V_{1j} \neq (0)$ then $V_1|_H$ is reducible. If $V_1|_H$ is reducible and W is an irreducible $K[H]$-submodule of V_1 then $W + yW$ is a proper $K[T]$-submodule of V. The only such submodules are U_1 and U_2 so that $(0) \neq W \leqq V_1 \cap U_j = V_{1j}$ for some j. Consequently, $V_1|_H$ is irreducible if and only if $V_{1j} = (0)$ for $j = 1, 2$.

Assume that $V_{ij} = (0)$ for all $i, j = 1, 2$. By our observations above, $V_i|_H$ and $U_j|_H$ are irreducible for $i, j = 1, 2$. Since $V_1 + V_2$ and $U_1 + U_2$ are two distinct decompositions of $V|_H$ into a sum of irreducible $K[H]$-modules, $V|_H$ could not have two nonisomorphic, hence unique, homogeneous components. We conclude that $V|_H$ is homogeneous and all of $V_i|_H$, $U_j|_H$, $i, j = 1, 2$, are isomorphic irreducible $K[H]$-modules proving (1).

After renumbering, we may now assume that $V_{11} \neq (0)$. By our previous comments, V_{11} is one of two irreducible constituents in $V|_H$. Since $xV_1 = V_1$, $yV_1 = V_2$, $xU_1 = U_2$, and since $V_1 \cap V_2 = U_1 \cap U_2 = (0)$, we have

$$
\begin{aligned}
V &= V_{11} + xV_{11} + yV_{11} + xyV_{11} \\
&= V_{11} + V_{12} + V_{21} + V_{22}
\end{aligned} \tag{7.10}
$$

where corresponding summands are equal (e.g., $V_{12} = xV_{11}$).

With $W_1 = V_{11} + V_{22}$ and $W_2 = V_{12} + V_{21}$, we know that W_1 and W_2 are $K[K]$-modules and $xW_1 = W_2$. Therefore $V \cong W_1|^G$ (from K).

We prove that $W_1^\perp \geqq W_2$. Notice that $V_{11} \leqq V_1 + U_1$ where V_1 and U_1 are totally isotropic. Consequently, $V_{11}^\perp \geqq V_1 + U_1 = V_{11} + V_{12} + V_{21} \geqq W_2$. Similarly, $V_{22}^\perp \geqq W_2$ so that $W_1^\perp \geqq V_{11}^\perp \cap V_{22}^\perp \geqq W_2$. We conclude that the modules W_1 and W_2 are nonsingular and orthogonal to each other. Thus W_1 form induces V from K.

Finally, assume that $W_1 \cong W_2$ as $K[K]$-modules. Since $V_{11} + V_{22} = W_1$ and $V_{12} + V_{21} = W_2$, V_{11} is isomorphic to one of V_{12} or V_{21} as a $K[H]$-module, because the modules V_{ij} are all irreducible $K[H]$-modules. From (7.10) we have

$$
\begin{aligned}
V_1 &= V_{11} + V_{12} \cong V_{11}|^S \text{ (from } H) \text{ , and} \\
V_2 &= V_{21} + V_{22} \cong V_{21}|^S \text{ (from } H) \cong V_{22}|^S \text{ (from } H) \text{ ,}
\end{aligned}
$$

as $\boldsymbol{K}[S]$-modules. Since V_1 and V_2 are nonisomorphic $\boldsymbol{K}[S]$-modules, V_{11} is *isomorphic to neither* V_{21} *nor* V_{22} as $\boldsymbol{K}[H]$-modules. Thus V_{11} is isomorphic to V_{12} as a $\boldsymbol{K}[H]$-module. Applying the same analysis to $U_1 = V_{11} + V_{21}$, $U_2 = V_{12} + V_{22}$, and T in place of V_1, V_2, and S we find that V_{11} cannot be isomorphic to V_{12} as a $\boldsymbol{K}[H]$-module. We conclude that W_1 and W_2 are nonisomorphic $\boldsymbol{K}[K]$-modules proving (2) and the theorem.

(7.11) COROLLARY. *In Theorem* (7.9), *if* V *is a minimal module then* $V_i|_N$ *is homogeneous for all* $N \leqq S$, $N \lhd G$.

Assume that $V_i|_N$ is not homogeneous so that $V|_N$ is not homogeneous. In the theorem we replace U_i, $i = 1, 2$, and T as follows. By the definition of minimal module, $V|_N = U_1 + U_2$ where U_i is totally isotropic and a homogeneous component. We set $T = \mathrm{stab}\,(G, U_1)$. Since $V_i|_N$ is not homogeneous, $S \neq T$. Let W be a component of $V_i|_N$ so that $(0) \neq W \leqq V_i \cap U_j$ for some j. Applying the theorem to this choice for S and T forces conclusion (2) to occur. But then $V|_K$ is the sum of two nonisomorphic nonsingular orthogonal homogeneous components violating the fact that V is a minimal module, and proving the corollary.

8. **Reduction theorems for minimal modules.** Situation (1) of Theorem (7.9) brings some complexity into the analysis of minimal modules. We set that situation as hypothesis and examine it in some detail.

(8.1) HYPOTHESIS. *Assume that* $\boldsymbol{K}$ *is a finite field;* G *is a group;* V *is an irreducible* $\boldsymbol{K}[G]$*-module; and* $g: V \times V \to \boldsymbol{K}$ *is a nonsingular classical bilinear form on* V *fixed by* G. *Suppose that* S *and* T *are distinct subgroups of* G *of index* 2 *such that* $V|_S = V_1 + V_2$ *and* $V|_T = U_1 + U_2$ *where* V_i *and* U_j, $i, j = 1, 2$, *are homogeneous totally isotropic components of* V. *If* $H = S \cap T$ *then* V_i *and* U_j, $i, j = 1, 2$, *are irreducible isomorphic* $\boldsymbol{K}[H]$*-modules such that* $V_i \cap U_j = (0)$ *for* $i, j = 1, 2$.

NOTATION. Choose $x \in S \backslash T$, $y \in T \backslash S$. Set $H = S \cap T$ and $K = \langle H, xy \rangle$. This notation conforms to that of Theorem (7.9).

(8.2) LEMMA. *Assume that* (8.1) *holds. Let* $L \geqq H$ *be a subgroup of* G *of index* 2 *such that* $V|_L = W_1 + W_2$ *where the* W_i *are homogeneous components.*

(1) $\mathrm{char}\,\boldsymbol{K} = p > 2$.

(2) *If for* $w \in W_1$ *we define* $z \cdot w = zw$ *for* $z \in H$ *or* $-zw$ *if*

$z \in L\backslash H$ then the $\cdot$-action of L on W_1 is isomorphic to the action of L on W_2 but not the action of L on W_1.

(3) $\mathrm{Hom}_{K[G]}(V, V) \cong \mathrm{Hom}_{K[L]}(W_1, W_1) = \mathrm{Hom}_{K[H]}(W_1, W_1)$.
$\cong \mathrm{Hom}_{K[H]}(V_1, V_1)$.

