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TAMELY RAMIFIED SUPERCUSPIDAL REPRESENTATIONS OF Gl_n

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Let F be a non-Archimedean local field of residual characteristic p; then conjecturally the supercuspidal representations of $Gl_n(F)$ are parameterized by admissible characters of extensions of F of degree n provided that n is prime to p. In this paper we establish the existence of the necessary representations if the conjecture is to be true. They will be realized as induced representations from certain subgroups, compact modulo the center. The more difficult question of whether all supercuspidal representations arise by this construction will not be treated. We will also leave aside the problem of computing the characters of these representations.

Let F be a locally compact non-Archimedean field of residual characteristic p. To simplify certain parts of the discussion, we take p to be odd. Let R be a maximal order, π a prime element. Let $F^\times$, $R^\times$ be the multiplicative groups of F and R, and $U = 1 + \pi R \subseteqq R^\times$. Let F' be an extension field of finite degree. We define R', π', $F'^\times$, $R'^\times$, U' in obvious analogy with F. Let $N(F'/F)\colon F'^\times \to F^\times$ be the norm map.

If ψ is a character of $F'^\times$, and $A \subseteqq F'^\times$ is a subgroup, we will say ψ is nondegenerate on A if there is no proper subextension F'', $F \subseteqq F'' \subsetneqq F'$, such that $\ker N(F'/F'') \cap A \subseteqq \ker \psi \cap A$.

Now suppose F' is tamely ramified over F. We will say a character ψ of $F'^\times$ is admissible if

(a) ψ is nondegenerate on $F'^\times$, and

(b) if on U', $\psi = \psi'' \circ N(F'/F'')$, where ψ'' is nondegenerate on $U'' \subseteqq F''^\times$, then F' is unramified over F''.

In particular, ψ is admissible if it is nondegenerate on U'.

Given extensions F'_1, F'_2 of F, and characters ψ_i of $F_i'^\times$, we say ψ_1 and ψ_2 are equivalent if there is an F-linear field isomorphism of F_1 onto F_2 which sends ψ_2 to ψ_1.

There are reasons for believing the following conjecture is true.

Conjecture: Suppose n is prime to p. Then the supercuspidal representations of $Gl_n(F)$ are parametrized by admissible characters of extensions of F of degree n. That is, given F' of degree n over F, and ψ an admissible character of $F'^\times$, then one may attach to ψ a supercuspidal representation $V(\psi)$ of $Gl_n(F)$. Two characters correspond to the same representation if and only if they are equivalent. Finally, all supercuspidal representations of $Gl_n(F)$ arise in this manner.

The evidence for this conjecture comes from the following sources:

(1) analogy with real groups,

(2) extension from Gl_2,

(3) expected connections with division algebras of degree n over F,

(4) Kirillov theory.

Here we will establish the existence of the correspondence indicated in the conjecture, and in particular, of the necessary representations. They will be realized as induced representations from certain subgroups, compact modulo the center. The more difficult question of whether all supercuspidal representations arise in this way, will not be treated. We will also leave aside the problem of computing the characters of these representations.

From now on F' will be a tamely ramified extension of F. In F, let B denote the multiplicative group of roots of unity, of order prime to p, and C the group generated by B and π. Similarly B' stands for the roots of unity of F' of order prime to p. Let e be the ramified degree of F' over F, and f the unramified degree, so $m = ef$ is the total degree. It is well known ([6]) that the prime π' of F' may be chosen so that $\pi'^e = \pi b'$, with $b' \in B'$. In this case, π' is determined by π up to an eth root of unity, and the multiplicative group C' generated by B' and π' is totally determined by π. Moreover, we note $C \subseteqq C'$ and $N(F'/F)(C') \subseteqq C$. Also, if $F \subseteqq F'' \subseteqq F'$, then $C'' \subseteqq C'$, and if F' is galois, C' is invariant under the galois action.

Let $|\ |_F$ be the natural ultrametric norm on F, so that if $\bar{F} = R/\pi R$ is the residue class field, with q elements, and ord_F is the valuation attached to R, then $|x|_F = q^{-\mathrm{ord}_F(x)}$. Define similarly $|\ |_{F'}$, $\bar{F}'$, q', $\mathrm{ord}_{F'}$. Note that, on F, $\mathrm{ord}_{F'} = e\,\mathrm{ord}_F$, and $q' = q^f$, so that $|x|_{F'} = |x|_F^m$ for $x \in F \subseteqq F'$.

We have $F = C \cdot U$, and $F' = C' \cdot U'$. Hence, given any $x \in F'$, there is a unique $c \in C'$, such that $c^{-1}x \in U$. Put another way, there is a unique $c \in C'$ such that $|c - x|_{F'} < |x|_{F'}$. We call c the standard representative of x, and write $c = \mathrm{s.r.}(x)$. From the above, and since the galois action fixes C', we see for any $g \in \mathrm{Gal}(F'/F)$, either $g(c) = c$, or $|g(c) - c|_{F'} = |c|_{F'}$.

Now consider $M_n(F)$. In $M_n(F)$, $M_n(R) = A$ is a maximal compact subring unique up to conjugacy, and $Gl_n(R) = K$ is a maximal compact subgroup, again the only one up to conjugacy. $M_n(R)$ is the set of matrices preserving the lattices $\pi^k R^n$, and these are the only lattices preserved by all of $M_n(R)$. Similarly K is the group of matrices g such that $g(\pi^k R^n) = \pi^k R^n$.

Now take F' of degree n. A choice of basis of R' over R

defines an injection $\alpha: F' \to Gl_n(F)$ by the regular action. Clearly $\alpha(R') \subseteqq M_n(R)$. We will identify F' and $\alpha(F')$. Under this identification, R' preserves precisely the lattices $\pi'^l R^n$. We associate to F' the order $A' = \bigcap_{x \in F'} xAx^{-1} = \bigcap_{l=0}^{e-1} \pi'^l A \pi'^{-l}$, which is characterized as the set of all matrices preserving the lattices $\pi'^l R^n$. We also associate to F' the group $K' = \bigcap_{x \in F'} xKx^{-1} = A' \cap Gl_n(F)$. Then clearly $R'^{\times} \subseteqq K'$, and F' normalizes K', so $F'^{\times} \cdot K'$ is an open subgroup of $Gl_n(F)$, compact modulo the center. A' may also be described as the intersection of all maximal orders of $M_n(F)$ containing R', and K' as the intersection of all maximal compact subgroups containing $R'^{\times}$.

This first lemma guarantees that this and succeeding constructions have the necessary invariance properties.

LEMMA 1. *Suppose $F'' = gF'g^{-1}$ is a subfield of $M_n(F)$ conjugate to F'. Then if $R'' = gR'g^{-1} \subseteqq A$, $g\pi'^l \in K$ for some l. If $R'' \subseteqq A'$, then $g\pi'^l \in K'$.*

Proof. The invariant lattices of R' are, as we have said, the lattices $\pi'^l R^n$. Also, g takes R'-invariant lattices to R''-invariant lattices. If $R'' \subseteqq A$, $R^n = g(\pi'^l R^n)$ for some l, so $g\pi'^l \in K$. If $R'' \subseteqq A'$, then $\pi'^m R^n = g(\pi'^{l(m)} R^n)$. Since $\pi'^{m+1} R^n$ is characterized as the largest proper A'-invariant sublattice of $\pi'^m R^n$, we see $l(m+1) = l(m) + 1$. Hence $g\pi'^l = g\pi'^{l(0)}$ is in K'.

Now we choose particular coordinates to get a very explicit description of A'. Let $F'_u \subseteqq F'$ be the maximal unramified subextension, and let $\{b_i\}_{i=1}^{f}$ be a basis of R'_u over R. We may assume $b_i \in B'$ if we wish. Now define a basis $\{z_k\}_{k=1}^{n}$ of R' over R by $z_{fj+i} = \pi'^j b_i$ for $0 \leqq j < e$. With respect to this basis, we see that if $m = ke + l$, then $\pi'^m R^n = \{\Sigma a_i z_i = (a_1, \cdots, a_n)\colon \mathrm{ord}_F(a_i) \geqq k$, and $\mathrm{ord}_F(a_i) \geqq k+1$ if $i \leqq fl\}$. Thus a basis for $\pi'^m R^n$ is $\{\pi^{k+1} z_i\}_{i=1}^{fl} \cup \{\pi^k z_i\}_{i=fl+1}^{n}$. Hence, we see that, in this basis $A' = \{T = (t_{ij})\colon \mathrm{ord}_F(t_{ij}) \geqq 0$, and $\mathrm{ord}_F(t_{ij}) \geqq 1$ if $[(i-1)/f] > [(j-1)/f]\}$. (Here [ ] denotes greatest integer.)

Now let $\mathrm{tr}(M_n(F)/F)$ denote the usual trace on $M_n(F)$. Then $\langle S, T\rangle = \mathrm{tr}(M_n(F)/F)(ST)$ is a nondegenerate symmetric bilinear form on $M_n(F)$. If $V \subseteqq M_n(F)$ is a subspace, $V^{\perp}$ will denote its orthogonal complement with respect to $\langle\ ,\ \rangle$. If $L \subseteqq M_n(F)$ is a lattice, (e.g., a compact open R-module) then $L^* = \{l \in M_n(F), \langle l, L\rangle \subseteqq R\}$ is also a lattice. L^* is naturally isomorphic with $\mathrm{Hom}_R(L, R)$. It is very easy to see that $M_n(R)^* = M_n(R)$. Moreover, the description of A', and the action of π' given above make it a simple calculation to verify this lemma.

LEMMA 2. $A'^* = \pi'^{1-e} A'$.

Now let $F'' \subseteq F'$ be any subextension of F.

By virtue of the action of F'' on F^n, we may identify F^n and F''^l, where $n = lk$ and k is the degree of F'' over F. In this identification, R^n becomes R''^l and the commuting algebra of F'' is just $M_l(F'')$. We note that $A_1 = M_l(R'') \subseteq M_n(R)$.

Of course we have $F' \subseteq M_l(F'')$, and from the definition of A', it is clear that $A_1' = A' \cap M_l(F'') = \bigcap_{x \in F'} xA_1x^{-1}$.

In this next lemma, $\oplus$ denotes direct sum.

LEMMA 3. *$A' = A_1' \oplus (M_l(F'')^\perp \cap A')$. In particular, $A' = R' \oplus (F'^\perp \cap A')$.*

REMARK. Whereas Lemmas 1 and 2 hold also for wildly ramified fields, Lemma 3 does not, and the resultant bad geometry makes analysis more difficult for that case.

Proof. This is a relation between various trace maps. $\alpha: A'^* \to \operatorname{Hom}_R(A_1', R)$, defined by $\alpha(x)(b) = \langle x, b\rangle$ has as kernel $M_l(F'')^\perp \cap A'^*$. Hence, if we can show $\alpha(M_l(F'') \cap A'^*) = \operatorname{Hom}_R(A_1', R)$, then $A'^* = (M_l(F'') \cap A'^*) \oplus (M_l(F'')^\perp \cap A'^*)$, and dualizing gives the decomposition of A'. By Lemma 2, $A'^* = \pi'^{1-e}A'$. Since π' preserves $M_l(F'')$ and $M_l(F'')^\perp$, $A'^* \cap M_l(F'') = \pi'^{1-e}A_1'$.

