PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 2, 1977

COMPUTING CHARACTERS OF TAMELY RAMIFIED p-ADIC DIVISION ALGEBRAS

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The purpose here is to prove the results announced in *Representation theory for division algebras over local fields* **(tamely ramified case), B.A.M.S. 17 (1971), 1063–1066, by R. Howe.**

1. We discuss here one aspect of the representation theory of division algebras over local fields. Since there is already an account of the construction of the representations of tamely ramified p-adic division algebras [1] by one of us, and an account of the analogous construction for Gl_n by the other [3], the emphasis of this paper is on character computations. The computations proceed inductively, and the main focus of interest is on the inductive step. This step is valid in a slightly wider context than the final result. The approach is pretty much head on, and makes no pretence to subtlety. It owes its success in this case to the extremely pleasant geometry of the conjugacy classes in tamely ramified division algebras (see for example Lemmas 1 and 2), a fact probably worth pointing out for itself.

2. Let F be a p-adic field, with p odd, R the integers of F, π a prime of F. Let D be a division algebra central over F, of degree n and dimension n^2. We take n prime to p, and odd. Let S be the integers of F. Let F_u be an unramified extension of F of degree n contained in D. Let Π be a prime of S normalizing F_u and such that

$$\Pi \times \Pi^{-1} = \rho(x) \quad \text{for} \quad x \in F_u \tag{1}$$

where ρ is a generator of the Galois group of F_u over F. We may arrange that $\Pi^n = \pi$. Let F^x, R^x, etc. be the multiplicative groups of F, R, etc. Let $\bar{F} = R/\pi R$ be the residue class field of F. Similarly $\bar{F}_u = R_u/\pi R_u$. Note that $S/\Pi S \simeq \bar{F}_u$. By $\bar{F}_u^x$ we understand the roots of unity of order prime to p in F_u^x, and similarly for $\bar{F}^x$. Note that $\bar{F}^x \subseteqq \bar{F}_u^x$.

Let C be the group in D^x generated by $\bar{F}_u^x$ and Π. Write

$$V_i = 1 + \Pi^i S\,. \tag{2}$$

The group V_1 is the maximal pro-p subgroup of D^x and we have the semidirect product decomposition

$$D^{\times} = C \cdot V_1 \,. \tag{3}$$

Let σ be an irreducible admissible representation of $D^{\times}$. We may as well assume that $\sigma(\pi) = 1$, so that σ factors through a compact group. Thus the Haar measures used later on will be understood taken modulo π. Thus entails no loss of generality, and will not be made explicit in the notation. If j is the smallest integer such that $V_j \subseteqq \ker \sigma$, then V_j is called the conductor of σ. We assume $j \geqq 2$. Since V_{j-1}/V_j is abelian, the restriction of σ to V_{j-1} breaks up into a sum of linear characters of V_{j-1}. Let ψ be one character of V_{j-1} which occurs in σ. All other characters of V_{j-1} which occur in σ will be equivalent to ψ under conjugation by C. We will give an inductive procedure for understanding the representations of $D^{\times}$ with conductor V_j whose restriction to V_{j-1} contains ψ.

Let $\mathrm{tr}\colon D \longrightarrow F$ be the reduced trace map. Let χ be an additive character of F such that $R \subseteqq \ker \chi$ but $\pi^{-1}R \not\subseteqq \ker \chi$. There is an unique element c_0 of C such that

$$\psi(1 + y) = \chi(\mathrm{tr}\,(c_0 y)) \qquad \text{for} \quad y \in \Pi^{j-1}S \,. \tag{4}$$

We will have $c_0 = \Pi^{-a} r$ with $r \in \bar{F}_u^{\times}$ and

$$a = n + j - 1 \,. \tag{5}$$

Let F' be the field generated over F by c_0. Let D' be the centralizer of F' in D.

THEOREM 1. (a) *There is a bijection between the following two sets.*

(1) *The set* $A = A(j, \psi)$ *of irreducible representations of* $D^{\times}$ *having conductor* V_j *and containing the character* ψ *of* V_{j-1}.

(2) *The set* $B = B(c_0, j)$ *of irreducible representations of* $D'^{\times}$ *which are trivial on* $D'^{\times} \cap V_{j-1}$.

(b) *Let* σ *be an element of* A *and* σ' *the corresponding element of* B. *Then* σ *is induced from a representation* $\check{\sigma}$ *of* $D'^{\times}$. V_1 *with the following properties. Let* $ch(\check{\sigma})$ *be the character of* $\check{\sigma}$, *and* $ch(\sigma')$ *be the character of* σ'. *Then*:

(i) $ch(\check{\sigma})(x) = 0$ *for* $x \in D^{\times}$ *unless* x *is conjugate to an element of* $D'^{\times}$ *modulo* V_{j-1}; *and*

(ii) *on* $D'^{\times}$, *there is a function* $\omega = \omega(c_0, j)$ *such that* $ch(\check{\sigma}) = \omega ch(\sigma')$.

Further particulars will emerge during the proof, which will occupy most of this section.

At first we will assume j is even because the argument is clearer then than for odd j. Afterward we will indicate the modifications necessary for odd j.

Let $D'^{\perp}$ be the orthogonal complement to D' with respect to the bilinear form $\operatorname{tr}(xy)$ on D. Let

$$U_i = (1 + D'^{\perp}) \cap V_i \,. \tag{6}$$

Let

$$V'_i = D'^{\times} \cap V_i \,. \tag{7}$$

Note that if f is the unramified degree of F' over F, and Π' is a prime of D', then

$$V'_j = 1 - \Pi'^{\alpha} S' \tag{8}$$

where α is the smallest integer such that $f\alpha \geqq j$.

An argument as in [3] (Lemma 5) shows that

$$V_i = V'_i \cdot U_i = U_i V'_i \,. \tag{9}$$

Note that the number a given in (5) must be divisible by f. Hence $j - 1$ is also divisible by f, so (4) shows that the character ψ essentially lives on V'_{j-1}, in the sense that $U_{j-1} \subseteqq \ker \psi$. Another argument as in [3] (Lemma 9) shows that we may fix a linear character $\tilde{\psi}$ of $D'^{\times}$ such that $\tilde{\psi}$ agrees with ψ on V'_{j-1}.