By hypothesis, $V|_H$ is homogeneous. In particular, $V_1|_H \cong W_1|_H$, and therefore $K_1 = \mathrm{Hom}_{K[H]}(W_1, W_1) \cong \mathrm{Hom}_{K[H]}(V_1, V_1)$. Since $W_1 \ncong W_2$ as $K[L]$-modules, there are two nonisomorphic extensions of $W_1|_H$ to L. These extensions are isomorphic to composition factors of $W_1|_H|^L \cong W_1 \otimes_k J$ where J is the regular $K[L/H]$-module [14, (51.7)]. Thus J has two nonisomorphic 1-dimensional composition factors (one with $z \in L\backslash H$ acting as -1 and the other with z acting as $+1$). Obviously we must have char $K = p > 2$, proving (1). The properties of the $\cdot$-action follow immediately since J is the sum of a trivial $K[L]$-module and a module on which $z \in L\backslash H$ acts as -1. From Proposition (1.7), since $W_1|_H$ is irreducible and has two distinct extensions to L, it follows that $\mathrm{Hom}_{K[L]}(W_1, W_1) = \mathrm{Hom}_{K[H]}(W_1, W_1)$. By Corollary (1.5) we conclude that $\mathrm{Hom}_{K[G]}(V, V) \cong \mathrm{Hom}_{K[L]}(W_1, W_1)$ proving (3) and the lemma.

(8.3) LEMMA. *Assume that* (8.1) *holds. There is a unique $K[H]$-isomorphism $\phi: V_1 \to V_2$ such that $\phi'(v) = v + \phi(v)$ and $\phi''(v) = v - \phi(v)$ define $K[H]$-isomorphisms of V_1 onto U_1 and U_2 respectively. If $x, y \in G$ are as in Theorem* (7.9) *then for $v \in V_1$,*

(1) $x\phi v = -\phi xv$, *and*

(2) $\phi y \phi v = yv$.

Since $\dim V_i = \dim U_i = 1/2 \dim V$ and $U_1 \cap V_1 = (0)$, for each $v \in V_1$ there is a unique $\phi v \in V_2$ such that $v + \phi v \in U_1$. It is straightforward to verify that ϕ is a uniquely defined $K[H]$-isomorphism from V_1 to V_2. From this the properties of ϕ' follow.

Consider the action of S on V_1 given by $z^* v = \phi^{-1} z \phi v$ for $z \in S$ and $v \in V_1$. This *-action is isomorphic to the action of S on V_2 but not on V_1. Up to isomorphism, there are two extensions of $V_1|_H$ to S: one represented by V_1 and the other represented by the $\cdot$-action of Lemma (8.2) with $L = S$ and $V_1 = W_1$. In particular, the $\cdot$- and *-actions are isomorphic. Since they are identical for H, there is a $\mu \in \mathrm{Hom}_{K[H]}(V_1, V_1)$ such that $z^* \mu v = \mu(z \cdot v)$. By Lemma (8.2), $\mu \in \mathrm{Hom}_{K[S]}(V_1, V_1)$ so that $\mu(z \cdot v) = z \cdot (\mu v)$ for all $z \in S$ and $v \in V_1$. We conclude that $z^* v = z \cdot v$ for all $z \in S$ and $v \in V_1$. Applying this with $z = x$ gives $\phi^{-1} x \phi v = -xv$ or $x\phi v = -\phi xv$ for all $v \in V_1$ proving (1).

Since $xU_1 = U_2$, since given $u \in U_2$ there are unique $v_1, v_2 \in V_1$ such that $u = v_1 + \phi v_2$, and since $x(v + \phi v) = xv + (x\phi x^{-1})(xv) = (xv) -$

$\phi(xv) \in U_2$, we have $u = v - \phi v$ for a unique $v \in V_1$. From this the properties of ϕ'' follow.

Since y stabilizes U_1, if $v \in V_1$ then for some unique $v' \in V_1$,

$$v' + \phi v' = y(v + \phi v) = y\phi v + yv .$$

Equating components in V_i, $i = 1, 2$, gives

$$y\phi v = v' \quad \text{and} \quad yv = \phi v' ,$$

from which we obtain

$$\phi y \phi v = yv$$

for all $v \in V_1$, proving (2) and the lemma.

(8.4) LEMMA. *Assume that* (8.1) *holds, and that* ϕ *is as in Lemma* (8.3). *Define* $\hat{\phi} \in GL(V, \boldsymbol{K})$ *by setting* $\hat{\phi}v = \phi v$ *if* $v \in V_1$ *or* $-\phi^{-1}v$ *if* $v \in V_2$ *and extending linearly to* V. *Then*

(1) $\hat{\phi}^2 = -1$,

(2) $\hat{\phi} \in \mathrm{Hom}_{K[K]}(V, V)$ *where* $K = \langle H, xy \rangle$,

(3) $x\hat{\phi} = -\hat{\phi}x$,

(4) $\hat{\phi}$ *is a* $\boldsymbol{K}[H]$*-isomorphism which interchanges* V_1 *and* V_2, *and also* U_1 *and* U_2, *and*

(5) $\hat{\phi}$ *fixes* g.

Using Lemma (8.3) we have

$$0 = g(u + \phi u, v + \phi v) = g(u, \phi v) + g(\phi u, v)$$

for all $u, v \in V_1$ so that

$$g(u, \phi v) = -g(\phi u, v) .$$

If $u, v, u', v' \in V_1$ then

$$\begin{aligned} g(\hat{\phi}(u + \phi v), \hat{\phi}(u' + \phi v')) &= g(-v + \phi u, -v' + \phi u') \\ &= -g(v, \phi u') - g(\phi u, v') = g(\phi v, u') + g(u, \phi v') \\ &= g(u + \phi v, u' + \phi v') \end{aligned}$$

proving (5).

If $v \in V_1$ then $x\hat{\phi}v = x\phi v = -\phi xv = -\hat{\phi}xv$, and $x\hat{\phi}(\phi v) = -xv = \phi^{-1}x\phi v = -\hat{\phi}x(\phi v)$ proving (3). Again if $v \in V_1$, $y\hat{\phi}v = y\phi v = \phi^{-1}yv = -\hat{\phi}yv$, and $y\hat{\phi}(\phi v) = -yv = -\phi y(\phi v) = -\hat{\phi}y(\phi v)$ proving that $y\hat{\phi} = -\hat{\phi}y$. Since $\hat{\phi}$ is clearly a $\boldsymbol{K}[H]$-isomorphism, and $xy\hat{\phi} = \hat{\phi}xy$ we conclude that (2) holds. It is obvious that $\hat{\phi}$ interchanges V_1 and V_2. Since $\hat{\phi}(v + \phi v) = -(v - \phi v)$, Lemma (8.3) implies that $\hat{\phi}$ interchanges U_1 and U_2, proving (4). Finally, if $v \in V_1$ we have $\hat{\phi}^2 v = -v$ and

$\hat{\phi}^2(\phi v) = -\phi v$ proving (1) and the lemma.

(8.5) PROPOSITION. *Assume that* (8.1) *holds. Recall the notation* $K = \langle H, xy \rangle$ *where* $x \in S\backslash T$ *and* $y \in T\backslash S$. *Set* $\hat{\boldsymbol{K}} = \mathrm{Hom}_{K[G]}(V, V)$.