Let e' be the degree of ramification of F' over F'' and let e'' be the degree of ramification of F'' over F. Then $e = e'e''$. Reasoning with F'' instead of F, we see that $\pi'^{1-e'}A_1' \cong \operatorname{Hom}_{R''}(A_1', R'')$. Now on $M_l(F'')$, we have $\operatorname{tr}(M_n(F)/F) = \operatorname{tr}(F''/F) \circ \operatorname{tr}(M_l(F'')/F'')$. Since F'' is tamely ramified over F, the different of F'' over F is $e''-1$, so that $\pi''^{1-e''}R'' \cong \operatorname{Hom}_R(R'', R)$. It follows that $\pi''^{1-e''}(\pi'^{1-e'}A_1') \cong \operatorname{Hom}_R(A_1', R)$. But now $\pi''^{1-e''}(\pi'^{1-e'}A_1') = \pi'^{e'(1-e'')+1-e'}A_1' = \pi'^{1-e}A_1'$, and the lemma is proved.

We now establish some facts about the geometry of the adjoint action of Gl_n on M_n. This study is suggested by Kirillov theory and has important implications for the representation theory of Gl_n.

For $T, S \in M_n(F)$, write $\operatorname{ad}_T(S) = [T, S] = TS - ST$. If T and W commute, then $\langle W, [T, S]\rangle = \operatorname{tr}(WTS) - \operatorname{tr}(WST) = \langle [W, T], S\rangle = 0$, so ad_T has image in the orthogonal complement of $\ker \operatorname{ad}_T$, the commuting algebra of T. By dimension counting $\operatorname{im} \operatorname{ad}_T = (\ker \operatorname{ad}_T)^\perp$. If $\langle\ ,\ \rangle$ is nonsingular on $\ker \operatorname{ad}_T$, then ad_T will be nonsingular on $\operatorname{im} \operatorname{ad}_T$.

Now suppose $T = c \in C' \subseteq F'$, and write $c = \pi'^m b$, $b \in B'$. Let F'' be the subfield of F' generated over F by c. Clearly $\operatorname{ad}_c(A') \subseteq \pi'^m A'$. Since also $\operatorname{ad}_c(M_l(F'')^\perp) \subseteq M_l(F'')^\perp$, we conclude that if $X = A' \cap M_l(F'')^\perp$, then $\operatorname{ad}_c(X) \subseteq \pi'^m X$.

LEMMA 4. *In fact* $\mathrm{ad}_c(X) = \pi'^m X$.

Proof. We see $\mathrm{ad}_c(y) = cy - yc = (cyc^{-1} - y)c$. Since $cX = \pi'^m X$, the lemma is equivalent to the statement that the map: $\beta: y \to cyc^{-1} - y$ maps X onto itself. But since $\beta(X) \subseteq X$, it suffices to show that β has as determinant an element in $R^\times$. But now the eigenvalues of β are all of the form $c^{-1}\tilde{c} - 1$, where $\tilde{c}$ is a conjugate to c by the galois group of F and not equal to c. From the properties of C' noted above, it follows that $c^{-1}\tilde{c} - 1$ has norm in $R^\times$, so also the determinant of β is in $R^\times$.

Lemma 4 has as a consequence one of the basic facts we will need. Before stating it, we need one more observation. We filter the group K' by a sequence of subgroups $K' = K'_0 \supseteqq K'_1 \supseteqq K'_2 \cdots$, where $K'_i = 1 + \pi'^i A'$ for $i \geqq 1$. For $i \geqq 1$, K'_i is a pro-p group.

F'' is still the subfield of F' generated by $c \in C'$.

LEMMA 5. *If* $k \in K'_i$, *then* $k = (1 + a)(1 + b)$, *where* $1 + a \in K'_i \cap M_l(F'')$ *and* $b \in M_l(F'')^\perp$. (Then also $1 + b \in K'_i$, and $a \in \pi'^i A'_1$.)

Proof. By definition of K'_i, $k = 1 + z$, with $z \in \pi'^i A'$. But by Lemma 3, $z = y + x$, with $y \in \pi'^i A'_1$ and $x \in \pi'^i X$ (X as in Lemma 4). But now put $a = y$, $b = (1 + y)^{-1}x$. Then since multiplication by elements from $M_l(F'')$ preserves $M_l(F'')^\perp$, this is the desired decomposition.

Let Ad denote the standard adjoint action of Gl_n on M_n. That is, $\mathrm{Ad}(S)(T) = STS^{-1}$. Then we have the following result on the geometry of this action.

LEMMA 6. *Take* $c = \pi'^m b \in C'$, *and* $S \in M_n(F)$. *Suppose* $S \in c + \pi'^j A'$, *with* $j > m$. *Then there is* $k \in K'_{j-m}$, *and* $T \in M_l(F'')$ *such that* $S = \mathrm{Ad}(k)(T)$. *In other words,* $c + \pi'^j A' = \mathrm{Ad}\, K'_{j-m}(c + \pi'^j A'_1)$.

Proof. We may write $S = c + y + x$, with $y \in \pi'^j A'_1$, $x \in \pi'^j X$. By Lemma 4, $x = \mathrm{ad}_c(z) = [c, z]$ with $z \in \pi'^{j-m} X$. Then $\mathrm{Ad}(1 + z)(S) = S + [z, S] + [z, S]z(1 + z)^{-1} = c + y + x + [z, c] + [z, y] + [z, x] + [z, S]z(1+z)^{-1} = c + y + [z, y] + [z, x] + [z, S]z(1 + z)^{-1} = c + y + \tilde{x}$ where $\tilde{x} \in \pi'^{2j-m} A'$. Thus $\mathrm{Ad}(1 + z)(S)$ is closer to $c + \pi'^j A'_1$ than S is. Continuing in this fashion, by a Hensel's lemma argument, the result follows.

We now begin to discuss representation theory. We will start by constructing certain representations of K', or more precisely of $F'^\times \cdot K'$, the normalizer of K'.

We notice that the commutator subgroup of K'_i and K'_j is contained in K'_{i+j}. In particular, K'_{i-1}/K'_i is in the center of K'_1/K_i, and if $2i > j$, K'_i/K'_j is abelian. In that case also, the mapping

$\nu\colon \pi'^i A'/\pi'^j A' \to K_i'/K_j'$ defined by $\nu(a) = 1 + a$ is an isomorphism of groups, and commutes with the action of K' by Ad on the two quotient groups.

Now suppose χ is a character of the additive group of F. Then as is well known χ defines an isomorphism $\theta\colon M_n(F) \to \widehat{M_n(F)}$, where $\frown$ denotes Pontryagin dual, by the formula $\theta(S)(T) = \chi(\langle S, T\rangle)$. The natural action of Gl_n on $\hat{M}_n$ is denoted Ad*, and is given explicitly by $\mathrm{Ad}^* S(\psi)(T) = \psi(\mathrm{Ad}\, S^{-1}(T))$. We see that θ is equivariant with respect to the actions Ad and Ad*. That is, $\theta(\mathrm{Ad}\, S(T)) = \mathrm{Ad}^* S(\theta(T))$. This property will be retained, insofar as it makes sense, by the various maps obtained below from θ.

We will suppose for simplicity that the largest lattice in F on which χ is trivial (the conductor of χ) is R itself. Then for any lattice $L \subseteq M_n(F)$, $L^\perp$, the annihilator of L in $\hat{M}_n$, is identified to L^* via θ; and for a subspace V, the orthogonal complement $V^\perp$ is identified to the annihilator, also to be written $V^\perp$. Thus, if $L_1 \subseteq L_2$ are two lattices, $\widehat{L_2/L_1} \cong L_1^*/L_2^*$. In particular, if we write $A'(j) = \pi'^j A'$, and $\lambda(j) = -j - e + 1$, then we have, if $i < j$, $\widehat{A'(i)/A'(j)} \cong A'(\lambda(j))/A'(\lambda(i))$. As mentioned above, these identifications commute with the obvious (sub-quotient) actions, Ad and Ad*, of K'.

Combining θ with ν, we get a map $\mu\colon \widehat{K_i'/K_j'} \to A'(\lambda(j))/A'(\lambda(i))$ when $2i \geqq j$. If $y + A'(\lambda(i)) = \mu(\psi)$ for some character ψ of K_i'/K_j', we will say y represents ψ. Again μ commutes with the obvious actions of K'.

Now take $j > 1$, and let ψ be a nontrivial character of K_{j-1}'/K_j'. Suppose ψ has a representative $y \in F'$. Then we see there is a unique $c \in C'$ which represents ψ. We will call c the standard representative of ψ. In this situation, we let F'' be the field generated by c over F, and retain the relevant previous notation. In particular $M_l(F'')$ is the commuting algebra of F''. For $i \geqq 0$, put $H_i = K_i' \cap M_l(F'')$. Then for $i \geqq 1$, $H_i = 1 + \pi'^i A_1'$.

ψ is invariant under $\mathrm{Ad}^* K_1'$. Suppose $2i \geqq j$ and φ is a character on K_i'/K_j' which agrees with ψ on K_{j-1}'. We will say φ lies over ψ.

LEMMA 7. *Notations as above.*

(a) *φ is conjugate by* $\mathrm{Ad}^* K_1'$ *to φ', which has a representative* $T \in M_l(F'')$.

(b) *If φ has a representative $T \in M_l(F'')$, then the isotropy group of φ under* $\mathrm{Ad}^* K'$ *is contained in* $H_0 \cdot K_{j-i}'$.

Proof. (a) Since c is the standard representative for ψ, and ψ is nontrivial on K_{j-1}', we see $c \in A'(\lambda(j)) - A'(\lambda(j-1))$, so $c = \pi'^{\lambda(j)} b'$, with $b' \in B'$. If S is a representative for φ, then since φ lies over ψ, $S \in c + A'(\lambda(j-1))$. Now it follows from Lemma 6 that $S = \operatorname{Ad} k(T)$, with $k \in K'_1$, and $T \in M_l(F'')$. Then also, by the equivariance of μ, $\operatorname{Ad}^* k^{-1}(\varphi) = \varphi'$ is represented by T.