We may now describe the correspondence of Theorem 1, part (1), assuming j is even. Set $j = 2i$. Choose σ' in B. Let us write $D'^{\times} \cdot V_k = E_k$. Define a representation $\tilde{\psi}\sigma'$ of E_i by the formula

$$\tilde{\psi}\sigma'(d'u) = \tilde{\psi}(d')\sigma'(d') \qquad \text{for} \quad d' \in D'^{\times} \quad \text{and} \quad u \in U_i \,. \tag{10}$$

Define σ as the induced representation

$$\sigma = \operatorname{ind}_{E_i}^{D^{\times}} \tilde{\psi}\sigma' \,. \tag{11}$$

We will show that the correspondence $\sigma' \to \sigma$ has the desired properties.

We will need some facts about the geometry of conjugacy classes in D. Let Ad denote action of $D^{\times}$ on D by conjugation; that is

$$\operatorname{Ad} x(y) = xyx^{-1} \qquad \text{for} \quad x \in D^{\times} \quad \text{and} \quad y \in D \,. \tag{12}$$

Another argument as in [3] (Lemmas 6 and 8) proves the following.

LEMMA 1. *For any y in $c_0 V'_1$ and $k \geqq 1$ we have*

$$\operatorname{Ad} U_k(y V'_k) = y V_k \,. \tag{13}$$

Furthermore, for any y_1 and y_2 in c_0V_1', if there is z in D^x such that $\operatorname{Ad} z(y_1) = y_2$, *then z belongs to D'^x.*

From this, we get a description of the conjugacy classes in D. This will be important in our computations. Let ord_D denote the standard valuation on D, i.e., $\operatorname{ord}_D(\Pi) = 1$.

LEMMA 2. *Given x in D, there is an unique element x' such that x' is conjugate to x by V_1 and*

$$z' = \sum_{l=1}^{\infty} c_l(x) \tag{14}$$

where each $c_l(x)$ is in C, all the c_l commute, and $\operatorname{ord}_D c_l(x) > \operatorname{ord}_D c_{l-1}(x)$.

Proof. From (3), there is an unique $c_1(x)$ in C such that $\operatorname{ord}_D(x - c_1(x)) > \operatorname{ord}_D x$. By Lemma 1, x is conjugate to x_1 in the centralizer of $c_1(x)$. Further if x is conjugate by V_1 to x_1 and $\tilde{x}_1$, both centralizing $c_1(x)$, then both x_1 and $\tilde{x}_1$ are conjugate in the centralizer of $c_1(x)$. Now replace x by $x_1 - c_1(x)$ and repeat.

Now return to consideration of representations. As above ψ is a character of V_{j-1}/V_j represented by c_0 and $j = 2i$. For any $k \geqq i$, the quotient group V_k/V_j is abelian. Let P be the set of unitary characters φ of V_k/V_j such that φ agrees with ψ on V_{j-1}. For such φ we can find z in c_0V_1 such that

$$\varphi(1 + x) = \chi(\operatorname{tr}(zx)) \qquad \text{for} \quad x \in \Pi^k S\,. \tag{15a}$$

Of course if z satisfies (15a) so will any element of zV_{j-k}. Thus (15a) yields a bijection

$$\beta\colon P \longrightarrow c_0V_1/V_{j-k}\,. \tag{15b}$$

Both domain and range of β are invariant under conjugation by E_1, and β is clearly equivariant with respect to conjugation by E_1. Therefore (13) implies that:

(a) any φ in P is conjugate by V_1 to φ' in P such that $\beta(\varphi')$ intersects c_0V_1'; and

(b) if φ_1 and φ_2 are conjugate by E_1, and $\beta(\varphi_i)$ intersects c_0V_1' for $i = 1, 2$, then the φ_i are conjugate by D'^x;

(c) for φ in P, if $\beta(\varphi)$ intersects $c_0(V_1')$, then the isotropy group of φ in E_1 is contained in E_{j-k}.

From these facts, for $k = i$, and standard Clifford theory [6], which describes the relation between representations of a finite group and a normal subgroup, part (a) of Theorem 1 is immediate.

We now turn to the proof of part (b) of Theorem 1. This will take longer. For $i \leqq k \leqq j$, let σ'_k denote the restriction of $\tilde{\psi}\sigma'$ to E_k. Let σ_k denote the induced representation

$$\sigma_k = \mathrm{ind}_{E_k}^{D'} \sigma'_k \ . \tag{16}$$

Consider also the representation

$$\check{\sigma}_k = \mathrm{ind}_{E_k}^{E_1} \sigma'_k \tag{17}$$

of E_1. We obviously have

$$\sigma_k = \mathrm{ind}_{E_1}^{D^x} \check{\sigma}_k \ . \tag{18}$$

Also the σ and $\check{\sigma}$ of Theorem 1 are now denoted σ_i and $\check{\sigma}_i$.

It is clear that $ch(\check{\sigma}_{j-1})(x) = 0$ unless x is conjugate by V_1 to an element of D'^x modulo V_{j-1}. According to Frobenius' formula for characters of induced representations we have, for $x \in D'^x$

$$ch(\check{\sigma}_{j-1})(x) = \sum_{x \in U_1/U_{j-1}} \dot{c}h(\sigma'_{j-1})(\mathrm{Ad}\, u(x)) \tag{19}$$

where $\dot{c}h$ is defined by

$$\dot{c}h(\sigma'_{j-1})(y) = \begin{cases} ch(\sigma'_{j-1})(y) & \text{if} \quad y \in E_{j-1} \\ 0 & \text{otherwise} \ . \end{cases} \tag{20}$$

We have the general formula

$$\mathrm{Ad}\,(1+z)(x) = x + [z, x](1+z)^{-1} = x(1 + x^{-1}[z, x](1+z)^{-1}) \ . \tag{21}$$

Suppose that $1 + z$ is in U_1 and x is in D'^x. Then $x^{-1}[z, x]$ belongs to $D'^{\perp} \cap \Pi S$, so $1+x^{-1}[z, x]$ belongs to U_1. Suppose $\mathrm{ord}_D(x^{-1}[z, x]) = m$. Then from (21), since $\mathrm{ord}_D(z) \geqq 1$, we see that $x^{-1}\,\mathrm{Ad}\,(1+z)(x)$ belongs to U_m modulo V_{m+1}, and does not belong to V'_1 modulo V_{m+1}. Therefore, in order for $\mathrm{Ad}\,(1+z)(x)$ to belong to E_{j-1}, it is necessary that $\mathrm{ord}_D(x^{-1}[z, x]) \geqq j - 1$. Since $U_{j-1} \subseteqq \ker \sigma'_{j-1}$, it follows from (19) that for $x \in D'^x$,

$$ch(\check{\sigma}_{j-1})(x) = \nu(x) ch(\sigma'_{j-1})(x) = \nu(x)\tilde{\psi}(x) ch\sigma'(x) \tag{22}$$

where $\nu(x)$ is the index of the centralizer of x modulo V_j in D'^x in the centralizer of x modulo $U_{j-1}V_j$ in E_1.