(1) -1 *is a square in* $\hat{\boldsymbol{K}}$ *if and only if* $V|_K$ *is reducible.*

(2) *If* $i \in \hat{\boldsymbol{K}}$ *and* $i^2 = -1$

then

$$W_1 = \{v + i\hat{\phi}v \mid v \in V_1\}$$

and

$$W_2 = \{v - i\hat{\phi}v \mid v \in V_1\}$$

are complementary nonisomorphic irreducible $\boldsymbol{K}[K]$*-modules.*

(3) *Assume that* -1 *is a square in* $\hat{\boldsymbol{K}}$. *Then* W_1 *and* W_2 *are nonsingular and orthogonal if and only if there is unitary* $\hat{\boldsymbol{K}}$*-form* f *on* V *fixed by* G *and* $\hat{\phi}$ *and an element* $\mu \in \hat{\boldsymbol{K}}$ *such that* $g = \tau\mu f$ *where* $\tau: \hat{\boldsymbol{K}} \to \boldsymbol{K}$ *is the trace, and* $\hat{\boldsymbol{K}} = GF(p^{2t})$ *where* $p \equiv 3 \pmod 4$ *and* $t \geqq 1$ *is odd.*

(*In all other cases* W_1 *and* W_2 *are totally isotropic.*)

(4) $\hat{\phi}$ *centralizes* $\hat{\boldsymbol{K}}$.

Assume first that $V|_K$ is irreducible. Set $\boldsymbol{K}_1 = \mathrm{Hom}_{K[K]}(V, V)$. By Lemma (8.4), $\hat{\phi} \in \boldsymbol{K}_1$. Since $x\hat{\phi} = -\hat{\phi}x$, $\hat{\phi} \notin \hat{\boldsymbol{K}} \leqq \boldsymbol{K}_1$. By Schur's lemma and Wedderburn's theorem on finite division algebras, $\boldsymbol{K}_1$ is a field. Since $\hat{\phi}$ and $-\hat{\phi}$ are the square roots to -1 in $\boldsymbol{K}_1$ and since $\hat{\phi} \notin \hat{\boldsymbol{K}}$, we conclude that -1 is not a square in $\hat{\boldsymbol{K}}$, proving part of (1).

Assume that $V|_K$ is reducible. Since V is irreducible and $[G: K] = 2$ we must have $V|_K = W_1 + W_2$ where W_i is irreducible. Since $x \notin K$, there is no loss in assuming that $W_2 = xW_1$. Redefine $\boldsymbol{K}_1 = \mathrm{Hom}_{K[K]}(W_1, W_1)$. We prove that -1 is a square in $\hat{\boldsymbol{K}}$. Suppose $W_1 \ncong W_2$ as $\boldsymbol{K}[K]$-modules. Since $\hat{\phi}$ induces a $\boldsymbol{K}[K]$-isomorphism of V, $\hat{\phi}$ stabilizes W_1, and therefore, $\hat{\phi}|_{W_1}$ is a square root of -1 in $\boldsymbol{K}_1$. By Lemma (8.2) $\boldsymbol{K}_1 \cong \hat{\boldsymbol{K}}$ so that -1 is a square in $\hat{\boldsymbol{K}}$. Assume next that $W_1 \cong W_2$. If $\alpha \in \boldsymbol{K}_1$ and $w_i \in W_i$ then we set $\alpha'(w_1 + w_2) = \alpha w_1 + x\alpha x^{-1}w_2$. In this way we obtain a field $\boldsymbol{K}_1' = \{\alpha' \mid \alpha \in \boldsymbol{K}_1\}$ isomorphic to $\boldsymbol{K}_1$. Since K is normal in G, since $xW_1 = W_2$, and since $\boldsymbol{K}_1$ centralizes K on W_1, it is straightforward to prove that K centralizes $\boldsymbol{K}_1'$ on V. But $x\alpha'(w_1 + w_2) = x\alpha x^{-1}(xw_1) + x^2\alpha x^{-1}w_2 = \alpha(xw_2) + x\alpha x^{-1}(xw_1) = \alpha' x(w_1 + w_2)$ since $x^2 \in K$ proving that $\boldsymbol{K}_1' \leqq \hat{\boldsymbol{K}}$. By Frobenius Reciprocity as in (1.4) we obtain $\boldsymbol{K}$-isomorphisms:

$$\hat{\boldsymbol{K}} = \mathrm{Hom}_{K[G]}(V, V) \cong \mathrm{Hom}_{K[K]}(W_1, W_1 \oplus W_1) \cong \boldsymbol{K}_1 \oplus \boldsymbol{K}_1$$

since $W_1|^G \cong V$ and $V|_K = W_1 + W_2 \cong W_1 \oplus W_1$, so that by dimension

count, $[\hat{K}: K_1'] = 2$. By Lemma (8.2) we have $\hat{K} = GF(p^{2t})$ for an odd prime p and an integer $t \geqq 1$. We have proven that -1 is a square in $\hat{K}$ when $V|_K$ is reducible, completing the proof of (1).

Assume that $i \in \hat{K}$ where $i^2 = -1$. Since $\hat{\phi}$ is a $K[H]$-isomorphism of V, and since H stabilizes V_1, W_1 and W_2 are $K[H]$-modules. But $xy(v+i\hat{\phi}v) = i\hat{\phi}xyv + xyv = i\hat{\phi}xyv - \hat{\phi}(\hat{\phi}xyv) = i((\hat{\phi}xyv) + i\hat{\phi}(\hat{\phi}xyv)) \in W_1$ because V_1 is a $\hat{K}$-module and because $\hat{\phi}xyv \in V_1$. We conclude that W_1 is a $K[K]$-module. A similar argument proves that W_2 is also.

Clearly W_1 and W_2 are $\hat{K}$-modules since V_1 and V_2 are. We show that $W_1 \cap W_2 + (0)$. Suppose that $v + i\hat{\phi}v = u - i\hat{\phi}u$ for $u, v \in V_1$. Then $i\hat{\phi}(v+u) = v - u \in V_1 \cap V_2 = (0)$ so that $v - u = 0 = v + u$ and $v = u = 0$. Thus $W_1 \cap W_2 = (0)$. Since $[G: K] = 2$, $V|_K = W_1 \dotplus W_2$ where W_i is an irreducible $K[K]$-module. By Frobenius Reciprocity as in (1.4), we have the following K-vector space isomorphisms.

$$\begin{aligned}\hat{K} = \mathrm{Hom}_{K[G]}(V, V) &\cong \mathrm{Hom}_{K[K]}(W_1, W_1 \dotplus W_2) \\ &\cong \mathrm{Hom}_{K[K]}(W_1, W_1) \oplus \mathrm{Hom}_{K[K]}(W_1, W_2)\,.\end{aligned}$$

Since $\hat{K}$ leaves W_1 invariant and centralizes the action of K on W_1, $\hat{K} \leqq \mathrm{Hom}_{K[K]}(W_1, W_1)$. Dimension considerations then give $\mathrm{Hom}_{K[K]}(W_1, W_2) = (0)$ proving that $W_1 \ncong W_2$.