(b) If $T \in M_l(F'')$ represents φ, then we have, as above, $T \in c + A'(\lambda(j-1))$. If $k \in K'_1$, and $\operatorname{Ad}^* k(\varphi) = \varphi$, then we have $\operatorname{Ad} k(T) \in T + A'(\lambda(i))$. Write $k = 1 + z$. Then $\operatorname{Ad} k(T) = (1+z)T(1+z)^{-1} = T + [z, T](1+z)^{-1}$. Hence $\operatorname{Ad} k(T) \in T + A'(\lambda(i))$ if and only if $[z, T] \in A'(\lambda(i))$. But now write $z = y + x$ with $y \in M_l(F'')$, $x \in M_l(F'')^{\perp}$. Then clearly $k \in H_0 \cdot K'_{j-i}$ if and only if $x \in \pi'^{j-i}X$, where X is defined as in Lemma 4. Write $X(\alpha) = \pi'^{\alpha}X$, analogously to $A'(\alpha)$. Now calculate $[z, T] = [y + x, c + T - c] = [x, c] + [x, T - c] + [y, T]$. Lemma 4 shows that if $x \in X(\alpha) - X(\alpha+1)$, then $[x, c] \in X(\alpha + \lambda(j)) - X(\alpha + \lambda(j) + 1)$, whereas it is immediate that $[x, T - c] \in X(\alpha + \lambda(j) + 1)$. Since $[y, T] \in M_l(F'')$, it is now clear from Lemma 3 that $[z, T] \in A'(\lambda(i))$ if and only if $x \in X(j-i)$. Following back through the argument, (b) is proved for $k \in K'_1$.

Now take any $k \in K'$. Then if $\operatorname{Ad}^* k$ fixes φ, it must also fix ψ on K'_{j-1}. But this means $\operatorname{Ad} k(c) \in c + A'(\lambda(j-1))$. Then by Lemma 6, $\operatorname{Ad} k(c) = \operatorname{Ad} k_1(T)$ for some $k \in K'_1$, and $T \in (c + A'(\lambda(j-1))) \cap M_l(F'')$. From the next lemma it follows that $kk_1^{-1} \in M_l(F'')$, or $k \in H_0 \cdot K'_1$. By reduction to the previous case, then, the whole of (b) is proved.

LEMMA 8. *Suppose T_1 and T_2 belong to $(c + A'(\lambda(j-1))) \cap M(F'')$, and suppose for some $g \in Gl_n(F)$, $\operatorname{Ad} g(T_1) = T_2$. Then $g \in M_l(F'')$, that is, $\operatorname{Ad} g(c) = c$.*

Proof. By assumption, $S_i = c^{-1}T_i c \in K'_1$. Hence $S_i^{p^m} \to 1$ as $m \to \infty$. Since c and T_i commute $S_i^a = c^{-a}T_i^a$. Since C' modulo the subgroup generated by π is p-regular, there exists a sequence m_α soing to infinity, such that $c^{-p^{m_\alpha}} = c^{-1}\pi^{v_\alpha}$. Then $(Y_i)_\alpha = \pi^{v_\alpha}T_i^{p^{m_\alpha}} \to c$ as $\alpha \to \infty$. Since $\operatorname{Ad} g((Y_1)_\alpha) = (Y_2)_\alpha$, we get in the limit $\operatorname{Ad} g(c) = c$.

REMARK. Lemma 8 provides an explicit proof of a fact that was implicit earlier and is worth noting: namely, if $x \in F'$, then the field generated over F by x contains s.r. (x), so that any subfield of F' is generated by its intersection with C'.

Take $\varphi \in \widehat{K'_i/K'_j}$, and suppose φ lies over ψ and is represented by $T \in M_l(F'')$. We want to make explicit the relation between φ and its restriction to H_i/H_j.

We still have $2i \geqq j$. Let $E(i, j)$ be the set of elements of the form $1 + y + x$, where $y \in A'(j)$ and $x \in X(i)$. Then in fact $E(i, j)$

is a group. Obviously, $K'_j \subseteq E(i, j)$. Moreover, Lemma 3 shows $K'_i = H_i \cdot E(i, j)$. In fact $E(i, j)$ is normal in K'_i, and $K'_i/K'_j = (H_i/H_j) \cdot (E(i, j)/K'_j)$ (direct product); furthermore, $E(i, j)$ is normalized by H_0, and $H_0 \cdot K'_i/K'_j = (H_0/H_j) \cdot (E(i, j)/K'_j)$ (semidirect product). As a corollary to this, and for future reference, we remark that any representation of H_0/H_j may be extended to $H_0 \cdot K'_i$ by letting it be trivial on $E(i, j)$.

In fact, φ arises in just this manner. For, since φ is represented by $T \in M_l(F''')$, it is trivial on $E(i, j)$, and therefore comes, by extension, from a character φ'' of H_i/H_j. φ'' may be described as follows.

On $M_l(F''')$ we have the F'''-bilinear form $\langle\, , \rangle''$, given by $\langle S, T\rangle'' = \operatorname{tr}(M_l(F''')/F''')(ST)$. If $\chi'' \in \hat{F}'''$, then $\theta'' \colon M_l(F''') \to \widehat{M_l(F''')}$ may be defined by $\theta''(S)(T) = \chi''(\langle S, T\rangle'')$. On the other hand, if $\tau \colon \widehat{M_n(F)} \to \widehat{M_l(F''')}$ is the natural projection, $\tau \circ \theta$ is also an isomorphism between $M_l(F''')$ and its dual. Since on $M_l(F''')$, $\langle S, T\rangle = \operatorname{tr}(F''')/F)(\langle S, T\rangle'')$, we see that if $\chi'' = \chi \circ \operatorname{tr}(F'''/F)$, then $\theta'' = \tau \circ \theta$. Therefore, with this choice of χ'', we may identify $\widehat{H_i/H_j}$ with $A'_1(\lambda(j))/A'_1(\lambda(i))$, where $\lambda(j) = -j - e + 1$, and we have written $A'_1(\alpha) = \pi'^{\alpha} A'_1$, in analogy with $A'(\alpha)$ and $X(\alpha)$. (Note that the annihilator of A'_1 is identified via θ'' with $A'_1(-e+1)$, and not with $A'_1(-e'+1)$, where e' is the degree of ramification of F' over F''', because the conductor of χ'' is not R'' but $\pi''^{1-e''}R''$, e'' being the ramified degree of F'' over F.) Finally, we see that in the above identification T becomes a representative for φ'' on H_i/H_j.

Now c itself represents some $\varphi_0 \in \widehat{K'_i/K'_j}$, and φ_0 clearly lies above ψ. Also, the isotropy group under $\mathrm{Ad}^* K'$ of φ_0 is clearly $H_0 \cdot K'_{j-i}$; and if φ''_0 is the restriction of φ_0 to H_i/H_j, then φ''_0 is again represented by c and is $\mathrm{Ad}^* H_0$-invariant.

LEMMA 9. *If $2i \geqq j + e' - 1$, then φ''_0 is the restriction to H_i of a linear character of $Gl_l(F''')$. Moreover, $j \geqq e' + 1$, so this always holds for $i = j - 1$.*

Proof. It suffices to show that $Sl_l(F''') \cap H_i \subseteq \ker \varphi''_0$. φ''_0 is given on H_i by $\varphi_0(1 + T) = \chi''(c \operatorname{tr}(M_l(F''')/F''')(T))$. Here $T \in A'_1(i)$ and $\operatorname{ord}_{F'} c = \lambda(j) = -j - e + 1$, and $c \in F''$, and the conductor of χ'' is $\pi''^{1-e''}R'' = \pi'^{e'-e}R''$. Thus, writing $\operatorname{tr}(M_l(F''')/F''') = \operatorname{tr}$ for this proof, we will have $1 + T \in \ker \varphi''_0$ if $\operatorname{ord}_{F'}(c) + \operatorname{ord}_{F'}(\operatorname{tr} T) \geqq e' - e$. Thus, we must show that, if $\det(1 + T) = 1$, then $\operatorname{ord}_{F'}(\operatorname{tr} T) \geqq e' + j - 1$. But since $c \in F''$, $\operatorname{ord}_{F'}(c)$ is divisible by e', and therefore so is $j - 1$; and since $j > 1$, certainly $j \geqq e' + 1$. Also, we are reduced to showing $\operatorname{ord}_{F''}(\operatorname{tr} T) \geqq (j - 1/e') + 1$.

Now $T \in A_1'(i) = \pi'^i A_1'$, so $T^{e'} \in A_1'(ie') \subseteqq \pi''^i M_l(R'')$. Let $\tilde{F}$ be an extension field of F'' containing the characteristic roots $\rho_1, \cdots, \rho_l$ of T. Let $\mathrm{ord}_{F''}$ denote the extension of $\mathrm{ord}_{F''}$ to $\tilde{F}$. Then $\pi''^{-i} T^{e'} \in M_l(R'')$ implies $\mathrm{ord}_{F''}(\rho_\alpha) \geqq i/e'$.

The condition $\det(1 + T) = 1$ means $\sum_{\beta=1}^{l} \sigma_\beta(\rho_1, \cdots, \rho_l) = 0$, where σ_β is the βth basic symmetric polynomial in the ρ_α's. In particular $\sigma_1(\rho_1, \cdots, \rho_l) = \sum_{\alpha=1}^{l} \rho_\alpha = \mathrm{tr}\, T$. Thus, the above relation implies $\mathrm{tr}\, T = -\sum_{\beta=2}^{l} \sigma_\beta(\rho_1, \cdots, \rho_l)$. Hence, $\mathrm{ord}_{F''}(\mathrm{tr}\, T) \geqq 2i/e'$. Thus, we require $2i/e' \geqq (j - 1/e') + 1$ or $2i \geqq j - 1 + e'$, as was to be proved.

We come now to a key result for this construction. The result actually holds in a wider context than that of $Gl_n(F)$. It is at least true for all central simple algebras over F, and probably has an analogue in any semisimple group where no wild ramification occurs. For division algebras, it yields an inductive method for the complete determination of the representations (when the degree is prime to p).

ψ'' is the character of H_{j-1} gotten by restricting ψ from K'_{j-1}. A representation of H_0 will be said to lie above ψ'' if its restriction to H_{j-1} is a multiple of ψ''. Similarly, if $O(\psi)$ is the $\mathrm{Ad}^*\, K'$ orbit of ψ in $\widehat{K'_{j-1}}$, a representation of K' will be said to lie over $O(\psi)$ if its restriction to K'_{j-1} contains precisely the characters in $O(\psi)$.

THEOREM 1. *There exists a one-to-one correspondence between the representations of H_0 lying over ψ and the representations of K' lying over $O(\psi)$.*

REMARK. The correspondence which is described below is very simple and functional, and would seem to deserve to be called canonical, though in what sense is at present unclear. One sense involves the characters of corresponding representations. This will be gone into elsewhere.

Proof. We divide the theorem into two cases, j even and j odd. The case of even j is very simple. Let W be a representation of K' lying above $O(\psi)$, and let W'' be the corresponding representation of H_0. We describe how to get W from W''. Since j is even, $i = j/2$ is an integer. Take W'' and extend it to W'' on $H_0 \cdot K_i'$, by letting it be trivial on $E(i, j)$, as described above. The induced representation of K' is then W.

We must show that each W lying over $O(\psi)$ arises uniquely in this fashion. This is easily done, using standard representation theory for finite groups. We briefly recall this.