The formula (22) shows that the quotient

$$\frac{ch(\check{\sigma}_{j-1})(x)}{ch(\sigma'_{j-1})(x)}$$

is independent of σ' in B. It also shows that the characters $ch(\check{\sigma}_{j-1})$ are linearly independent. Since the $ch(\check{\sigma}_{j-1})$ are obviously linear combinations of the irreducible characters $ch(\check{\sigma}_i)$, by part (a) of

Theorem 1, and since there are the same number of $ch(\check{\sigma}_{j-1})$ as $ch(\check{\sigma}_i)$, it follows that these two sets of characters have the same span. In particular, each $ch(\check{\sigma}_i)$ is a linear combination of the $ch(\check{\sigma}_{j-1})$. Thus $ch(\check{\sigma}_i)(x)$ vanishes unless x is conjugate in E_1 to an element of D'^x modulo V_{j-1}. Similarly $ch(\sigma_i)(x)$ vanishes unless x is conjugate in D^x to element of D'^x modulo V_{j-1}. Thus statement (i) of part (b) of Theorem 1 is proved.

To complete Theorem 1 we study the intertwining between certain of the $\check{\sigma}_k$ for $i \leqq k \leqq j$. Observe that the set

$$\{\sigma'_k: \sigma' \in B\}$$

may be characterized as the set of representations ρ of E_k such that

(i) the restriction of ρ to V_{j-1} is a multiple of ψ, and

(ii) $U_k \subseteqq \ker \rho$.

With this in mind, choose σ' and τ' in B. We will compute $I(\check{\sigma}_k, \check{\tau}_{k+1})$, the intertwining number of $\check{\sigma}_k$ and $\check{\tau}_{k+1}$. Let $\dot{c}h(\sigma'_k)$ be the function on E_1 which equals $ch(\sigma'_k)$ on E_k and vanishes off E_k. Let $\dot{c}h(\tau'_{k+1})$ be defined similarly. For any function f on E_1, define

$$\mathrm{Ad}^*\, y(f)(x) = f(\mathrm{Ad}\, y^{-1}(x)) \qquad \text{for} x,\ y \text{ in } E_1\,. \tag{23}$$

Frobenius reciprocity and the Frobenius formula for induced characters combine to give

$$I(\check{\sigma}_k, \check{\tau}_{k+1}) = \sum_{u^{-1}\in U_1/U_k} \int_{E_{k+1}} \mathrm{Ad}^*\, u(\dot{c}h\sigma'_k)\overline{ch(\tau'_{k+1})}de\,. \tag{24}$$

Here de is normalized Haar measure on E_{k+1}, and $^{-}$ indicates complex conjugate. Denote the integrals in (24) by $m(u)$.

Define

$$\mathrm{Ad}\, u(E_k) = \{\mathrm{Ad}\, u(x): x \in E_k\}\,, \tag{25a}$$

and

$$\mathrm{Ad}^*\, u(\sigma'_k)(\mathrm{Ad}\, u(x)) = \sigma'_k(x) \qquad \text{for} \quad x \text{ in } E_k\,. \tag{25b}$$

Thus $\mathrm{Ad}^*\, u(\sigma'_k)$ is a representation of $\mathrm{Ad}\, u(E_k)$. Evidently

$$\dot{c}h(\mathrm{Ad}^*\, u(\sigma'_k)) = \mathrm{Ad}^*\, u(\dot{c}h\sigma'_k)\,. \tag{26}$$

The group $U_{k+1}\cdot V_j$ is normal in E_{k+1} and belongs to $\ker \tau'_{k+1}$ as noted above. Clearly $U_{k+1}\cdot V_j \subseteqq \mathrm{Ad}\, u(E_k)$. The restriction of $\mathrm{Ad}^*\, u(\sigma'_k)$ to $U_{k+1}\cdot V_j$ will be a sum of linear characters. From Lemma 1 we can see none of these characters can be trivial on $U_{k+1}\cdot V_j$ unless u is in U_{j-k-1}. Therefore $m(u)$ will vanish unless u belongs to U_{j-k-1}. From now on we assume this is true. Then U_{k+1} is in the kernel of $\mathrm{Ad}^*\, u(\sigma'_k)$, so $\mathrm{Ad}^*\, u\dot{c}h(\sigma'_k)$ is constant on cosets of $U_{k+1}\cdot V_j$. Therefore we may write

$$m(u) = \int_{D'^x} \mathrm{Ad}^*\, u(\check{c}h\sigma'_k)\overline{ch(\tau')}dx \tag{27}$$

where dx is normalized Haar measure on D'^x (modulo π, as always). Let us compute $\mathrm{Ad}^*\, u(\check{c}h\sigma'_k(x))$ for x in D'^x. Write $u^{-1} = 1 + z$. Then $\mathrm{Ad}\, u^{-1}(x)$ is given by (21). In order that $\mathrm{Ad}\, u^{-1}(x)$ be in E_k it is clearly necessary and sufficient that $\mathrm{ord}_D\, (x^{-1}[z, x]) \geqq k$. If this holds, then since z is in $\Pi^{j-k-1}S$, we have

$$\mathrm{Ad}\, u^{-1}(x) = x(1 + x^{-1}[z, x] - x^{-1}[z, x]z) \quad \text{modulo} \quad V_j\ . \tag{28}$$

We conclude that

$$\mathrm{Ad}^*\, u\sigma'_k(x) = \sigma'_k(\mathrm{Ad}\, u^{-1}(x)) = \sigma'(x)\psi(1 - x^{-1}[z, x]z) \tag{29}$$

for x in D'^x such that $\mathrm{Ad}\, u^{-1}(x)$ is in E_k.