By Lemma (8.4) $\hat{\phi} \in \mathrm{Hom}_{K[K]}(V, V)$. By Lemma (8.2), $\hat{K} \cong \mathrm{Hom}_{K[K]}(W_i, W_i)$. Since $W_1 \ncong W_2$, $K_1 = \mathrm{Hom}_{K[K]}(V, V) = \mathrm{Hom}_{K[K]}(W_1 + W_2, W_1 + W_2) \cong \mathrm{Hom}_{K[K]}(W_1, W_1) \oplus \mathrm{Hom}_{K[K]}(W_2, W_2) \cong \hat{K} \oplus \hat{K}$. Since $\hat{\phi}$ and $\hat{K}$ lie in the commutative ring K_1, $\hat{\phi}$ centralizes $\hat{K}$, proving (4).

Let G^* be the linear group on V generated by $\hat{\phi}$ and G. By Lemma (8.4) G^* fixes g. Now $\hat{K} \leqq \mathrm{Hom}_{K[G^*]}(V, V) \leqq \hat{K}$ so that $\hat{K} = \mathrm{Hom}_{K[G^*]}(V, V)$. By Theorem (6.7) there is a classical $\hat{K}$-form f fixed by G^* and a $\mu \in \hat{K}$ such that $g = \tau\mu f$ where $\tau: \hat{K} \to K$ is the trace mapping. Let ν be the automorphism of $\hat{K}$ associated with f if f is unitary, or $\nu = 1$ otherwise. By Proposition (6.8) U_i and V_j are totally isotropic for f if $i, j = 1, 2$.

Thus if $u, v \in V_1$ then

$$\begin{aligned}\omega = f(u + i\hat{\phi}u, v + i\hat{\phi}v) &= f(u, i\hat{\phi}v) + f(i\hat{\phi}u, v) \\ &= i^\nu f(u, \hat{\phi}v) + if(\hat{\phi}u, v) \\ &= (i^\nu - i)f(\hat{\phi}u, v)\end{aligned}$$

because $\hat{\phi}$ fixes f. If $i^\nu = i$ then $\omega = 0$ so that W_1 is totally isotropic for f hence also g. Given $u \in V_1$, we may find $v \in V_1$ so that $g(\hat{\phi}u, v) \neq 0$ since V_2 is totally isotropic and g is nonsingular. Thus $f(\hat{\phi}u, v) \neq 0$. By choosing a scalar multiple of v we may take

$f(\hat{\phi}u, v)$ to be equal to any preassigned value in $\hat{K}$. If $i^{\nu} \neq i$ then we may choose $f(\hat{\phi}u, v)$ so that

$$g(u + i\hat{\phi}u, v + i\hat{\phi}v) = \tau\mu f(\hat{\phi}u, v) \neq 0 ,$$

proving that W_1 is not totally isotropic. If W_1 is not totally isotropic then W_1 is nonsingular, because it is irreducible.

From this we conclude that W_1 is nonsingular for g if and only if $i^{\nu} \neq i$, that is, f is unitary, and $\hat{K} = GF(p^{2t})$ where $p \equiv 3 \pmod 4$ and $t \geqq 1$ is odd, since i is a fourth root of unity.

Since $W_1 \ncong W_2$ and $W_1^{\perp}$ is a $K[K]$-module, if W_1 is nonsingular then $W_1^{\perp} = W_2$. If W_1 is totally isotropic then W_2 is also, completing the proof of the proposition.

We may now prove a uniqueness result for the subgroup H.

(8.6) PROPOSITION. *Assume that* (8.1) *holds;* -1 *is a square in* $\hat{K} = \mathrm{Hom}_{K[G]}(V, V)$; *and* V *is a minimal module. If* N *is normal in* G, $V|_N = X_1 + X_2$ *where* X_i *is a homogeneous component, and* $R = \mathrm{stab}\,(G, X_1)$ *then* $R \geqq H$.

Assume that $R \ngeq H$ so that $RH = G$. Applying Theorem (7.9) and the fact that V is a minimal module to the pairs (S, T), (R, S) we find that $V|_H$ and $V|_{R\cap S}$ each is the sum of two isomorphic irreducible modules. In particular, $V_1|_H \cong V_2|_H$ and $V_1|_{R\cap S} \cong V_2|_{R\cap S}$. Choose $x \in R \cap S \backslash R \cap S \cap T$, $w \in S \cap T \backslash R \cap S \cap T$, $y \in R \cap T \backslash R \cap S \cap T$, and set $L = \langle R \cap S \cap T, xw \rangle$. By Proposition (1.8) applied to S, $V_1|_L \ncong V_2|_L$.

Set $K_1 = \langle R \cap T, xw \rangle$. Applying Theorem (7.9), and the fact that V is a minimal module to the pair (R, T) proves that (8.1) holds for that pair. In applying Proposition (8.5) to the pair (R, T) we find that $V|_{K_1} = W_1 + W_2$ where the W_i are homogeneous components. Note now that $K_1 \cap S = L$. Since V is a minimal module, applying Theorem (7.9) to the pair (K_1, S) we find that $V|_L$ is the sum of two isomorphic irreducible modules against the fact that $V_1|_L \ncong V_2|_L$. We conclude that $R \geqq H$, completing the proof.

As a corollary, we obtain the following classification of minimal modules when -1 is a square in $\hat{K}$.

(8.7) THEOREM. *Assume that* (7.1) *holds,* V *is a minimal module, and* -1 *is a square in* $\hat{K} = \mathrm{Hom}_{K[G]}(V, V)$. *Precisely one of the following occurs.*

(1) $V|_N$ *is homogeneous for all* N *normal in* G.

(2) *There is a unique subgroup* S *of index* 2 *in* G *such that*

(i) $V|_S = V_1 + V_2$ *where* V_i *is a totally isotropic homogeneous*

component,

(ii) $V_i|_N$ *is homogeneous for every normal subgroup* $N \leqq S$ *of* G, *and*

(iii) $V|_N$ *is homogeneous for every normal subgroup* $N \not\leqq S$ *of* G.

(3) *There is a unique normal subgroup* H *of* G *where* G/H *is a four group such that*

(i) $V|_L$ *is not homogeneous for any* $H < L < G$, *and*

(ii) $V|_N$ *is homogeneous for any normal subgroup* N *of such that* $N \leqq H$ *or* $NH = G$.

Assume that (1) does not hold. Then there is a normal subgroup of K of G such that $V|_K = V_1 + V_2$ where the V_i are homogeneous totally isotropic components. Let $S = \mathrm{stab}(G, V_1)$. By Corollary (7.11), $V_1|_N$ is homogeneous for every normal subgroup $N \leqq S$ of G. Therefore, if $N \leqq S$ is normal in G and $V|_N$ is inhomogeneous, then the homogeneous components are precisely V_1 and V_2. If (iii) of part (2) holds then S is obviously unique proving (2). Assume that (iii) of (2) fails so that there is a normal subgroup $L \not\leqq S$ of G such that $V|_L = U_1 + U_2$ where the U_i are totally isotropic homogeneous components. Set $T = \mathrm{stab}(G, U_1)$ so that $S \neq T$. By Theorem (7.9) Hypothesis (8.1) holds. By Proposition (8.5) condition (i) of (3) holds with $H = S \cap T$. Part (ii) of (3) holds when $N \leqq H$ since $V|_H$ is homogeneous by Theorem (7.9), and since parts (i) and (ii) of (2) hold for $V|_S$, as we have already observed. Part (ii) of (3) when $NH = G$ and the uniqueness of H are implied by Proposition (8.6) completing the proof of the theorem.