Let G be a finite group, N a normal subgroup. Let $\hat{N}$ be the set of representations of N; $\hat{G}$, those of G. Conjugation by G induces an action of G/N on $\hat{N}$, denoted $\mathrm{Ad}^* G/N$ or $\mathrm{Ad}^* G$. A representation W of G restricted to N is a direct sum of a certain number of copies of the representations in some $\mathrm{Ad}^* G/N$ arbit O. W is said to lie above O. To find all representations lying above O, proceed as follows. Fix $Y \in O$, and let G_1 be the isotropy group of Y under $\mathrm{Ad}^* G/N$. If $Z_1, \cdots, Z_m$ are all the representations of G_1 lying above ψ then $Z_1, \cdots, Z_m$ induce distinct irreducible representations of G, and all representations of G lying above O are obtained in this way.

Applying this to our situation, we have seen in Lemma 7 that every $\mathrm{Ad}^* K'$ orbit in K'_i/K'_j (where now $2i = j$) which lies above $O(\psi)$ contains an element φ which lies over ψ and whose isotropy group I_φ is contained in $H_0 \cdot K'_i$. Furthermore, if Z is any representation of I_φ lying over φ, then Z is trivial on $E(i, j)$, and so then will W'', the representation of $H_0 \cdot E(i, j)$ induced from I_φ, be trivial on $E(i, j)$. Evidently, then, inducing further on up to K' yields an irreducible representation W of K'. It is evident by this that all representations W of K' lying above $O(\psi)$ arise in this manner, and furthermore, that distinct W'''s lying above the same $\mathrm{Ad}^* H_0$ orbit in $\widehat{H_i/H_j}$ yield distinct W's. Finally, Lemma 8 guarantees that a subset of $\widehat{K'_i/K'_j}$ which lies over ψ, has representatives in $M_l(F'')$, and belongs to a single $\mathrm{Ad}^* K'$ orbit actually belongs to a single $\mathrm{Ad}^* H_0$ orbit. Hence any two distinct W'''s yield distinct W's, and the theorem is established for j even.

When j is odd the procedure is more complicated. Let $j = 2i + 1$. Our first goal will be to construct a certain representation on $H_0 \cdot K'_i$. When this is done, we may proceed just as for even j.

Let $\tilde{\varphi}''$ be a linear character of $Gl_l(F'')$ lying above ψ'' on H_{j-1}/H_j. By restricting to H_0, then extending to $H_1 \cdot K'_{i+1}$, we get a character $\tilde{\varphi}$ on this group. Of course, by definition $E(i+1, j) \subseteq \ker \tilde{\varphi}$. We see that $H_1/\ker \tilde{\varphi}''$ is central in $(H_1 \cdot K'_i)/\ker \tilde{\varphi} = \mathscr{H}$, and that $(H_1 \cdot K'_i)/(H_1 \cdot K'_{i+1}) \cong \mathscr{H}/\mathscr{Z}$ ($\mathscr{Z}$ = center of $\mathscr{H}$) is isomorphic to $(\boldsymbol{Z}/p\boldsymbol{Z})^{2\alpha}$ for some α. We also observe that $\mathrm{Ad}\, H_0$ factors to an action by automorphisms on $\mathscr{H}$, again denoted by Ad. H_1 of course acts trivially. Since $\mathrm{Ad}\, Gl_l(F'')$ preserves $M_l(F'')^\perp$, we see that this action has the following property: for $x \in H_0$, $y \in \mathscr{H}$, $\mathrm{Ad}\, x(y) = y$ if and only if $\mathrm{Ad}\, x(y) = y \bmod \mathscr{Z}$.

Since the commutator group of K'_i is contained in K'_{j-1}, the function $\alpha(x, y) = \psi(xyx^{-1}y^{-1})$ is well defined on $K'_i \times K'_i$.

LEMMA 10. *$\alpha(x, y)$ factors to a nondegenerate antisymmetric biadditive form $\bar{\alpha}$: $(\mathscr{H}/\mathscr{Z}) \times (\mathscr{H}/\mathscr{Z}) \to \boldsymbol{T}$, $\boldsymbol{T}$ being the unit circle.*

Proof. If $x = 1 + a$, $y = 1 + b$, then, modulo K'_j we have

$xyx^{-1}y^{-1} = 1 + [a, b]$. Therefore $\alpha(x, y) = \psi(1 + [a, b]) = \chi(\langle c, [a, b]\rangle)$. This immediately gives antisymmetry and biadditivity of α. If either x or $y \in K'_{i+1}$, then $xyx^{-1}y^{1} \in K'_j$, so $\alpha(x, y) = 1$. Also, if, say, $x \in H_i$, then write $b = b_1 + b_2$, with $b_1 \in A'_1(i)$, $b_2 \in X(i)$. Then $\alpha(x, y) = \chi(\langle c, [a, b_1 + b_2]\rangle) = \chi(\langle c, [a, b_1]\rangle) \cdot \chi(\langle c, [a, b_2]\rangle)$. The first factor is 1 because $\tilde{\varphi}''$ is a character on H_i, and the second factor is 1 because $\langle c, [a, b_2]\rangle = 0$, since $[a, b_2] \in M_i(F'')^{\perp}$. Thus we see α factors to a form $\bar{\alpha}$ on $K'_i/H_i \cdot K'_{i+1} \cong \mathscr{H}/\mathscr{Z}$. It remains to show this factored form is nondegenerate. But we have $\alpha(x, y) = \chi(\langle c, [a, b]\rangle) = \chi(\langle [c, a], b\rangle)$. If x does not represent zero in $\mathscr{H}/\mathscr{Z}$, then Lemma 4 shows $[c, a] \in A'(\lambda(j) + i) - A'(\lambda(j) + i + 1) = A'(\lambda(i) - 1) - A'(\lambda(i))$. On the other hand, b is arbitrary in $A'(i)$. Since $A'(i)^* = A'(\lambda(i))$, we conclude that for some y, $\alpha(x, y) \neq 0$, so $\bar{\alpha}$ is indeed nondegenerate.

REMARK. It is precisely here that our assumption of p odd makes its impact. We are dealing with the representation theory of a 2-step nilpotent p-group. The extra complications in this theory that arise when $p = 2$ could be handled, but at the expense of a long digression.

We want to find a representation of $H_0 \cdot K'_i$ that lies over $\tilde{\varphi}$ on $H_i \cdot K'_{i+1}$. Let $\tilde{\mathscr{H}}$ be the image in $\mathscr{H}$ of $E(i, j - 1)$, and $\tilde{\mathscr{Z}} = \mathscr{Z} \cap \tilde{\mathscr{H}}$. Then it is not hard to see that $\tilde{\mathscr{H}}/\tilde{\mathscr{Z}} \cong \mathscr{H}/\mathscr{Z}$. Also, it is easily verified that $H_0 \cdot K'_i/\ker \tilde{\varphi}$ is a homomorphic image of the semidirect product $H_0/H_j \times_s \tilde{\mathscr{H}}$, where the first factor acts on the second by Ad. (Of course H_1/H_j acts trivially.) ψ becomes a faithful character $\tilde{\psi}$ on $\tilde{\mathscr{Z}}$, which is isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$.

A Heisenberg p-group is a 2-step nilpotent p-group P such that: (1) the center $\mathscr{Z}(P)$ is isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$; (2) the center and the commutator subgroup of P coincide; and (3) every element of P has order p. A quick check shows that $\tilde{\mathscr{H}}$ is a Heisenberg p-group. Here one uses Lemma 10.

A Heisenberg p-group is determined by its order, which is p^{γ} for any odd $\gamma > 1$. Owing to the celebrated Weil representation ([9]), the representation theory of Heisenberg groups is very well known. We summarize what we need.

Besides one-dimensional representations, P has exactly $p - 1$ irreducible representations, each of dimension $p^{(\gamma-1)/2}$ (where p^{γ} is the order of P), and each one determined by the character $\tilde{\psi}$ it defines on $\mathscr{Z}(P)$. Call such a representation $Y(\tilde{\psi})$. $Y(\tilde{\psi})$ is induced from any character of any maximal abelian subgroup which agrees with $\tilde{\psi}$ on $\mathscr{Z}(P)$.

The automorphism group of P which acts trivially on $\mathscr{Z}(P)$ is isomorphic to $\mathrm{Sp}(p, \gamma - 1) \times_s P/\mathscr{Z}(P)$ (semidirect product), where the second factor is the inner automorphism group, and the first factor is the group preserving a symplectic form on a $\boldsymbol{Z}/p\boldsymbol{Z}$-module of dimension $\gamma - 1$. It is known that for each (nontrivial) $\tilde{\psi}$, $Y(\tilde{\psi})$ extends in a unique way to $\mathrm{Sp}(p, \gamma - 1) \times_s P$. An arbitrary group G of automorphisms will belong to a conjugate of $\mathrm{Sp}(p, \gamma - 1)$ in $\mathrm{Aut}(P)$ if and only if (1) G acts trivially on $\mathscr{Z}(P)$; and (2) G commutes with an automorphism of P trivial on $\mathscr{Z}(P)$ and having no fixed points modulo $\mathscr{Z}(P)$.

Applying these facts to our situation, we see immediately that there is a representation $V(\psi)$ of $H_0/H_j \times_s \tilde{\mathscr{H}}$, of dimension $(\tilde{\mathscr{H}}/\tilde{\mathscr{Z}})^{1/2} = p^\alpha$ lying above $\tilde{\psi}$ on $\tilde{\mathscr{Z}}$. $V(\psi)$ is completely determined by requiring that its restriction to H_0/H_j be the pullback of the extension of $Y(\tilde{\psi})$ to $\mathrm{Sp}(p, 2\alpha)$ via the homomorphism $\mathrm{Ad}\colon H_0/H_j \to \mathrm{Aut}(\tilde{\mathscr{H}})$. In particular $V(\psi)$ will be trivial on H_1/H_j. Now consider the representation $V(\tilde{\varphi}'') = \tilde{\varphi}'' \otimes V(\psi)$, where $\tilde{\varphi}''$ here denotes the character of $H_0/H_j \times_s \tilde{\mathscr{H}}$ which is trivial on $\tilde{\mathscr{H}}$ and factors to $\tilde{\varphi}''$ on H_0/H_j. Tracing back through the above constructions shows that $V(\tilde{\varphi}'')$ actually factors to a representation of $H_0 \cdot K_i'/\ker \tilde{\varphi}$, which then of course lifts to a representation of $H_0 \cdot K_i'$. We denote this representation by $V(\tilde{\varphi}'')$ also.

Now we may describe the correspondence between W and W''. Given a representation W'' of H_0 lying over ψ, consider the representations $W'' \otimes \tilde{\varphi}''^{-1}$. This is trivial on H_{j-1}. Hence it may be extended to a representation of $H_0 \cdot K_i'$ (denoted by the same symbol). Now form the representation $V(\tilde{\varphi}'') \otimes (W'' \otimes \tilde{\varphi}''^{-1})$ of $H_0 \cdot K_i'$, and induce up to K'. The resulting representation is W.

The proof that this correspondence sends irreducible W'' to irreducible W and is bijective is similar in essence to the proof for j even, but is again more complicated. It involves the same facts about Heisenberg groups used in the construction of $V(\tilde{\varphi}'')$. We omit the rather tedious details.