Write

$$\nu(x, u) = \begin{cases} \psi(1 - x^{-1}[z, x]z) \\ 0 \quad \text{otherwise} \end{cases} \quad \text{if} \quad \mathrm{Ad}\, u^{-1}(x) \text{ is in } E_k\ . \tag{30}$$

Note that $\nu(x, u)$ depends on x, u, and ψ, but is independent of σ' in B. We may rewrite (27) as

$$m(u) = \int_{D'^x} ch(\sigma')\overline{ch\tau'}\nu(x, u)dx\ . \tag{31}$$

Plugging (31) into (24) and summing we conclude:

There is a function ν_k defined on D'^x such that

$$I(\check{\sigma}_k, \check{\tau}_{k+1}) = \int_{D'^x} ch(\sigma')\overline{ch}(\tau')\nu_k dx\ . \tag{32}$$

On the other hand, both $\check{\sigma}_k$ and $\check{\tau}_{k+1}$ are linear combinations of the $\check{\sigma}_i$ and we have seen that $ch(\check{\sigma}_i(x)) = 0$ unless x is conjugate in E_1 to an element of E_{j-1}. Therefore we may write

$$I(\check{\sigma}_k, \check{\tau}_{k+1}) = \int_{D'^x} ch(\check{\sigma}_k)\overline{ch}(\check{\tau}_{k+1})\mu dx \tag{33}$$

where $\mu(x)$ is a factor expressing the difference in volume between conjugacy classes in D'^x and in E_1. From (22) we know that

$$ch(\check{\sigma}_l(x)) = \omega_l(x)ch(\sigma'(x)) \tag{34}$$

for $l = j - 1$ and a suitable function ω_l. Assume by induction that (34) holds for $l = k + 1$. Assume also that the function ν_k in (32) vanishes nowhere. Then comparing (32) and (33), assuming inductively that the functions

$$\{ch(\check{\tau}_{k+1}): \tau' \in B\}$$

span the sames pace as the $\check{\tau}_i$, we see that (34) must also hold for $l = k$, with

$$\omega_k = \nu_k(\mu\bar{\omega}_{k+1})^{-1} . \tag{35}$$

Then (34) and (35), plus nonvanishing of ν_k shows that

$$\{ch(\check{\sigma}_k): \sigma' \in B\}$$

also has the same span as the $\check{\sigma}_i$. Thus the induction may continue. Evidently when we reach $l = 1$, Theorem 1 is proved. Thus our computation below of ν_k, from which we will see in particular that ν_k is never zero, will complete the proof of Theorem 1, when j is even.

REMARKS. (a) The above argument, which evidently could be axiomatized, may be regarded as a justification of the analogy drawn in [4] between the character of the Weil representation and the Weyl character formula. Formula (35) is also suggestive in this regard.

(b) The above arguments in fact go through when n is divisible by p so long as c_0 generates a tamely ramified extension of F.

Before continuing, let us briefly indicate what happens when j is odd. Retain the notations of (1) through (9). If j is odd, put $i = (j-1)/2$. Then $V_{j-1}\cdot U_i = H$ is a group, and $(\ker\psi)\cdot U_{i+1}$ is a normal subgroup and the quotient is a finite Heisenberg group. Thus there is an essentially unique representation ρ of H which is trivial on U_{i+1} and which is a multiple of ψ on V_{j-1}. The group H is normalized by D'^x. Note the equalities

$$D'^x\cdot H = E_i = D'^x\cdot V_i = C'\boldsymbol{x}_s(V_1'\cdot H)$$

where $C' = C \cap D'$.

Choose an extension $\tilde{\psi}$ of ψ from V'_{j-1} to D'^x. Assume for simplicity that $\tilde{\psi}$ is trivial on C'. We may immediately extend ρ to a representation of $V_1'\cdot H$ by defining

$$\rho(v) = \tilde{\psi}(v) \qquad \text{for } v \text{ in } V_1' .$$

The extended ρ is still essentially a representation of an Heisenberg group. Since E_i is the semidirect product of C' and $V_1'\cdot H$, we may extend ρ to E_i by defining

$$\rho(c) = \omega(c) \qquad \text{for } c \text{ in } C$$

where ω is the oscillator or Weil representation. See for example [4].

Now if σ' belongs to the set B of Theorem 1, we may extend σ' to a representation of E_i by letting σ' be trivial on H. Then we know $\sigma' \otimes \rho$ is an irreducible representation of E_i. We define

$$\sigma = \operatorname{ind}_{E_i}^{D^\times} \sigma' \otimes \rho .$$

The proof that this works is very much like the case of even conductor, with slight additional complications. The situation is not nearly so bad as in [3] where it was necessary to cope with the nonnormality of the maximal compact subgroup. The induction by means of the $\check{\sigma}_k$ goes precisely as before up to level $k = i + 1$. To pass from the $\check{\sigma}_{i+1}$ to the $\check{\sigma}_i$ is similar, and can be done with the use of the formulas [4].

3. We now focus still closer on the computation of $ch(\sigma)$, with the goal of proving Theorem 2 of [2]. In this paragraph, we concentrate on computing $ch(\check{\sigma})$ as a function on $D'^\times$. To do this essentially amounts by (35) and Theorem 1 to computing the ν_k, so that is our current goal.

According to the discussion preceding (32) we have

$$\nu_k(x) = \sum_{\nu} \nu(x, u) \tag{36}$$

where the sum is over the set $U(k, x)$ given by

$$U(k, x) = \{u^{-1} \in U^{j-k-1}/U^k,\ u^{-1} = 1 + z, \text{ and } \operatorname{ord}(x^{-1}[z, x]) \geqq k\} .$$

Also $\nu(x, u)$ is given by (30).