(8.8) DEFINITION. Let G be a group with normal subgroup H where G/H is a four group. Suppose that $\hat{K}$ is a field of characteristic $p > 2$ and $i \in \hat{K}$ satisfies $i^2 = -1$. Fix $x, y \in G$ such that $G = \langle H, x, y \rangle$. Let U be a 2-dimensional $\hat{K}$-space with basis $\{u_1, u_2\}$. Define an action of G on U by setting:

$$\begin{array}{llll} xzu_1 = iu_1 & yzu_1 = iu_2 & xyzu_1 = u_2 & zu_1 = u_1 \\ xzu_2 = -iu_2 & yzu_2 = iu_1 & xyzu_2 = -u_1 & zu_2 = u_2 \end{array}$$

where $z \in H$. If $\alpha, \alpha', \beta, \beta' \in \hat{K}$ then set

$$g_0(\alpha u_1 + \beta u_2, \alpha' u_1 + \beta' u_2) = \alpha\beta' - \alpha'\beta \, .$$

(8.9) PROPOSITION. *U of Definition* (8.8) *is an absolutely irreducible projective* $\hat{K}[G]$*-module with kernel* H *and factor set* γ *inflated from* G/H *and tabulated below. The form* g_0 *is nonsingular,*

symplectic, and fixed by the action of G.

	xH	yH	xyH
xH	-1	1	-1
yH	-1	-1	1
xyH	1	-1	-1

REMARK. The proof is easy once one notes that the extension of G/H by γ is a quaternion group of order 8, and that U is an irreducible module for this quaternion group.

(8.10) THEOREM. *Assume that* (8.1) *holds and that* -1 *is a square in* $\hat{\boldsymbol{K}} = \mathrm{Hom}_{K[G]}(V, V)$.

(1) *There is a projective extension* V_1^* *of* $V_1|_H$ *to* G *such that* $\Gamma\colon V_1^* \otimes_{\hat{K}} U \to V$ *given by*

$$\Gamma(v_1 \otimes u_1 + v_2 \otimes u_2) = v_1 + \hat{\phi} v_2$$

(*where* $v_i \in V_1^* (= V_1)$, U *is as in Definition* (8.8), *and* $\hat{\phi}$ *is as in Lemma* (8.4)) *is a* $\hat{\boldsymbol{K}}[G]$*-isomorphism, and the factor set* γ *of* V_1^* *is inflated from* G/H *and is given by the table of* (8.9).

(2) *There is a classical* $\hat{\boldsymbol{K}}$*-form* f *on* V_1^* *fixed by* G *and a* $\mu \in \hat{\boldsymbol{K}}$ *such that* $g = \tau\mu(f \otimes g_0)$ *where* $\tau\colon \hat{\boldsymbol{K}} \to \boldsymbol{K}$ *is the trace.*

(3) *If* $\boldsymbol{K} = \hat{\boldsymbol{K}}$ *and* V *is a minimal module then* $V_1^*|_N$ *is homogeneous for all* N *normal in* G.

(4) *If* $\boldsymbol{K} = \hat{\boldsymbol{K}}$, V *is form primitive, and* G^* *is the extension of* G *afforded by* γ *then* V_1^* *is a form primitive* $\hat{\boldsymbol{K}}[G^*]$*-module for* f.

Define an action of G on $V_1^* (= V_1)$ as follows:

$$xz^*v = -ixzv\,, \quad yz^*v = -iyz\hat{\phi}v\,, \quad xyz^*v = -xyz\hat{\phi}v\,, \quad z^*v = zv$$

where $v \in V_1$ and $z \in H$. Since $\hat{\phi}$ maps V_1 to V_2 and both yz and xyz also map V_1 to V_2, the *-action of any $w \in G$ is a well-defined $\hat{\boldsymbol{K}}$-linear transformation of V_1^*. It is a straightforward calculation to prove that V_1^* is a projective $\hat{\boldsymbol{K}}[G]$-module with factor set γ tabulated in (8.9).

Obviously Γ defines a $\hat{\boldsymbol{K}}[H]$-isomorphism of $V_1^* \otimes U$ onto V. Since $V_1^* \otimes U$ has factor set $\gamma^2 = 1$, $V_1^* \otimes U$ is nonprojective. Thus one only need prove that x and y commute with Γ in order to complete the proof of (1).

By Theorem (6.7), there is a classical $\hat{\boldsymbol{K}}$-form $\tilde{g}$ on V fixed by G and an element $\omega \in \hat{\boldsymbol{K}}$ such that $g = \tau\omega\tilde{g}$. If $\tilde{g}$ is unitary then we let ν be the associated automorphism of order 2, otherwise we

let $\nu = 1$. By Lemma (8.4), $\hat{\phi}$ fixes g so that we may choose $\tilde{g}$ such that $\tilde{g}$ is fixed by $\hat{\phi}$ also.

If $v_1, v_2 \in V_1^*$ then we set

$$\begin{aligned} f(v_1, v_2) &= \eta\tilde{g}(v_1, \hat{\phi}v_2) \\ &= \eta\tilde{g}(-\hat{\phi}v_2, v_1) \\ &= -\varepsilon\eta^\nu\tilde{g}(v_1, \hat{\phi}v_2)^\nu \\ &= -\varepsilon(\eta^\nu\eta^{-1})f(v_2, v_1)^\nu , \end{aligned}$$

where $\varepsilon = -1$ if $\tilde{g}$ is symplectic or 1 if otherwise, and $\eta^\nu = -\eta \neq 0$ if $\tilde{g}$ is unitary or 1 otherwise. Since $\tilde{g}$ is bilinear, f is also. For any $v_1 \in V_1$, $v_1 \neq 0$, there is a $v_2 \in V_1$ such that $f(v_1, v_2) = \eta\tilde{g}(v_1, \hat{\phi}v_2) \neq 0$ because V is nonsingular for $\tilde{g}$, and because V_1 and $V_2 = \hat{\phi}V_1$ are totally isotropic for $\tilde{g}$ by Proposition (6.8). Consequently, f is nonsingular. Using 0 for unitary, $+$ for symmetric, and $-$ for symplectic; we have the following type table for $\tilde{g}$ and f.

$\tilde{g}$	0	$+$	$-$
f	0	$-$	$+$

If $v_i \in V_i$ for $i = 1, 2, 3, 4$ then

$$\begin{aligned} & f \otimes g_0(v_1 \otimes u_1 + v_2 \otimes u_2, v_3 \otimes u_1 + v_4 \otimes u_2) \\ &\quad = f(v_1, v_3)g_0(u_1, u_1) + f(v_1, v_4)g_0(u_2, u_2) \\ &\qquad + f(v_2, v_3)g_0(u_2, u_1) + f(v_2, v_4)g_0(u_2, u_2) \\ &\quad = f(v_1, v_4) - f(v_2, v_3) \\ &\quad = \eta\tilde{g}(v_1, \hat{\phi}v_4) - \eta\tilde{g}(v_2, \hat{\phi}v_3) \\ &\quad = \eta\tilde{g}(v_1, \hat{\phi}v_4) + \eta\tilde{g}(\hat{\phi}v_2, v_3) \\ &\quad = \eta\tilde{g}(v_1 + \hat{\phi}v_2, v_3 + \hat{\phi}v_4) . \end{aligned}$$

Setting $\mu = \omega\eta^{-1}$ we have $g = \tau\mu(f \otimes g_0)$ proving (2). Using Theorem (6.7) we may make the following table of possibilities.