With Theorem 1 proved, we can begin the construction of supercuspidal representations of $Gl_n(F)$. Since these will be induced from the groups $F'^{\times} \cdot K'$, we first construct the representations of these groups, from which we will be inducing.

Now let ψ be a character of $F'^{\times}$. Recall that $U' = U_1' = 1 + \pi'R$. For all $i \geqq 1$, put $U_i' = 1 + \pi'^i R'$. The conductor of ψ is the largest of the U_i' contained in $\ker \psi$.

We can set up in a consistent way on F' the same structures we set up on $M_n(F)$ for passing from the multiplicative to the additive situation. Thus when F' is regarded as a subalgebra of $M_n(F)$, $\mathrm{tr}(M_n(F)/F)$ coincides with $\mathrm{tr}(F'/F)$. Thus there is no ambiguity if we write $\langle x, y\rangle = \mathrm{tr}(F'/F)(xy)$ for $x, y \in F'$. Using χ, we get an isomorphism $\theta' \colon F' \to \hat{F}'$ given by $\theta'(x)(y) = \chi(\langle x, y \rangle)$. θ' is just the composition of θ and the projection of $\widehat{M_n(F)}$ onto $\hat{F}'$. Since the conductor of χ is R, it is a simple computation to show $\theta'^{-1}(R'^{\perp}) = R'^* = \pi'^{1-e}R'$.

Put, as with A', A'_1, X, $R'(i) = \pi'^i R'$, and retain the notation $\lambda(i) = -i - e + 1$. If $2i \geqq j$, we again have the isomorphism ν': $R'(i)/R'(j) \to \widehat{U'_i/U'_j}$. ν' is just the restriction of ν defined previously. We also get μ': $\widehat{U'_i/U'_j} \to R'(\lambda(j))/R'(\lambda(i))$. If $i = j - 1$, we can choose a unique $c \in C'$ to represent a nontrivial character ψ_{j-1} of U'_{j-1}/U'_j, and this c will be called the standard representative ψ_{j-1}.

LEMMA 11. *Let U'_j be the conductor of ψ, and let ψ_{j-1} be the restriction of ψ to U'_{j-1}. Let c be the standard representative of ψ_{j-1}. Let F'' be the field generated by c over F. Then*

(i) *$\psi = \psi_1 \cdot \psi_2$ where $\psi_2 = \psi'' \cdot N(F'/F'')$, with $\psi'' \in \widehat{F''^{\times}}$, and ψ_1 is trivial on U'_{j-1}.*

(ii) *If ψ is of the form $\psi = \psi''' \circ N(F'/F''')$ for some subextension F''', then $F'' \subseteqq F'''$.*

(iii) *ψ is admissible if and only if ψ_1 is admissible when F' is considered as an extension of F''.*

Proof. Clearly (i) and (ii) imply (iii). On the other hand, (i) is a consequence of Lemma 9 because of the consistency of the identifications ν and ν', μ and μ', θ and θ'. It remains to prove (ii).

Suppose $\psi = \psi''' \circ N(F'/F''')$. Let e''' be the degree of ramification of F' over F'''. Then $N(F'/F''')$ maps $U'_{ie'''}$ onto U'''_i, and maps $U'_{ie'''+1}$ into U'''_{i+1}. Thus if the conductor of ψ''' is U'''_i, the conductor of ψ is $U_{(i-1)e'''+1}$, that is, $j-1=(i-1)e'''$. Now $\mathrm{ord}_{F'}(c) = -j - e + 1 = -e - (j-1)$, so $\mathrm{ord}_{F'}(c)$ is a multiple of e'''.

Let e'' be the smallest integer such that $e'' \, \mathrm{ord}_{F'}(c)$ is a multiple of e, the ramified degree of F' over F. Then $c^{e''} = \pi^{\alpha} b'$ for some integer α and $b' \in B'$. Hence we see e'' is the ramified degree of F'' over F. If e' is the ramified degree of F' over F'', then $e = e'e''$. Thus, from the previous paragraph we see that e''' divides e'.

Now let $F^{(4)}$ be the compositum of F'' and F'''. Define $\psi^{(4)} = \psi''' \circ N(F^{(4)}/F''')$. Then $\psi = \psi^{(4)} \circ N(F'/F^{(4)})$. Thus it suffices to prove (ii) when $F^{(4)} = F'$. But by the previous paragraph, $F^{(4)}$ is unramified over F'''. So we may assume F' is unramified over F'''. Then F' is a cyclic galois extension of F''', and $\psi = \psi''' \circ N(F'/F''')$ if and only if ψ is invariant by $\mathrm{Gal}(F'/F''')$ by Hilbert's Theorem 90. In particular, ψ_{j-1} on U'_{j-1} must be invariant by $\mathrm{Gal}(F'/F''')$. If $\sigma \in \mathrm{Gal}(F'/F''')$, then since either $\sigma(c) = c$ or $|\sigma(c) - c|_{F'} = |c|_{F'}$, we see that c must be invariant by $\mathrm{Gal}(F'/F''')$, since our mappings

μ', ν', θ' are all galois-equivariant for galois extensions. Thus we see $c \in F'''$, and so $F'' \subseteqq F'''$, since c generates F''.

COROLLARY. *Given $\psi \in F'^{\times}$, there is a well-defined sequence of integers $j_1 > j_2 > \cdots > j_\alpha$, and subfields $F_1 \subset F_2 \subset \cdots \subset F_\alpha$, such that $\psi = \psi_1 \cdot \psi_2 \cdots \psi_\alpha$, where the conductor of ψ_i is $j_i = (l_i - 1)e_i + 1$ where e_i is the degree of ramification of F' over F_i, and $\psi_1 \cdot \psi_2 \cdots \psi_i = \psi^{(i)} \circ N(F'/F_i)$ and $\psi^{(i)}$ is nondegenerate on $(U_i)_{l_i - 1}$.*

Proof. We may induce on the conductor of ψ. Then (i) of Lemma 11 gives $\psi = \psi_1(\psi_1^{-1}\psi)$, with ψ_1 having the desired properties (the relation between conductors being given in the proof of (ii)) and $\psi_1^{-1}\psi$ having conductor containing strictly the conductor of ψ. Then (ii) allows us to continue the induction, with F' now being regarded as an extension of $F'' = F_1$.

We now show how to obtain the representations of K' from which we will be inducing.

LEMMA 12. *Let ψ' be an admissible character of $F'^{\times}$, with conductor U'_j. Let c be the standard representative of ψ' on U'_{j-1}. Let ψ be the character of K'_{j-1}/K'_j represented by c. Then there is an irreducible representation $W(\psi')$ of K', with conductor K'_j, and lying over $\mathrm{Ad}^* K'(\psi)$ on K'_{j-1}, corresponding to ψ'. $W(\psi')$ in fact depends only on the restriction of ψ' to R'.*

Proof. Let F'' be the field generated over F by c. Let $\psi' = \psi'_1 \cdot \psi'_2$ be the decomposition given by (i) of Lemma 11. By induction, and (iii) of Lemma 11, we may assume we have constructed $W''(\psi'_1)$. $W''(\psi'_1)$ will then be trivial on H_{j-1}. Now let ψ'' be the character of $Gl_l(F'')$ defined by $\psi''(T) = \psi''(N(M_l(F'')/F'')T)$ (N in this case being determinant). Then ψ'' agrees with ψ'_2 on F', is trivial on H_j, and has standard representative c on H_{j-1}. Now simply let $W(\psi')$ be the representation of K' corresponding to $W''(\psi'_1) \otimes \psi''$ by Theorem 1. $W(\psi')$ clearly has the properties required of it.

Finally, we must deal with the case when ψ' is trivial on U'. When this happens, if ψ' is to be admissible, F' must be unramified over F, and $K' = K = Gl_n(R)$. Moreover, the image $\bar{F}'$ of R' in $Gl_n(\bar{F}) = K/K_1$, is the multiplicative group of the extension field of $\bar{F}$ of degree n — in other words, is a "minisotropic" Cartan subgroup of $Gl_n(\bar{F})$. Also ψ factors to a nondegenerate character $\bar{\psi}$ of $\bar{F}'$. Now it is known (see [1]) that to each nondegenerate character of $\bar{F}$, there is associated a cuspidal representation $W(\bar{\psi})$ of $Gl_n(\bar{F})$. We associate to ψ the lift of $W(\bar{\psi})$ to $Gl_n(R)$. This finishes Lemma 12.

To construct our representations, we need to recall some basic facts ([7]) about induced representations. Let G be a separable locally compact group, $I \subseteq G$ an open compact subgroup. Let V_1, V_2 be two finite dimensional representations of I, and write $V_i = \sum m^i_\alpha W_\alpha$, where the m^i_α's are the multiplicities of the irreducible representations W_α occurring in V_i. Then the intertwining number of V_1 and V_2, which is the dimension of the space of intertwining operators (or I-morphisms) from V_1 to V_2 is $\sum_\alpha m^1_\alpha m^2_\alpha$.

Now let W_1 and W_2 be two irreducible representations of subgroups I_1, I_2. For $g \in G$, put $\mathrm{Ad}\,(g)I_2 = gI_2g^{-1}$, and let $\mathrm{Ad}^*\, g(W_2)$ be the representation on $\mathrm{Ad}\,(g)(I_2)$ defined by $\mathrm{Ad}^*\, g(W_2)(x) = W_2(g^{-1}xg)$, for $x \in \mathrm{Ad}\,(g)(I_2)$. We say g intertwines W_1 and W_2 i times if the intertwining number of the restrictions of W_1 and $\mathrm{Ad}^*\,(g)(W_2)$ to $I_1 \cap \mathrm{Ad}\,(g)I_2$ is i. If $i > 0$, we say g intertwines W_1 and W_2. The number of times g intertwines W_1 and W_2 depends only on the (I_1, I_2) double coset of g and is symmetric in W_1 and W_2. It is known that if only a finite number of (I_1, I_1) double cosets of G contain elements which intertwine W_1 with itself, then the representation of G induced from W_1 on I_1 decomposes into finitely many irreducible components; and in particular, if only $g \in I_1$ intertwine W_1 with itself, then the induced representation is irreducible. It is also known that if W_1 and W_2 both induce irreducible representations, then these representations are inequivalent if and only if no $g \in G$ intertwines W_1 and W_2. All these remarks also apply if I_1, I_2 are compact modulo the center of G.

Now let F' and $\tilde{F}'$ be two tamely ramified extensions of F of degree n. Let K' and $\tilde{K}'$ be the corresponding compact subgroups. Let ψ, $\tilde{\psi}$ be nontrivial characters of K'_{j-1}/K'_j and $\tilde{K}'_{\tilde{j}-1}/\tilde{K}'_{\tilde{j}}$ which have standard representatives c, $\tilde{c}$ in F', $\tilde{F}'$ respectively. Let F'', $\tilde{F}''$ be the subfields of F', $\tilde{F}'$ generated by c and $\tilde{c}$ over F. Take i, $\tilde{i}$ satisfying $2i \geqq j$, $2\tilde{i} \geqq \tilde{j}$. Let φ, $\tilde{\varphi}$ be characters of K'_i/K'_j and $\tilde{K}'_{\tilde{i}}/\tilde{K}'_{\tilde{j}}$, respectively, which have representatives T and $\tilde{T}$ belonging to $M_l(F'')$ and $M_{\tilde{l}}(\tilde{F}'')$ respectively.