Up to conjugation in $D'^\times$, we may assume x has the form given by Lemma 2. Let the $c_l(x)$ for $l \geqq 1$ be as in that lemma. Recall D' was defined as the centralizer of c_0, with c_0 as in (4). Let $D = D^0$, and for $m \geqq 1$, let D^m be the centralizer of $c_l(x) = c_l$ for $1 \leqq l \leqq m$. Let

$$Y^m = D'^\perp \cap D^{m\perp} \cap D^{m-1} \qquad \text{for} \quad m \geqq 1 .$$

Let

$$Y^\infty = D'^\perp \cap (\bigcap_{m=1}^{\infty} D^m) .$$

Then, except that some of the summands might vanish, we have

$$D'^\perp = (\bigoplus_{m=1}^{\infty} Y^m) \oplus Y^\infty .$$

It is clear that the map

$$z \longrightarrow x^{-1}[z, x]$$

preserves each Y^m. For z in $D'^\perp$, write

$$z = \sum z_m$$

with z_m in Y^m. The argument giving (9) extends to imply $\operatorname{ord}_D(z_m) \geqq \operatorname{ord}_D(z)$ for all $m \geqq 1$. Further the argument used for (13) in Lemma 1 shows that

$$x^{-1}[z_m, x] = c_1^{-1}[z_m, c_m] + r \tag{37}$$

with $\operatorname{ord}_D(r) > \operatorname{ord}_D(c_1^{-1}[z_m, x_m])$.

Combining this with (4) and (30), we have for u in $U(k, x)$,

$$\nu(x, u) = \prod_{m=1}^{\infty} \bar{\chi}(\operatorname{tr}(c_0 c_1^{-1}[z_m, c_m]z_m)) . \tag{38}$$

Here $\bar{\chi}$ is the complex conjugate of χ. Moreover we see that $u = (1+z)^{-1}$ belongs to $U(k, x)$ if and only if each $u_m = (1 + z_m)^{-1}$ does so. Thus if we put

$$U(m, k, x) = \{u \in U(k, x),\ u = (1+z)^{-1},\ \text{and } z \in Y^m\} ,$$

we have the factorization

$$\nu_k(x) = \prod_{m=1}^{\infty} \nu_{km}(x) \tag{39}$$

where

$$\nu_{km}(x) = \sum_{u \in U(m,k,x)} \nu(x, u) = \sum_{z=u-1} \bar{\chi}(\operatorname{tr}(c_0 c_1^{-1}[z, c_m]z)) . \tag{40}$$

Consider the number $\bar{\chi}(\operatorname{tr}(c_0 c_1^{-1}[z, c_m]z)) = \nu(x, (1+z)^{-1})$, for some z in Y^m. For $\nu(x, (1+z)^{-1})$ to be defined, we need

$$\begin{aligned} &(\mathrm{a}) \quad \operatorname{ord}_D(z) \geqq j - k - 1 \quad \text{and} \\ &(\mathrm{b}) \quad \operatorname{ord}_D(c_1^{-1}[z, c_m]) = \operatorname{ord}_D c_m + \operatorname{ord}_D z - \operatorname{ord}_D b_1 \geqq k . \end{aligned} \tag{41}$$

If either inequality in (41) is strict, then $\nu(x, (1+z)^{-1}) = 1$. Therefore if $\nu(x, (1+z)^{-1}) \neq 1$, we have the relations

$$\begin{aligned} &(\mathrm{a}) \quad \operatorname{ord}_D z = j - k - 1 \quad \text{and} \\ &(\mathrm{b}) \quad \operatorname{ord}_D c_m = \operatorname{ord}_D c_1 + 2k - j + 1 . \end{aligned} \tag{42}$$

Since $\operatorname{ord}_D c_m$ increases strictly with m, we see there is at most one m for which there can exist z in Y^m such that $\nu(x, (1+z)^{-1})$ is not equal to 1. For m not satisfying (42) we will have

$$\nu_{km}(x) = {}^{\#}(U(m, k, x)) \tag{43}$$

where as before ${}^{\#}(\ )$ indicates cardinality.

Suppose now that c_m satisfies (42b). Then we see that the map

$$z \longrightarrow \pi \operatorname{tr}(c_0 c_1^{-1}[z, c_m]z) \tag{44}$$

may be regarded as a quadratic form from $(Y^m \cap \prod^{j-k-1} S)/\prod^{j-k} S$ to $\bar{F}$. Let γ be the Gauss sum attached to this quadratic form and to the character $\chi(x) = \chi^\circ(\pi^{-1}x)$ on $\bar{F}$. Then we clearly have

$$\nu_{km}(x) = \gamma^\#(U(m, k, x) \cap U_{j-k}) . \tag{45}$$

Thus to complete the computation of $\nu_k(x)$ it remains to determine the quadratic form given by (44).

It is at this point that we invoke the assumption that n is odd. It was superfluous up to this point and even here could doubtless be eliminated with some extra work (the work for $n = 2$, quaternion algebras, is rather small, and, indeed, that case is beside the present purpose); but at this point assuming n to be odd does simplify the computation.

Combining (42b) and (5) we get

$$2k = \mathrm{ord}_D\, c_m - \mathrm{ord}_D\, c_1 + \mathrm{ord}_D\, c_0 + n . \tag{46}$$

Let F_m be the field generated by c_0 and the c_l for $1 \leqq l \leqq m$: Since n is odd, the equation (46) implies there is y in F_m such that $\mathrm{ord}_D\,(y) = k$. More systematically, let $\tilde{\pi}$ be a prime of F_m. We can assume $\tilde{\pi}$ belongs to C. Write

$$c_l = \tilde{\pi}^{a_l} r_l \qquad \text{with} \quad r_l \in \bar{F}_m . \tag{47}$$

Also write

$$j - k - 1 = \mathrm{ord}_D\,(\tilde{\pi}) b \qquad \text{with} \quad b \in \boldsymbol{Z} .$$

Put $\prod^{j-k-1} S/\prod^{j-k} S = L$. We may parametrize L by the elements

$$\alpha(r) = \tilde{\pi}^b r \qquad \text{with} \quad r \in \bar{F}_u . \tag{48}$$

That is, since $\bar{F}_u \simeq S/\prod S$, the map $\alpha: r \longrightarrow \tilde{\pi}^b r$ is an isomorphism from $\bar{F}_u$ to L. Of course $\bar{F}_m$ is a subfield of $\bar{F}_u$.