REMARK. If $\hat{K}$ has an automorphism of order 2 then let ν be it, and choose η so that $\eta^\nu = -\eta \neq 0$. Denoting unitary by 0, symmetric by $+$, and symplectic by $-$ we have the following possibilities.

g	0	$+$	$+$	$-$	$-$
f	0	$-$	0	$+$	0
μ	η	1	η	1	1

In order to prove (3), we first show that if $N \lhd G$ and $NH < G$ then $V_1^*|_N$ is homogeneous. Let $S, T, K = \langle H, xy \rangle$ be the maximal subgroups of G which contain H. Now

$$U|_S = \hat{K}u_1 + \hat{K}u_2 ,$$
$$U|_T = \hat{K}(u_1 + u_2) + \hat{K}(u_1 - u_2) , \quad \text{and}$$
$$U|_K = \hat{K}(u_1 + iu_2) + \hat{K}(u_1 - iu_2) .$$

That is, if L is one of S, T, or K then $V|_L \cong V_1^*|_L \otimes (X_1 + X_2)$ where $U|_L = X_1 + X_2$ is the sum of 1-dimensional modules. Since we assume that $NH < G$, we may take $N \leqq L$. If $\hat{X}_1$ is the dual of X_1 then $X_1 \otimes \hat{X}_1$ is the trivial module, so that $V_1^* \otimes X_1 \otimes \hat{X}_1 \cong V_1^*$. Since $V_1^* \otimes V_1|_N$ is homogeneous by Theorem (8.7), $V_1^*|_N$ is homogeneous.

Now suppose that $N \lhd G$, $NH = G$, and $V_1^*|_N = W_1 + \cdots + W_t$ where the W_i are homogeneous components. Suppose that $t \geqq 1$ and W_1' is an irreducible $\hat{K}[N]$-constituent of W_i for $i = 1, 2$. By our supposition on N, $N_1 = S \cap N$, $N_2 = T \cap N$, and $N_3 = K \cap N$ are three distinct subgroups of index two in N. Since $V_1^*|_{N_i}$ is homogeneous for $i = 1, 2, 3$ we conclude that $W_1'|_{N_i} \cong W_2'|_{N_i}$, $i = 1, 2, 3$. By Proposition (1.8) we have $W_1' \cong W_2'$ contradicting the fact that $W_1 \ncong W_2$. The proof of (3) is complete.

Suppose that M^* is a subgroup of G^*, W is a proper $\hat{K}[M^*]$-submodule of V_1^* which form induces V_1^*. There is no loss in assuming that M^* is a maximal subgroup of G^*. Under the natural mapping of G^* onto G, let M be the image of M^*. Certainly W is a direct summand of $V_1^*|_{M^*}$. If $G > M > H$ then $U|_{M^*}$ is the sum of two modules so that $V|_M$ would have more than two components contrary to (8.1). Consequently we must have $MH = G$. Thus $W \otimes U$ is nonsingular for $\hat{K}[M]$ and form induces V. The proofs of (4) and the theorem are complete.

We turn now to the case where -1 is not a square in $\hat{K}$, and set the following additional hypotheses.

(8.11) HYPOTHESIS. *Assume that* (8.1) *holds, and that* -1 *is not a square in* $\hat{K} = \mathrm{Hom}_{K[G]}(V, V)$. *Let* $\tilde{K}$ *be the splitting field of* $X^2 + 1$ *over* K *and* $\bar{K} = \tilde{K} \otimes_K \hat{K}$. *If* W *is a* K*-subspace of* V *then set* $\tilde{W} = \hat{K} \otimes_K W$. *Define*

$$\tilde{g}(\alpha \otimes u, \beta \otimes v) = \alpha\beta g(u, v)$$

for $\alpha, \beta \in \tilde{K}$ *and* $u, v \in V$.

(8.12) LEMMA. (1) $\hat{K} = GF(p^t)$ *where* $p \equiv 3 \pmod 4$ *and* t *is odd.*

(2) $\bar{K}$ *is the splitting field of* $X^2 + 1$ *over* $\hat{K}$.

(3) *Let* $\tau: \bar{K} \to \tilde{K}$ *be the trace. Then* $\tau|_{\hat{K}}: \hat{K} \to K$ *is the trace.*

(4) $\bar{K}$ *acts upon* $\tilde{V}$ *and* $\bar{K} = \mathrm{Hom}_{\tilde{K}[G]}(\tilde{V}, \tilde{V})$.

(5) g *is not unitary, and there is a classical* $\hat{K}$*-form* $\hat{g}$ *on* V

fixed by G *of the same type as* g *such that* $g = \tau\hat{g}$.

(6) *If we set* $g^*(\alpha \otimes u, \beta \otimes v) = \alpha\beta\hat{g}(u, v)$ *for* $\alpha, \beta \in \tilde{K}$ *and* $u, v \in V$ *then* g^* *and* $\tilde{g}$ *are well defined forms of the same type as* g *and* $\tilde{g} = \tau g^*$.

(7) *With respect to* S, T, $\tilde{V}$, $\tilde{g}$, $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{U}_1$, *and* $\tilde{U}_2$ *the Hypothesis* (8.1) *holds.*

Part (1) follows from the fact that -1 is not square in $\hat{K}$. Parts (2) and (3) are obvious. If $\alpha, \alpha' \in \tilde{K}$, $\beta \in \hat{K}$, and $v \in V$ then $(\alpha \otimes \beta)(\alpha' \otimes v) = (\alpha\alpha') \otimes (\beta v)$ defines an action of $\bar{K}$ on $\tilde{V}$ which commutes with G on $\tilde{V}$. Since V is an absolutely irreducible $\hat{K}[G]$-module [14, (29.3)], $\tilde{V}$ is an absolutely irreducible $\bar{K}[G]$-module [14, (29.21)] so that (4) follows [14, (29.3)], and in particular, $\tilde{V}$ is an irreducible $\tilde{K}[G]$-module. Part (5) follows from (1) and Theorem (6.7) since neither K nor $\hat{K}$ has an automorphism of order two. Part (6) follows by (3), (5) and the fact that neither g nor $\hat{g}$ is unitary. Finally, (7) follows from Theorem (7.9) since $\tilde{V}_i \cap \tilde{U}_j = (0)$ for $i, j = 1, 2$, and since $\tilde{V}$ is an irreducible $\tilde{K}[G]$-module.

REMARK. This lemma allows us to apply the reduction of (8.10) to the module $\tilde{V}$ in place of V whenever -1 is not a square in $\hat{K}$. The difficulty with this approach is that if V is a minimal module, then $\tilde{V}$ may not be. Further, if $K = \hat{K}$ and V is a form primitive, then $\tilde{V}$ may not be.

(8.13) THEOREM. *Assume that* (8.1) *holds and that* -1 *is not a square in* $\hat{K} = \mathrm{Hom}_{K[G]}(V, V)$. *If* $H = S \cap T$ *and* $K = \langle H, xy \rangle$ *where* $x \in S \backslash T$ *and* $y \in T \backslash S$ *then the following hold.*

(1) $\hat{\phi}$ *of Lemma* (8.4) *has square* -1 *and lies in the field* $\bar{K} = \mathrm{Hom}_{K[K]}(V, V)$. *Further* $[\bar{K}: \hat{K}] = 2$.