LEMMA 13. *If $g \in Gl_n(F)$ intertwines φ and $\tilde{\varphi}$, then g belongs to a double coset $K'_{j-i}g_0\tilde{K}'_{\tilde{j}-\tilde{i}}$ with* $\mathrm{Ad}\, g_0(c) = \tilde{c}$. *In particular, if g intertwines φ with itself, then $g \in K'_{j-i}Gl_l(F'')K'_{j-i}$.*

Proof. For $1 + x \in K'_i$, we have $\varphi(1 + x) = \chi(\langle T, x\rangle)$. Similarly, for $1 + y \in \tilde{K}'_{\tilde{i}}$, we have $\tilde{\varphi}(1 + y) = \chi(\langle \tilde{T}, y\rangle)$. Also $\mathrm{Ad}\,(g)(1 + y) = 1 + \mathrm{Ad}\, g(y)$. Recalling that $K'_i = 1 + A'(i)$ and $\tilde{K}'_{\tilde{i}} = 1 + \tilde{A}'(\tilde{i})$, we see φ and $\mathrm{Ad}^*\, g(\tilde{\varphi})$ agree on $K'_i \cap \mathrm{Ad}\, g\tilde{K}'_{\tilde{i}}$ if and only if $\theta(T)$ and $\theta(\mathrm{Ad}\,(g)(\tilde{T}))$ agree on $A'(i) \cap g\tilde{A}'(\tilde{i})g^{-1}$. This means $T - \mathrm{Ad}\, g(\tilde{T})$ is in $(A'(i) \cap g\tilde{A}'(\tilde{i})g^{-1})^* = A'(i)^* + g\tilde{A}'(\tilde{i})^*g^{-1}$. Thus, we can find $S \in$

$A'(i)^*$, $\widetilde{S} \in \widetilde{A}'(\widetilde{i})^*$, such that $T + S = \mathrm{Ad}\, g(\widetilde{T} + \widetilde{S})$.

Now, since φ lies over ψ, $T \in c + A_1'(\lambda(j) + 1)$, and $c^{-1}T \in H_1$. Therefore $\mathrm{ad}\, c - \mathrm{ad}\, T(X) \subseteq \pi' X$, and Lemmas 4 and 6 are true for T as well as for c. That is, $T + S = \mathrm{Ad}\, k(T')$ for some $T' \in T + A_1'(\lambda(i))$, and $k \in K_{j-i}'$. Similarly $\widetilde{T} + \widetilde{S} = \mathrm{Ad}\, \widetilde{k}(\widetilde{T}')$ for $\widetilde{T}' \in T + \widetilde{A}_1'(\lambda(\widetilde{i}))$ and $\widetilde{k} \in \widetilde{K}_{\widetilde{j}-\widetilde{i}}'$. Thus we have $T' = \mathrm{Ad}\,(k^{-1}g\widetilde{k})(\widetilde{T}')$. Put $g_0 = k^{-1}g\widetilde{k}$. Since $T' \in c + A_1'(\lambda(j) + 1)$ and $\widetilde{T}' \in \widetilde{c} + A_1'(\lambda(\widetilde{j}) + 1)$, a slight modification of the reasoning in Lemma 8 shows $\mathrm{Ad}\,(g_0)(\widetilde{c}) = c$, and the lemma is proved.

Now, notations as above, we again consider ψ on K_{j-1}/K_j, or on H_{j-1}/H_j. Let W'' be a representation of H_0 lying above ψ and let W be the representation of K' corresponding to it by Theorem 1.

If j is even, put $i = j/2$. If j is odd, put $i = (j-1)/2$. Then we know W is induced from a representation, which we shall denote by Y, of $H_0 \cdot K_i'$. Moreover, recalling $X(i) = \pi'^i X$, where X is as in Lemma 4, we may see from Lemma 5 that the set $1 + X(i)$ is a set of (right or left) coset representatives for H_0 in $H_0 \cdot K_i'$. Thus $H_0 \cdot K_i' = H_0 \cdot (1 + X(i)) = (1 + X(i)) \cdot H_0$. If j is even, then $1 + X(i)$ is contained in the kernel of Y; however, for odd j this is (unfortunately) not true.

LEMMA 14. (i) *If $g \in Gl_n(F)$ intertwines W with itself, then $g \in K'g_0K'$ with $g_0 \in Gl_l(F'')$.*

(ii) *If j is even, then there is a one-to-one correspondence between intertwining operators for W and for W''. Specifically, if $g_0 \in Gl_l(F'')$ intertwines W'' on H_0 γ times, then it intertwines W'' on K' γ times.*

(iii) *Let j be odd. Suppose $g_0 \in Gl_l(F'')$ is such that we may write $M_l(F'')^\perp = S_1 \oplus S_2 \oplus S_3$ where S_i is an invariant subspace for $\mathrm{Ad}\, g_0$, $X(i) = (S_1 \cap X(i)) \oplus (S_2 \cap X(i)) \oplus (S_3 \cap X(i))$, and $\mathrm{Ad}\, g_0(S_1 \cap X(i)) \subseteq X(i+1)$, $\mathrm{Ad}\, g_0(S_2 \cap X(i)) = S_2 \cap X(i)$, and $\mathrm{Ad}\, g_0(S_3 \cap (X(i) - X(i+1))) \cap X(i) = \phi$.* (That is, $\mathrm{Ad}\, g_0$ shrinks S_1, is isometric on S_2, and stretches S_3.) *Then if g_0 intertwines W'' on H_0 γ times, it intertwines W on K' γ times.*

Proof. Statements (i) and (ii) are quite easy. We observe that since W is induced from Y, in order to compute intertwining operators for W, it suffices to compute them for Y. But since Y lies over ψ *on* K_{j-1}, it is easily seen from Lemma 13 that if g intertwines Y with itself, then $g \in (H_0 \cdot K_i')g_0(H_0 \cdot K_i')$where $g_0 \in Gl_l(F'')$. Statement (i) follows a fortiori.

If j is even, Y is simply the extension of W'' from H_0 to $H_0 \cdot K_i' = H_0 \cdot E(i, j)$ which is trivial on $E(i, j)$. If $g_0 \in Gl_l(F'')$, and $z = hx$, with $h \in H_0$, $x \in 1 + X(i)$, then it is easy to see $g_0 z g_0^{-1} \in H_0 \cdot K_i'$

if and only if $g_0hg_0^{-1} \in H_0$ and $g_0xg_0^{-1} \in 1 + X(i)$. That is $B = (H_0 \cdot K_i') \cap g_0(H_0 \cdot K_i')g_0^{-1} = (H_0 \cap g_0H_0g_0^{-1}) \cdot (1 + (X(i) \cap g_0X(i)g_0^{-1}))$. Thus we see that the representations Y and $\mathrm{Ad}^* g_0(Y)$ of B are determined completely by their restrictions to $H_0 \cap g_0H_0g_0^{-1}$. Thus g_0 intertwines Y with itself γ times if and only if it intertwines W'' with itself γ times. This establishes the first part of (ii). For the second part, all we need verify is that if $g_0 \in Gl_l(F'')$, then $K'g_0K' \cap Gl_l(F'') = H_0g_0H_0$. This follows fairly easily from the work of Iwahori-Matsumoto ([4]). Since the precise statement of (ii) is not needed for the rest of this paper, we leave the indicated verification to the reader.

Now we turn to (iii). Take $g_0 \in Gl_l(F'')$, and let $S_1 \oplus S_2 \oplus S_3 = M_l(F'')^\perp$ be the posited decomposition of $M_l(F'')^\perp$. Put $S_j(i) = S_j \cap X(i)$, so $X(i) = S_1(i) \oplus S_2(i) \oplus S_2(i)$. I claim first that $1 + S_1(i)$ and $1 + S_2(i)$ are isotropic with respect to the form $\alpha(\ ,\ )$ of Lemma 10, and that both are orthogonal to $1 + S_2(i)$. For if $s_j \in S_j(i)$, then we have, as calculated in Lemma 10, $\alpha(1 + s_j, 1 + s_k) = \chi(\langle c, [s_j, s_k]\rangle) = \chi(\langle \mathrm{Ad}\, g_0^{-1}(c_0), [s_j, s_k]\rangle) = \chi(\langle c, [\mathrm{Ad}\, g_0(s_j), \mathrm{Ad}\, g_0(s_k)]\rangle)$. If now $j = 1$ and $k = 1$ or 2, then $\mathrm{Ad}\, g_0(s_j) \in X(i + 1)$, so $\mathrm{Ad}\,(g_0)(1 + s_j) \in K_{i+1}$, and similarly $\mathrm{Ad}\, g_0(1 + s_k) \in K_i$. Hence $\alpha(1 + s_j, 1 + s_k) = 1$ by Lemma 10. Replacing g_0 by g_0^{-1} gives the result for $j = 2, 3$, $k = 3$. (We note that by similar but more complicated arguments, we could show S_1 and S_3 are isotropic with respect to $\langle\ ,\ \rangle$, and are orthogonal to S_2. We do not need this, however.)

As noted before, we have $B = (H_0 \cdot K_i') \cap g_0(H_0 \cdot K_i')g_0^{-1} = (H_0 \cap g_0H_0g_0^{-1}) \cdot (1 + (X(i) \cap g_0X(i)g_0^{-1}))$, and $X(i) \cap g_0X(i)g_0^{-1} = (S_1(i) \cap g_0S(i)g_0^{-1}) \oplus S_2(i) \oplus S_3(i) \subseteqq X(i + 1) + S_2(i) + S_3(i)$. Moreover, $g_0^{-1}S_3(i)g_0 \subseteqq X(i + 1)$. Recall that $\mathscr{H}$ is the Heisenberg group constructed in the proof of Theorem 1 for j odd. Let I_1, I_3, be the images in $\mathscr{H}$ of $1 + S_1(i)$, $1 + S_2(i)$, and let I_2 be the inverse image in $\mathscr{H}$ of the image of $1 + S_2(i)$ in $\mathscr{H}/\mathscr{Z}$. Then I_1, I_3 are abelian, and $I_2 \cap \mathscr{Z} = I_3 \cap \mathscr{Z} = \{\text{identity}\}$, and $\mathscr{Z} \subseteqq I_2$, and I_1 and I_3 centralize I_2. Moreover, the inverse image in $\mathscr{H}$ of the image in $\mathscr{H}/\mathscr{Z}$ of $1 + (X(i) \cap g_0X(i)g_0^{-1})$ is $I_2 \cdot I_3$. It is also clear that the restriction to I_3 of the representation $\mathrm{Ad}^* g_0(Y)$ is a multiple of the identity representation, since $1 + g_0^{-1}S_3(i)g_0 \subseteqq 1 + X(i + 1) \subseteqq \ker Y$. Thus in computing the intertwining number between Y and $\mathrm{Ad}^* g_0(Y)$ on B, it suffices to compute the intertwining number between $\mathrm{Ad}^* g_0(Y)$ and the subrepresentation Y_1 of Y on which $1 + S_3(i)$, or the inverse image of I_3 in $E(i, j - 1)$, acts trivially. Similarly, since $1 + \mathrm{Ad}\, g_0(S_1(i)) \subseteqq \ker Y$, we need only compute the intertwining number between Y_1 and the subrepresentation of $\mathrm{Ad}^* g_0(Y)$ on which $1 + \mathrm{Ad}\, g_0(S_1(i))$ acts trivially.