For r in $\bar{F}_u$ and a suitable τ in $\mathrm{Gal}\,(\bar{F}_u/\bar{F})$ we have

$$\tilde{\pi}^{-1} r \tilde{\pi} = \tau(r) . \tag{49}$$

If ρ is defined by (1), then $\tau = \rho^d$ with $d = -\mathrm{ord}_D\,(\pi)$. However, this will not matter to us.

Denote the form defined by (44) by Q. With the notation of the preceding paragraph, we have

$$Q(\alpha(r)) = \mathrm{tr}(t\tau^b(\tau^{a_m} - 1)(r)r) \tag{50}$$

where $t = \pi\tilde{\pi}^{a_0} r_0 \tilde{\pi}^{-a_1} r_1^{-1} \tilde{\pi}^{a_m} r_m \tilde{\pi}^{2b}$, and $\mathrm{tr}: \bar{F}_u \longrightarrow \bar{F}$ is the reduction modulo $\prod S$ of $\mathrm{tr}: S \longrightarrow R$. Observe that t belongs to $\bar{F}_m^\times$. Also tr is the usual trace map from $\bar{F}_u$ to $\bar{F}$.

We see from (50) that we may consider $Q' = Q \circ \alpha$ as a quadratic form on $\bar{F}_u$. We are not interested in Q' on all of $\bar{F}_u$, but only on the inverse image $\alpha^{-1}(Y^m \cap \prod^{j-k-1} S)$. It is not hard to see that this is the subspace

$$\operatorname{im}(\tau^{a_0} - 1) \cap \left(\bigcap_{l=1}^{m-1} \ker(\tau^{a_l} - 1)\right) \cap \operatorname{im}(\tau^{a_m} - 1)\,. \tag{51}$$

Of course $\ker(\tau^a - 1)$ is just the fixed field of τ^a. Thus the situation is as follows. Let

$$\begin{aligned}
K_1 &= \text{fixed field of } \tau^{a_l} \text{ for } 0 \leqq l \leqq m\,.\\
K_2 &= \text{fixed field of } \tau^{a_l} \text{ for } 1 \leqq l \leqq m\,.\\
K_3 &= \text{fixed field of } \tau^{a_l} \text{ for } 0 \leqq l \leqq m-1\,.\\
K_4 &= \text{fixed field of } \tau^{a_l} \text{ for } 1 \leqq l \leqq m-1\,.
\end{aligned}$$

Then we have a diamond.

$$\begin{tikzcd}
 & K_4 \arrow[dl, dash] \arrow[dr, dash] & \\
K_2 \arrow[dr, dash] & & K_3 \arrow[dl, dash] \\
 & K_1 &
\end{tikzcd}$$

We have the trace maps

$$\operatorname{tr}_{ij}\colon K_i \longrightarrow K_j\,,$$

defined K_i lies above K_j in the diamond. The space in which we are interested is

$$J = \ker \operatorname{tr}_{42} \cap \ker \operatorname{tr}_{43}\,. \tag{52}$$

Note that $\bar{F}_m$ is contained in K_1. The main point about Q' and J is the following lemma.

LEMMA 3. *Let G be a finite abelian group of order prime to p acting on a vector space Y over a finite field $\bar{F}$ of characteristic p (p odd). Suppose there are no vectors y such that $g(y) = \pm y$ for all g in G. Let Q' be a nondegenerate quadratic form on Y invariant under G. Then Q' is unique, up to isomorphism.*

Proof. Consider first the case when G acts irreducibly. Then reasoning just as in [4] (Lemma, p. 296) we see there is an extension $\bar{F}'$ of $\bar{F}$ such that if $\bar{F}''$ is the quadratic extension of $\bar{F}'$, then there is an embedding

$$\alpha\colon G \longrightarrow \bar{F}''^{\times}$$

and an isomorphism

$$\beta: Y \longrightarrow \bar{F}''$$

such that $\beta(g(y)) \rightarrow \alpha(g)\beta(y)$ for g in G and y in Y. Further Q' will then have the form

$$Q'(u, v) = \operatorname{tr}(\bar{F}''/\bar{F})(c\tau(u)v)$$

where c belongs to $\bar{F}'$ and τ is the Galois automorphism of $\bar{F}''$ over $\bar{F}'$. If z is in $\bar{F}''x$, then the linear map $u \rightarrow zu$ commutes with the action of G and

$$Q'(zu, zv) = \operatorname{tr}(\bar{F}''/\bar{F})((c\tau(z)z)\tau(u)v) .$$

Since $z \rightarrow \tau(z)z$ is surjective from $\bar{F}''^{x}$ to $\bar{F}'^{x}$ for finite fields, we see that all possible Q' are isomorphic in this case.

In general write $Y = \sum Y_i$ as the decomposition of Y into isotypic components for G. If Y_j is the component of Y contragedient to Y_i, then Y_i must be orthogonal to every Y_k except Y_j under Q'. In particular if $Y_i \neq Y_j$, then $Y_i \oplus Y_j$ is a Q'-orthogonal direct summand of Y and Q' is split, i.e., a sum of hyperbolic planes, on $Y_i \oplus Y_j$.

Consider the possibility that Y_i is self-contragredient. Then V_i is the direct sum of irreducible G-modules, all mutually isomorphic and self-contragredient. If V_1 is an irreducible submodule, then either Q' is nondegenerate on V_1, or Q' is trivial. To finish, it will suffice to show that the orthogonal direct sum of two irreducible modules on which Q' is nondegenerate is equivalent to a split form. Let the sum be represented by pairs (u, v) with u and v in $\bar{F}''$. Then

$$Q'((u_1, v_1), (u_2, v_2)) = \operatorname{tr}(\bar{F}''/F)(c_1\tau(u_1)u_2 + c_2\tau(v_1)v_2) ,$$

with c_1 and c_2 in $\bar{F}'$. Choose α in $\bar{F}''$ such that $\alpha\tau(\alpha) = -c_1c_2^{-1}$. Then if $V_+ = \{(u, \alpha u); u \in \bar{F}''\}$ and $V_- = \{(u, -\partial u); u \in \bar{F}''\}$, we see our module is $V_+ \oplus V_-$ and that both V_+ and V_- are isotropic for Q'. This completes the lemma.