(2) *There is a nonsingular unitary* $\bar{K}$*-form* f *on* V *fixed by* K *such that* $g = \tau\mu f$ *where* $\tau: \bar{K} \to K$ *is the trace and where* $\mu = 1$ *if* g *is symmetric or* $\mu = \hat{\phi}$ *if* g *is symplectic.*

(3) *If* $2^t \| p + 1$ *where* $\mathrm{char}\, K = p$ *and if* ω *is a primitive* 2^t *root of unity in* $\bar{K}$ *then there is a projective extension* $V^*(= V)$ *of the* $\bar{K}[H]$*-module* V *to a projective* $\bar{K}[G]$*-module with factor set* γ *inflated from* G/H *and tabulated below.*

	xH	yH	xyH
xH	ω	$-\omega$	-1
yH	ω	$-\omega$	-1
xyH	1	1	1

(4) *The action of G on V^* fixes f.*
Let G^* be the central extension of G afforded by γ.

(5) *If $\boldsymbol{K} = \bar{\boldsymbol{K}}$ and V is form primitive then V^* is a form primitive $\boldsymbol{K}[G^*]$-module for f.*

By Schur's lemma and Wedderburn's theorem on finite division rings, $\bar{\boldsymbol{K}}$ is a field because $V|_K$ is irreducible by Proposition (8.5). The first part of (1) now follows from Lemma (8.4). The second part follows from Proposition (1.7).

We may apply Theorem (6.7) to the $\boldsymbol{K}[K]$-module V. By Lemma (8.13) g is not unitary. Since $\hat{\phi}$ fixes g by Lemma (8.4), we may choose f so that $\hat{\phi}$ fixes f. If f is not unitary then

$$f(u, v) = f(\hat{\phi}u, \hat{\phi}v) = \hat{\phi}^2 f(u, v) = -f(u, v)$$

for $u, v \in V$. We conclude that f is unitary. Since $\hat{\phi}^{-1} = -\hat{\phi}$, and since the automorphism of order two of $\bar{\boldsymbol{K}}$ inverts $\hat{\phi}$, μ of Theorem (6.7) (ii) may be taken to be $\hat{\phi}$ if g is symplectic or 1 otherwise, completing the proof of (2).

By Lemma (8.12) (1), $\bar{\boldsymbol{K}} = GF(p^{2s})$ where $p \equiv 3 \pmod 4$ and s is odd. Therefore, the highest power of 2 in $p^{2s} - 1$ is twice the highest power of 2 in $p + 1$, i.e., 2^{t+1}. Choose $\xi \in \bar{\boldsymbol{K}}$ of order 2^{t+1} such that $\xi^2 = \omega^{-1}$. Let ν be the automorphism of order two of $\bar{\boldsymbol{K}}$. Since $\hat{\phi}V_1 = V_2$, we may define a $\hat{\boldsymbol{K}}$-linear ($\bar{\boldsymbol{K}}$-semilinear) action for ν on V given by

$$\nu(v_1 + \hat{\phi}v_2) = v_1 - \hat{\phi}v_2 ,$$

where $v_i \in V_1$. Since ν fixes V_1 and V_2, the action of ν clearly commutes with the action of H on V. But for $v_1, v_2 \in V_1$ we have

$$\nu x(v_1 + \hat{\phi}v_2) = \nu(xv_1 - \hat{\phi}xv_2) = xv_1 + \hat{\phi}xv_2 = x(v_1 - \hat{\phi}v_2) = x\nu(v_1 + \hat{\phi}v_2) ,$$

and

$$\begin{aligned} \nu y(v_1 + \hat{\phi}v_2) &= \nu(y\hat{\phi}v_2 - \hat{\phi}(\hat{\phi}yv_1)) = y\hat{\phi}v_2 + \hat{\phi}(\hat{\phi}yv_1) \\ &= y(-v_1 + \hat{\phi}v_2) = -y\nu(v_1 + \hat{\phi}v_2) \end{aligned}$$

proving that on V we have

$$x\nu = \nu x \quad \text{and} \quad y\nu = -\nu y . \tag{8.14}$$

By Proposition (1.7),

$$x\xi = \xi^{\nu}x \quad \text{and} \quad y\xi = \xi^{\nu}y . \tag{8.15}$$

For elements of G we define the following action on V:

$$\Theta(xz)v = \nu\xi xzv , \quad \Theta(yz)v = \nu\xi yzv , \quad \Theta(xyz) = -xyzv , \quad \Theta(z)v = zv$$

where $z \in H$ and $v \in V$. From (8.14) and (8.15) it is straightforward to verify that $\Theta(z)$ is $\bar{K}$-linear on V for all $z \in G$. Further, direct calculation shows that Θ is a projective representation of G with factor set γ inflated from G/H and tabulated in (3).

REMARK. Let G_1 be the group $\langle u, v \mid u^2 = v^{2^t+1} = 1, uvu = v^{1+2^t} \rangle$ and equate $v^2 = \omega$, $v^{2^t} = -1$. Idenifying xH with v, yH with uv and xyH with u allows one to consider $G/H \cong G_1/\langle v^2 \rangle$ as having a central extension by $\langle v^2 \rangle$ with factor set γ. Keeping this in mind simplifies factor set computations.

Since f is fixed by K, it is fixed by $\Theta(z)$, $z \in K$. If $u, v \in V$ then

$$\begin{aligned} f(\nu\xi xu, \nu\xi xv) &= f(\xi^\nu \nu xu, \xi^\nu \nu xv) = \xi^\nu \xi f(\nu xu, \nu xv) \\ &= -f(\nu xu, \nu xv) = -f(x\nu u, x\nu v) . \end{aligned}$$

By Lemma (8.12) (5) $\hat{g} = \tau_0 uf$ is a classical $\hat{K}$-form of the same type as g and fixed by G where $\tau_0 : \bar{K} \to \hat{K}$ is the trace. Applying Proposition (1.3) yields:

$$\begin{aligned} f(x\nu u, x\nu v) &= (2\mu)^{-1}\hat{g}(x\nu u, x\nu v) + (2\mu\hat{\phi})^{-1}\hat{g}(\hat{\phi}x\nu u, x\nu v) \\ &= (2\mu)^{-1}g(\nu u, \nu v) - (2\mu\hat{\phi})^{-1}\hat{g}(\hat{\phi}\nu u, \nu v) . \end{aligned}$$

If $u = u_1 + \hat{\phi}u_2$, $v = v_1 + \hat{\phi}v_2$ where $u_i, v_j \in V_1$ then

$$\begin{aligned} f(x\nu u, x\nu v) &= (2\mu)^{-1}\hat{g}(\nu u, \nu v) - (2\mu\hat{\phi})^{-1}\hat{g}(\hat{\phi}\nu u, \nu v) \\ &= (2\mu)^{-1}\hat{g}(u_1 - \hat{\phi}u_2, v_1 - \hat{\phi}v_2) - (2\mu\hat{\phi})^{-1}\hat{g}(u_2 + \hat{\phi}u_1, v_1 - \hat{\phi}v_2) \\ &= -(2\mu)^{-1}[\hat{g}(u_1, \hat{\phi}v_2) + g(\hat{\phi}u_2, v_1)] \\ &\quad - (2\mu\hat{\phi})^{-1}[-\hat{g}(u_2, \hat{\phi}v_2) + g(\hat{\phi}u_1, v_1)] \\ &= -(2\mu)^{-1}\hat{g}(u_1 + \hat{\phi}u_2, v_1 + \hat{\phi}v_2) \\ &\quad - (2\mu\hat{\phi})^{-1}\hat{g}(\hat{\phi}(u_1 + \hat{\phi}u_2), v_1 + \hat{\phi}v_2) \\ &= -f(u, v) . \end{aligned}$$

Combining our calculations gives

$$f(\nu\xi xu, \nu\xi xv) = f(u, v) .$$

Now $\nu\xi y = \nu\xi x \cdot xyz$ where $z = y^{-1}x^{-2}y$, and both $\nu\xi x$ and xyz fix f so that

$$f(\nu\xi yu, v\xi yv) = f(u, v) .$$

From this (4) follows.