Since B is a group, we see that $H_0 \cap g_0H_0g_0^{-1}$, acting on $\mathscr{H}$, normalizes I_3 and $I_2 \cdot I_3$. Also I_2 is a Heisenberg group, with center

$\mathscr{Z}$ (unless it reduces to $\mathscr{Z}$).

We must now recall the precise structure of Y. Let $\tilde{\varphi}''$ be an extension of ψ to a linear character of H_0. Since W'' lies over ψ, $W'' \otimes \tilde{\varphi}''^{-1}$ is a representation of H_0 trivial on H_{j-1}. We extend it to a representation, also denoted $W'' \otimes \tilde{\varphi}''^{-1}$, of $H_0 \cdot K'_i$, trivial on $E(i, j-1)$. We then take the representation $V(\tilde{\varphi}'')$ of $H_0 \cdot K'_i$, lying over ψ on K_{j-1} and constructed from the Weil representation, using $\tilde{\varphi}''$. Then $Y = (W'' \otimes \tilde{\varphi}''^{-1}) \otimes V(\tilde{\varphi}'')$. The restriction of Y to B is thus the tensor product of the restrictions of $W'' \otimes \tilde{\varphi}''^{-1}$ and $V(\tilde{\varphi}'')$, and similarly for $\mathrm{Ad}^* g_0(Y)$.

Let V_1 be the restriction to B of the subrepresentation of $V(\tilde{\varphi}'')$ on which the inverse image in I_3 in $E(i, j-1)$ acts trivially. Then the subrepresentation Y_1 of Y defined above is just $(W'' \otimes \tilde{\varphi}''^{-1}) \otimes V_1$. (Here we restrict $W'' \otimes \tilde{\varphi}''^{-1}$ to B.)

Now $I_2 \cdot I_3/I_3 \cong I_2$ is a Heisenberg group, on which $H_0 \cap g_0 H_0 g_0^{-1}$ acts, and $\mathscr{Z}$ is the center of I_2, and the action preserves ψ on $\mathscr{Z}$. Also V_1 is the lift of a representation from an extension to $H_0 \cap g_0 H_0 g_0^{-1} \times_s I_2$ (semidirect product), of the unique representation of I_2 lying over ψ on $\mathscr{Z}$. Thus V_1 is (essentially) simply the Weil representation of B deriving from the action of $H_0 \cap g_0 H_0 g^{-1}$ on I_2.

Now from our remarks above, it follows that the intertwining number of Y and $\mathrm{Ad}^* g_0(Y)$ on B is the same as the intertwining number of $(W'' \otimes \tilde{\varphi}''^{-1}) \otimes V_1$, and $\mathrm{Ad}^* g_0(W'' \otimes \tilde{\varphi}''^{-1}) \otimes V_1$. But now it follows from standard theory (see for example, the discussion of Proposition 2 of [3]) that this is the same as the intertwining number of $W'' \otimes \tilde{\varphi}''^{-1}$ and $\mathrm{Ad}^* g_0(W'' \otimes \tilde{\varphi}''^{-1})$ on $H_0 \cap g_0 H_0 g_0^{-1}$. But since $\tilde{\varphi}''$ is simply the restriction to H_0 of a character of $Gl_l(F'')$, this is the same as the intertwining number of W'' and $\mathrm{Ad}^* g_0 W''$ on $H_0 \cap g_0 H_0 g_0^{-1}$. This concludes the important part of (iii). To completely finish (iii), we should verify the same facts about double cosets as for (ii). But since we are here mainly interested in nonexistence of intertwining operators, and since for the double cosets in which we are particularly interested, the verification is especially simple, we again omit this point. Lemma 14 is now concluded.

The purpose of this next lemma is to provide an important class of g_0 which verify the conditions of (iii) in Lemma 14.

LEMMA 15. *Let $F'' \subseteq F'$ be a subfield, and let $\mathscr{A}'' \subseteq Gl_l(F'')$ be a Cartan subgroup, split over F'', and such that $\mathscr{A}''_0$, the maximal compact subgroup of $\mathscr{A}''$, is contained in K'. Then $A'(i) = \bigoplus_j (A'(i) \cap S_j)$, where the S_j are irreducible subspaces for* $\mathrm{Ad}\,\mathscr{A}''$ *acting on $M_n(F)$.*

Proof. We prove the result in stages. First we take $F'' = F$, then galois over F, then finally we reduce to the case of F'' galois.

If $F'' = F$, then $\mathscr{A}'' = \mathscr{A}$ is just a split Cartan, which we may assume, by an analogue of Lemma 1, valid for all Cartan subgroups of $Gl_n(F)$, that, up to conjugation by K', $\mathscr{A}$ is the diagonal matrices. Then let E_{jk} be the one-dimensional subspaces of $M_n(F)$ spanned by the matrix units. It will certainly suffice to show $A'(i) = \bigoplus_{j,k}(A'(i) \cap E_{jk})$. If F is unramified over F, then $A' = M_n(R)$, and the desired conclusion is obvious. In general, $A'(i) = \bigcap_m \pi'^{m+i}M_n(R)\pi'^{-m}$. We can find $y \in K'$ such that $\pi' y = x$ normalizes $\mathscr{A}$. Then $A'(i) = \bigcap_m x^{m+i}M_n(R)x^{-m} = \bigcap_m x^{m+i}(\bigoplus_{j,k} M_n(R) \cap E_{jk})x^{-m}$. But now $x^{m+i}E_{jk}x^{-m} = E_{j'k'}$, for some j', k', since x normalizes $\mathscr{A}$. Now if $z \in A'(i)$, $z = x^{m+i}z_m x^{-m}$ for each m, with $z_m \in M_n(R)$. Also $z = \sum e_{jk}$, with $e_{jk} \in E_{jk}$, and for each m, $z_m = \sum e_{jk}^{(m)}$, with $e_{jk}^{(m)} \in E_{jk} \cap M_n(R)$. Since each of the above decompositions is unique, $x^{m+i}e_{jk}^{(m)}x^{-m} = e_{j'k'}$. Therefore $e_{j'k'} \in A'(i)$, and we have established the lemma when $F'' = F$.

Now take F'' galois over F. Then we may choose a set $\{\tilde{\sigma}\} \subseteq Gl_n(F)$ of representatives for the galois group $\mathrm{Gal}(F''/F)$, such that each $\tilde{\sigma}$ is in K', and normalizes $\mathscr{A}''$. Then the $\tilde{\sigma}$ are then determined up to their $\mathscr{A}_0''$ cosets, satisfy $\tilde{\sigma}a\tilde{\sigma}^{-1}a^{-1} \in \mathscr{A}_0''$ for any $a \in \mathscr{A}''$.

We have the decomposition $M_n(F) = \bigoplus_{\sigma \in \mathrm{Gal}(F''/F)} \tilde{\sigma}M_l(F'')$. Since $\tilde{\sigma}M_l(F'') \times \tilde{\tau}M_l(F'') = \tilde{\sigma}\tilde{\tau}M_l(F)$, we see $\tilde{\sigma}M_l(F'')$ and $\tilde{\tau}M_l(F'')$ are orthogonal with respect to $\langle\, , \rangle$ unless $\sigma\tau = 1$. Thus $M_l(F'')^{\perp} = \bigoplus_{\sigma \neq 1} \tilde{\sigma}M_l(F'')$. We know that $A'(i) = (M_l(F'') \cap A'(i)) \oplus (M_l(F'')^{\perp} \cap A'(i))$. Since $\tilde{\sigma} \in K$, we see that multiplying this decomposition by $\tilde{\sigma}$ yields $A'(i) = (\tilde{\sigma}M_l(F'') \cap A'(i)) \oplus ((\bigoplus_{\tau \neq \sigma} \tilde{\tau}M_l(F'')) \cap A'(i))$ for each σ, which in turn implies $A'(i) = \bigoplus_{\sigma}(\tilde{\sigma}M_l(F'') \cap A'(i))$. Now each $\tilde{\sigma}M_l(F'')$ is left invariant by $\mathrm{Ad}\,\mathscr{A}''$, and so is a sum of irreducible spaces for $\mathrm{Ad}\,\mathscr{A}''$. Moreover, if $M_l(F'') = \bigoplus_{j,k} E''_{jk}$ is the decomposition of $M_l(F'')$ into matrix units, then, taking $\mathscr{A}''$ to be the diagonal matrices of $Gl_l(F'')$, E''_{jk} is invariant by right and left multiplication by $\mathscr{A}''$. Therefore since $\tilde{\sigma}$ normalizes $\mathscr{A}''$, $\tilde{\sigma}M_l(F'') = \bigoplus_{j,k} \tilde{\sigma}E''_{jk}$ is a decomposition of $\tilde{\sigma}M_l(F'')$ into $\mathrm{Ad}\,\mathscr{A}''$-invariant, irreducible subspaces. But now, again since $\tilde{\sigma} \in K'$, we have $\tilde{\sigma}(M_l(F'') \cap A'(i)) = \tilde{\sigma}M_l(F'') \cap A'(i)$. Since, by reduction to the case $F'' = F$, we have $A'(i) \cap M_l(F'') = \bigoplus_{j,k}(A'(i) \cap E''_{jk})$, we see $A'(i) \cap \tilde{\sigma}M_l(F'') = \bigoplus_{j,k}(A'(i) \cap \tilde{\sigma}E''_{jk})$ for all $\sigma \in \mathrm{Gal}(F''/F)$, and so finally $A'(i) = \bigoplus_{\sigma,j,k}(A'(i) \cap \tilde{\sigma}E''_{jk})$.

Now we pass to the general case, when F'' is any subextension of F'. Since in any case F'' is tamely ramified, its galois closure F''' is unramified over it. Therefore, for a suitable unramified extension F_u of F, the algebra $F'' \otimes_F F_u$ breaks up into a direct sum of subalgebras isomorphic to F'''.

We consider the matrix algebra $M_n(F_u) = M_n(F) \otimes_F F_u$, and in it $A'_u(i) = A'(i) \otimes R_u$, where of course R_u denotes the integers of F_u. $\mathscr{A}''$ is just the multiplicative group of an abelian subalgebra $\mathfrak{A}''$ of rank n of $M_n(F)$, and $\mathfrak{A}''$ is isomorphic of F'''^l. We let $\mathscr{A}'''$ be the multiplicative group of $\mathfrak{A}'' \bigotimes_F F_u = \mathfrak{A}'''$. Then $\mathfrak{A}'''$ is a direct sum of a certain number of copies of F'''.