Lemma 3 obviously applies to $Q' = Q \circ \alpha$ as defined by (50). This Q' is clearly invariant by $\operatorname{Gal}(\bar{F}_u/\bar{F}_m)$, hence by $\operatorname{Gal}(K_4/K_1)$ which, being cyclic of odd order prime to p and having no fixed points in J, satisfies the hypotheses of the lemma. Thus Q' is what it has to be, according to how $\operatorname{Gal}(K_4/K_1)$ acts on it. Thus its specific form (50) is not important. Similarly γ in (45) is what it must be, as determined by the diamond of the K_i.

We note at this point formulas (39), (43), and (45) together imply that ν_k is nowhere zero and thus effectively completes the proof of Theorem 1. Furthermore, the above discussion shows that the functions $\nu_k(x)$, hence the function ω of Theorem (1b), depend on x only as an element of the field it generates, and does not depend on the Hasse invariant of D.

4. We now consider the passage from $\check{\sigma}$ to σ, with a view to extending the final statement of §3 in suitable fashion. Such an extension will be tantamount to proving Theorem 2 of [2]. Since $\sigma = \mathrm{ind}_{E_1}^{D^x} \check{\sigma}$, and $E_1 \supseteqq V_1$, we have by Frobenius' formula and (3) that

$$ch(\sigma(x)) = \sum_{z \in C/C'} \dot{c}h\check{\sigma}(\mathrm{Ad}\, z(x)) \qquad \text{for } x \text{ in } E_1 . \tag{53}$$

Here $C' = C \cap D'$. We know that $ch\check{\sigma}$ vanishes off E_1 conjugacy classes which do not intersect E_{j-1}, so we may as well take x actually belonging to E_{j-1}. Moreover, we may as well take x to be the canonical member of its V_1 conjugacy class, as described by Lemma 2. Then since z belongs to C, we know $\mathrm{Ad}\, z(x)$ is the canonical element in its V_1 conjugacy class. It is easy to see that such an element is conjugate in E_1 to an element of E_{j-1} only if it already belongs tc E_{j-1}. Hence the nonzero terms in the sum on the right of (53) come from those z such that $\mathrm{Ad}\, z(x)$ is again in E_{j-1}.

To begin, let us consider the case when x belongs to V_{j-1}. Since we are only interested in x modulo V_j, we may assume $x = 1 + c_2$, where $\mathrm{ord}_D(c_2) = j - 1$. Formulas (4) and (53) combine to give

$$ch\sigma(x) = \sum_{z \in C/C'} \chi(\mathrm{tr}\,(c_0 z c_2 z^{-1})) . \tag{54}$$

The sum in (54) will actually be invariant under $\mathrm{Ad}\, C$, since c_0 is invariant under $\mathrm{Ad}\, C'$. To make this explicit we could if we wish replace the sum over C/C', with $^{\#}(C'/(C \cap F))^{-1}$ times a sum over $C/(C \cap F^x)$. But let just keep it in mind. From (5), we see that

$$\mathrm{ord}_D\, c_2 = j - 1 = \mathrm{ord}_D\,(c_0\pi)^{-1} .$$

Thus we may write

$$z c_2 z^{-1} = (c_0\pi)^{-1} r(c_2, z) \qquad \text{with } r(c_2, z) \text{ in } \bar{F}_n^x .$$

If we do so, (54) then becomes

$$ch\sigma(x) = (C'/(C \cap F))^{-1} \sum_{z \in C/C \cap F} \chi(\pi^{-1}\, \mathrm{tr}\, r(c_2, z)) . \tag{55}$$

What we want to do is to show the set

$$\mathrm{tr}\, Z = \{\mathrm{tr}\, r(c_2, z) : z \in C\}$$

depends only on the minimal polynomials of c_0 and of c_2, and not on the Hasse invariant of D, i.e., not on the specific ρ occurring in (1).

Let b be the smallest positive integer such that n divides $b(j - 1)$, say $b(j-1) = nd$. Let $\bar{F}'_u = \bar{F}'_u \cap D'$ be the fixed field of $\operatorname{Ad} c_0$. We may write for r in $\bar{F}'_u$,

$$((c_0\pi)^{-1}r)^b = \pi^d s_0 N(r)$$

where $N: \bar{F}_u \to \bar{F}'_u$ is the usual norm map, and s_0 belongs to $\bar{F}'_u$. The minimal polynomial of c_0 determines s_0 up to the action of $\operatorname{Gal}(\bar{F}'_u/\bar{F})$. Similarly, the minimal polynomial of c_2 determines $s_0 N(r(c_2, 1)) = s_2$ up to the action of $\operatorname{Gal}(\bar{F}_u/\bar{F})$. It is well-known and easy to check that the $\operatorname{Ad} C$ orbit of c_2 is the set

$$\{u \in \bar{F}_u: N(u) = \gamma(s_2) \text{ for some } \gamma \text{ in } G\} .$$

Therefore, for a fixed choice of c_0, we have

$$Z = \{r(c_2, z): z \in C\} = \{\pi c_0 \tau(c_2) v: \tau \in \operatorname{Gal}(\bar{F}_u/\bar{F}) \text{ and } x \in \ker N\} .$$

If we replace c_0 by some conjugate of c_0, this has the effect of either (a) multiplying c_0 by some element in $\ker N$ or (b) replacing c_0 by $\mu(c_0)$ with μ in $\operatorname{Gal}(\bar{F}_u/\bar{F})$; or (c) both (a) and (b). The operation (a) does not change the set Z at all. The operation (b) replaces Z by $\mu(Z)$, as does operation (c). Taking traces, we find the set $\operatorname{tr} Z$ does in fact depend only on the irreducible polynomials of c_0 and c_2, as desired.