Suppose that M^* is a subgroup of G^* and that U^* is a non-singular $\bar{K}[M^*]$-submodule of V^* which form induces V^*. Since $\langle \omega \rangle \leqq \bar{K}$ and G^* has a central element acting as ω on V^*, M^* contains this central element acting as ω. Therefore, $\mathrm{Hom}_{\hat{K}[M^*]}(U^*, U^*) \geqq$

$\bar{K}$. Now by Frobenius Reciprocity as in Proposition (1.4) we have the following $\hat{K}$-vector space isomorphisms

$$\bar{K} = \mathrm{Hom}_{\hat{K}[G^*]}(V^*, V^*) \cong \mathrm{Hom}_{\hat{K}[M^*]}(U^*, V^*) .$$

But $\dim_{\hat{K}} \mathrm{Hom}_{\hat{K}[M^*]}(U^*, V_*) \geqq \dim_{\hat{K}} \mathrm{Hom}_{\hat{K}[M^*]}(U^*, U^*) \geqq \dim_K \bar{K}$. Consequently, as $\hat{K}$-vector spaces

$$\mathrm{Hom}_{\hat{K}[M^*]}(U^*, V^*) \cong \mathrm{Hom}_{\hat{K}[M^*]}(U^*, U^*) . \tag{8.16}$$

Since $\nu V^* = V^*$, νU^* is a $\bar{K}$-submodule of V^*. Since U^* is a $\bar{K}[M^*]$-module, νU^* is also. Further $\Gamma(u) = \nu u$ defines a $\hat{K}[M^*]$-isomorphism of U^* onto νU^* by Proposition (1.9). Since $\bar{K} = \mathrm{Hom}_{\hat{K}[M^*]}(U^*, U^*)$, U^* is an irreducible $\hat{K}[M^*]$-module. Thus $\nu U^* = U^*$ or $U^* + \nu U^*$ is a direct sum in V^*. By (8.16) the latter cannot occur. Therefore, $\nu U^* = U^*$. Since U^* is ν- and $\bar{K}$-invariant, if M is the image of M^* in G under the natural homomorphism of G^* onto G then U^* is M-invariant, as may be seen by checking the action of elements given by Θ. Recall that $\hat{g} = \tau_0 \mu f$ where $\tau_0 : \bar{K} \to \hat{K}$ is the trace. Since U^* is ν- and $\bar{K}$-invariant, so are all the G^*-translates. Since the kernel of the natural homomorphism $G^* \to G$ has kernel acting as a subgroup of $\bar{K}$, a transversal of M^* in G^* maps onto a transversal of M in G. Thus a G^*-translate of U^* is a G-translate of U^*. Therefore, U^* is a nonsingular $\hat{K}[M]$-module which form induces $V^*(=V)$ for $\hat{g}(=g$, since $\hat{K} = K)$. Therefore, $M = G$ so that $M^* = G^*$ completing the proof of the theorem.

REMARK. This theorem is an analog to, but not as strong as, Theorem (8.10). For example, if $K = \hat{K}$ and V is a minimal $\hat{K}[G]$-module for g then V^* may fail to be a minimal $\bar{K}[G^*]$-module for f. Proposition (8.5) (3) describes a way in which this situation can occur. Applying this theorem to V gives a module over a field in which -1 is a square, and therefore, shifts consideration to modules of previously considered type.

(8.17) COROLLARY. *If $K = \hat{K}$ and V is form primitive in Theorem* (8.13), *then with respect to f and as a $\bar{K}[G^*]$-module, precisely one of the following occurs.*

(1) *$V^*|_{N^*}$ is homogeneous for every N^* normal in G^*.*

(2) *There is a unique subgroup S^* of index 2 in G^* such that*

(i) *$V^*|_{S^*} = V_1^* + V_2^*$ where V_i^* is a totally isotropic homogeneous component,*

(ii) *$V_i^*|_{N^*}$ is homogeneous for every normal subgroup $N^* \leqq S^*$ of G^*, and*

(iii) $V|_{N^*}$ *is homogeneous for every normal subgroup* $N^* \trianglelefteq S^*$ *of* G^*.

The proof of (8.17) follows exactly that of Theorem (8.7). In that proof, we discovered that if (1) and (2) fail then Hypothesis (8.1) must hold. By Lemma (8.12) (1) and Theorem (8.13) (1), (2), in applying Proposition (8.5) (3), the modules W_i are nonsingular and V^* is form induced. Therefore, Hypothesis (8.1) cannot hold, completing the proof of the theorem.

We add one final proposition so that Theorem (8.10) (4) and Theorem (8.13) (5) become more useful.

(8.18) PROPOSITION. *In* (7.1) *if* V *is form primitive, if* $\hat{\boldsymbol{K}} = \mathrm{Hom}_{K[G]}(V, V)$ *and if* $\tilde{g}$ *is a classical* $\hat{\boldsymbol{K}}$*-form on* V *fixed by* G *such that* $g = \tau\mu\tilde{g}$ *for some* $\mu \in \hat{\boldsymbol{K}}$ *where* $\tau: \hat{\boldsymbol{K}} \to \boldsymbol{K}$ *is the trace, then* V *is form primitive for* $\tilde{g}$.

Assume that U is a nonsingular $\hat{\boldsymbol{K}}[S]$-module of V for some subgroup H of G which form induces V with respect to $\tilde{g}$. Since $\tilde{g}$ is nonsingular on U, $g = \tau\mu\tilde{g}$ is nonsingular on U. Further, if xU and yU are orthogonal for $\tilde{g}$, they are orthogonal for g. Thus U form induces V as a $\boldsymbol{K}[G]$-module with respect to g.

REMARK. (1) Unfortunately, the converse of this proposition is false.

(2) For form primitive modules, application of Theorems (8.13), (8.10), and Proposition (8.18) reduces the structure of such $\boldsymbol{K}[G]$-modules with form g to the structure of absolutely irreducible form primitive $\bar{\boldsymbol{K}}[G^*]$-modules V^* with form f where $V^*|_{N^*}$ is homogeneous for all N^* normal in G^*, and where $\bar{\boldsymbol{K}}$, G^*, f respectively bear a fixed relationship to $\boldsymbol{K}$, G, g.

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