Now we pass to the general case, where x is in E_{j-1}, not necessarily in V_{j-1}. As stated above, we assume x is the canonical member of its V_1 conjugacy class, as specified by Lemma 2. Thus we may write

$$x = \sum_{i=1}^{\infty} c_i = c_1(1 + \sum_{i=2}^{\infty} \tilde{c}_i)$$

where c_i is in C, all the c_i commute, $\operatorname{ord}_D(c_{i+1}) > \operatorname{ord}_D(c_i)$, and $\tilde{c}_i = c_i c_1^{-1}$ for $i \geqq 2$. Since we are working modulo V_j, we may as well assume the sum is finite, say up to l, and that $\operatorname{ord}_D(\tilde{c}_l) = j-1$. (If $\operatorname{ord}_D(\tilde{c}_i)$ is never equal to $j-1$, we agree to put $\tilde{c}_l = 0$.) Then modulo V_j we may write

$$x = c_1(1+y)(1+\tilde{c}_l)$$

where

$$y = \sum_{i=2}^{l-1} \tilde{c}_i .$$

Put $\hat{x}_1 = c_1(1+y)$. It is clear that for $\operatorname{Ad} z(x)$ to belong to E_{j-1}, it is necessary and sufficient that $\operatorname{Ad} z(x_1)$ belong to $D'^{\times}$. Let $\hat{x}_1, \cdots, \hat{x}_m$ be representatives for the C' conjugacy classes of $\operatorname{Ad} C(\hat{x}_1) \cap D'$. If $\hat{x}_i = \operatorname{Ad} z_i(\hat{x}_1)$, put $1 + s_i = \operatorname{Ad} z_i(1+\tilde{c}_l)$. Denote by $\hat{C}_i$ intersection of C with the centralizer of $\hat{x}_i$, and write $\hat{C}'_i = \hat{C}_i \cap D'$. In these terms we may rewrite (53) in the form

$$ch\sigma(x) = \sum_i ch\check{\sigma}(\hat{x}_i)\Big(\sum_{z \in \hat{C}_i/\hat{C}'_i} \psi(\mathrm{Ad}^* z(1 + s_i))\Big) . \tag{56}$$

It remains to investigate the individual terms in (56). Evidently each $\hat{x}_i$, as an element of D', generates an extension of F'. The degree of this extension must divide the degree m of D' over F', which satisfies $m \dim(F'/F) = n = \deg(D/F)$. The nature of the extension $\hat{x}_i$ generates over F' is determined by the irreducible polynomial q_i over F' satisfied by $\hat{x}_i$. This irreducible polynomial q_i also determines the conjugacy class of $\hat{x}_i$ in D'. As is well-known, q_i is a factor of the irreducible polynomial of $\hat{x}_1$ over F. Thus we see, the $\hat{x}_i$ are in one-one correspondence with the F'-irreducible factors of degree dividing m of the irreducible polynomial of $\hat{x}_1$ over F. (Here we use the fact [5] that p-adic division algebras contain all fields of the appropriate degrees.)

There is another way of viewing the $\hat{x}_i$ that makes things more symmetric between $\hat{x}_1$ and c_0. Let us suppose that $\hat{x}_1$ and c_0 are embedded in some algebraic closure K of F, and let G be the Galois group of K over F. Let H_1 be the isotropy group of $\hat{x}_1$ in G and let H_2 be the isotropy group of c_0. The irreducible polynomial of $\hat{x}_1$ over F is

$$q(X) = \prod_{g \in G/H_1} (X - g(x_1)) .$$

Each F'-irreducible factor of q is a product of the $X - g(\hat{x}_1)$ over an H_2-orbit in G/H_1. In other words, the irreducible polynomials over F' that conjugates of $\hat{x}_1$ satisfy are in natural bijection with the (H_1, H_2) double cosets. From this point of view, the relation between x_1 and c_0 is obviously symmetric, and there is thus a natural bijection between the irreducible polynomials satisfied by conjugates of $\hat{x}_1$ over F' and the irreducible polynomials satisfied by conjugates of c_0 over the field generated by $\hat{x}_1$ over F. The degree of the extension over F generated by c_0 and $g(\hat{x}_1)$, or by $\hat{x}_1$ and $g^{-1}(c_0)$ is the index of $gH_1g^{-1} \cap H_2$ in G.

From this discussion we may conclude that for each $\hat{x}_i$ in (56), the element c_0 lies in a well-defined conjugacy class in the centralizer of $\hat{x}_i$. Furthermore, it is not hard to see that the conjugacy class of $\tilde{c}_l$ in the centralizer of $\hat{x}_1$ is determined by x. Hence for each s_i occurring in (56), the irreducible polynomial over the field generated by $\hat{x}_i$ is determined by x, or more precisely, the irreducible polynomial of x. Hence the considerations of the earlier part of this section show that the sums

$$\sum_{z \in \hat{C}_i/\hat{C}'_i} \psi(\mathrm{Ad}^* z(1 + s_i))$$

occurring in (56) are determined by the irreducible polynomials of x over F and $\hat{x}_i$ over F'. Since we know from §3, together with an obvious induction hypothesis that the quantities $ch\check{\sigma}(\hat{x}_i)$ depend only on the irreducible polynomials of the $\hat{x}_i$ over F', we find that (56) is entirely determined by the irreducible polynomial of x over F. This establishes Theorem 2 of [2].

5. We will finish by making two remarks. First, if the formula (35) is drawn out over several steps, it reads

$$\omega_k = (\nu_k\nu_{k+2}\nu_{k+4}\cdots\nu_{k+2l})(\nu_{k+1}\nu_{k+3}\cdots\nu_{k+2l+1})^{-1}\omega_{k+2l+2}\,.$$

Thus the ω's are alternating products of the ν's. This circumstance has an homological flavor.

Second, it should be pointed out how much the foregoing analysis simplifies if F' happens to be unramified over F. In that case, we have $E_j = E_{j-1}$, or in other words V'_{j-1} maps onto V_{j-1}/V_j. It follows that the Gauss sums of §3 defining the ν_k are simply volumes of certain groups, and are positive numbers in particular, and relatively simple to compute. No quadratic forms need be considered. Furthermore, (53) reduces to a sedate sum over the $\mathrm{Gal}(F'/F)$ conjugates of x in D'^{x}, and consideration of the character sums of (54) is completely unnecessary. Written out explicitly, one finds

$$ch\sigma(x) = (\sum_{g\in \mathrm{Gal}(F'/F)} ch(\sigma'\otimes\psi)(g(x))\omega(x))$$

where $\omega(x)$ is a factor which accounts for the difference of volume between conjugacy classes in D and D'. The analogy with the Weyl character formula, also suggested in [4], is clear. On the other hand, the formula (56) would seem to depart in certain ways from the Weyl formula. This may be attributed to the existence of non-Galois extensions.

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Received May 21, 1977.

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