It is clear that $A'_u = \bigcap_m \pi'^m M_n(R_u)\pi'^{-m}$ where $\pi' = \pi' \bigoplus 1 \subseteqq M_n(F_u)$. Thus we may find a field F'_u so that A'_u is the order associated to it by the discussion preceding Lemma 1. In fact, we may choose F'_u so that it contains F'''. (Let F'_u be any of the conjugate fields which are the summands of $F' \bigotimes_F F_u$.) Suppose for a moment that the maximal order of $\mathfrak{A}'''$ is contained in A'_u. Then, since F''' is galois F_u, we have $A'_u(i) = \bigoplus_j (A'_u(i) \cap S^u_j)$, where S^u_j are isotypic subspaces for $\operatorname{Ad} \mathscr{A}'''$. Now $\operatorname{Gal}(F_u/F)$ acts on $M_n(F) \otimes F_u$ and on $\mathfrak{A}'''$, and this action permutes the S^u_j. Let $\{T_k\}$ be the collection of subspaces which are direct sums of S^u_j's-invariant by $\operatorname{Gal}(F_u/F)$, and minimal with respect to these properties. Then certainly $A'_u(i) = \bigoplus_k (A'_u(i) \cap T_k)$. Moreover $S_k = T_k \cap M_n(F)$ will be an isotypic component of $\operatorname{Ad} \mathscr{A}''$ acting on $M_n(F)$. Now if $z \in A'(i) = A'_u(i) \cap M_n(F)$, we have $z = \sum t_k$. This decomposition is unique, and since z is $\operatorname{Gal}(F_u/F)$-invariant, and the T_k are, each t_k must be, so $t_k \in S_k \cap A'_u(i)$, and so $A'(i) = \bigoplus_k (A'(i) \cap S_k)$.

The above reasoning was carried out under the assumption that the maximal order of $\mathfrak{A}'''$ was contained in A'_u. This will be a consequence of the next lemma, which will then complete the proof of Lemma 15.

LEMMA 16. *If F_1 is any extension of F, and F_u is an unramified extension of F, then the maximal order of $F_1 \bigotimes_F F_u$ is the image of $R_1 \otimes R_u$.*

Proof. Let $F_2 \subseteqq F_1$ be the maximal unramified subfield over F. Then we may write $F_1 \bigotimes_F F_u = F_1 \bigotimes_{F_2} (F_2 \bigotimes_F F_u)$. Now $F_2 \bigotimes_F F_u$ will be a direct sum of unramified extensions of F_2. Therefore, we may reduce the lemma to the two extreme cases when F_1 is either unramified or totally ramified over F.

If F_1 is totally ramified over F, then $F_1 \bigotimes_F F_u$ is still a field, since F_1 and F_u are linearly disjoint over F (see Serre [8]). We see $R_1 \otimes R_u$ will contain all roots of unity of $F_1 \otimes F_u$ of order prime to p, and will contain a prime element of $F_1 \otimes F_u$. Hence it must equal the entire maximal order.

In the second case, $F_1 \otimes F_u$ is a direct sum of fields unramified over F. Therefore, the maximal order of $F_1 \otimes F_u$ is its own dual lattice with respect to the bilinear form induced by the trace on $F_1 \otimes F_u$. On the other hand, this bilinear form is just the tensor product of the bilinear forms $\operatorname{tr}(F_1/F)(xy)$ and $\operatorname{tr}(F_u/F)(xy)$ on F_1 and F_u, and R_1 and R_u are their own dual lattices with respect to these forms. Therefore $R_1 \otimes R_u \subseteqq F_1 \otimes F_u$ is its own dual lattice. Since it is contained in the maximal order, it must be equal to it. This finishes Lemma 16.

We want to make note of the following result, which is immediate from Lemmas 14 and 15. Notations are as in those lemmas.

COROLLARY. *Suppose $F \subseteqq F'' \subseteqq F''' \subseteqq F'$, and $g_0 \in Gl_k(F''') \subseteqq Gl_l(F'')$ is in a split Cartan subgroup of $Gl_k(F''')$, whose compact subgroup is contained in K'. Then g_0 intertwines W'' if and only if g_0 intertwines Y* (if and only if g_0 intertwines W, providing $K'g_0K' \cap Gl_l(F'') = H_0g_0H_0$).

Now we are ready to construct our supercuspidal representations.

THEOREM 2. *For every admissible character ψ' of $F'^{\times}$, for every tamely ramified extension F' of F of degree n, there exists a supercuspidal representation $V(\psi')$ of $Gl_n(F)$, induced from a representation of $F'^{\times} \cdot K'$, agreeing with $W(\psi')$ on K'. $V(\psi'_1)$ and $V(\psi'_2)$ are equivalent if and only if ψ'_1 and ψ'_2 are equivalent.*

Proof. To begin, let us note that $F'^{\times} \cdot H_0 = \tilde{H}_0$ is actually the semidirect product of H_0 and the cyclic group generated by π'. Similarly for $F'^{\times} \cdot K' = \tilde{K}'$. Thus it is easy to see that Theorem 1 and Lemmas 12 and 14 apply equally well to these groups as to H_0 and K'. Therefore, we may assume we have defined representations $\tilde{W}(\psi')$ on $\tilde{K}'$ in the manner of Lemma 12. Of course, the restriction of $\tilde{W}(\psi')$ to K' is just $W(\psi')$. We will show the representations $V(\psi')$ induced from $\tilde{W}(\psi')$ have the desired properties.

To prove the theorem, we merely examine which g can possibly intertwine $\tilde{W}(\psi')$ with itself, or $\tilde{W}(\psi'_1)$ with $\tilde{W}(\psi'_2)$.

Put $\tilde{K}'_i = K'_i$. Let U'_j be the conductor of ψ. We know $\tilde{W}(\psi')$ is induced from a representation, lying above ψ (ψ is the extension to $\tilde{K}_{j-1}$ of the restriction to U'_{j-1}) of ψ' of the subgroup $\tilde{H}_0 \cdot K'_i$, where $2i = j$ if j is even, or $2i + 1 = j$ if j is odd. We call this inducing representation $Y(\psi')$.

It is clear from the constructions of Theorem 1 and Lemma 12 that the restriction of $Y(\psi')$ to H_1 is just a multiple of $W''(\psi')$ restricted to H_1. Also, $Y(\psi')$ lies above ψ on K'_{j-1}. Therefore, Lemma 13 shows that the only double cosets which can support nontrivial intertwining operators for $Y(\psi')$ are those of the form $(\tilde{H}_0 \cdot K'_i)g_0(\tilde{H}_0 \cdot K'_i)$ where $g_0 \in Gl_l(F'')$. Moreover, since, as we mentioned $Y(\psi')$ on H_1 is a multiple of $W''(\psi')$ on H_1, g_0 cannot intertwine $Y(\psi')$ unless it intertwines W'' on H_1.

Now let $F''' \subseteqq F'$ be the subfield of F', such that the restriction of ψ' to U'_1 is of the form $\psi''' \circ N(F'/F''')$, where ψ''' is non-degenerate on U'''_1. Then $F'' \subseteqq F'''$, and F' is unramified over F''' by the definition of admissibility. By induction, we may assume that g_0 cannot intertwine $W''(\psi')$ on H_1 with itself unless g_0 is in a double coset $H_1 g_1 H_1$, with $g_1 \in Gl_k(F''')$, the centralizer of F'''. Therefore, the only double cosets which can support intertwining operators for $Y(\psi')$ are of the form $(H_0 \cdot K'_i) g_1 (H_0 \cdot K'_i)$ with $g_1 \in Gl_k(F''')$.

Now since F' is unramified over F''', $K' \cap Gl_k(F''') = Gl_k(R''')$ is a maximal compact subgroup of $Gl_k(F''')$. It is well known (see [4]) that in this case, one may choose, as a set of double coset representatives for $K' \cap Gl_k(F''')$ in $Gl_k(F''')$, elements from an F'''-split Cartan subgroup $\mathscr{A}'''$ in $Gl_k(F''')$, whose maximal compact subgroup is contained in K'. Therefore, applying Lemma 15, (ii) or (iii) according as j is even or odd, we may conclude that $a \in \mathscr{A}'''$ can intertwine $Y(\psi')$ with itself if and only if it intertwines $\widetilde{W}''(\psi')$ on $\widetilde{H}_0$ with itself. Then by induction we may assume this can only happen if a intertwines $\widetilde{W}'''(\psi')$, the representation of $\widetilde{K}' \cap Gl_k(F''')$ associated to ψ', with itself. Therefore, we are reduced to the case when F' is unramified over F.

If F' is unramified over F, then $\widetilde{K}' = \widetilde{K} = F^{\times} \cdot Gl_n(R)$, and $\widetilde{W}(\psi')$ is a character on $F^{\times}$ times the pullback to $Gl_n(R)$ of a cuspidal representation of $Gl_n(\bar{F})$. A set of double coset representatives for $\widetilde{K}$ consists of diagonal matrices with entries $(1, \pi^{\alpha_2}, \pi^{\alpha_3}, \cdots, \pi^{\alpha_n})$ with $0 \leqq \alpha_2 \leqq \alpha_3 \leqq \cdots \leqq \alpha_n$.

Let N_0 be the intersection of $Gl_n(R)$ with the group N of upper triangular unipotent matrices. Then, if g is one of the above double coset representatives $N_0 \subseteqq \widetilde{K} \cap g\widetilde{K}g^{-1}$. Moreover, if $g \neq 1$, then there is some parabolic subgroup P, containing N, such that $N_0(P)$, the intersection of the unipotent radical of P with N_0 satisfies $g^{-1}N_0(P)g \subseteqq \widetilde{K}_1$. Therefore, since $\widetilde{W}(\psi')$ is trivial on $K_1 \cap N_0$, $N_0(P) \subseteqq \ker \mathrm{Ad}^* g(\widetilde{W}(\psi'))$. On the other hand, the fact that $\widetilde{W}(\psi')$ on $Gl_n(R)$ is the pullback of a cuspidal representation of $Gl_n(\bar{F})$ means that the restriction of $\widetilde{W}(\psi')$ to $N_0(P)$ for any P does not contain the trivial representation. Therefore, if $g \neq 1$, g does not intertwine $\widetilde{W}(\psi')$ with itself. This shows $V(\psi')$ is irreducible and completes the construction of the $V(\psi')$.

It remains to establish the facts on the equivalence and non-equivalence of $V(\psi')$. Let $F'^{(1)}$ and $F'^{(2)}$ be two fields, of degree n, and tamely ramified over F. Let $K'^{(1)}$ and $K'^{(2)}$ be the corresponding compact groups. Let $\psi'^{(1)}$ and $\psi'^{(2)}$ be admissible characters of $F'^{(1)\times}$ and $F'^{(2)\times}$. Let $U'^{(i)}_{j_i}$ be the conductors of the $\psi'^{(i)}$, and let $c^{(i)}$ be the standard representatives for the $\psi'^{(i)}$ on $U'^{(i)}_{j_i-1}$. Let $F''^{(i)}$ be the subfields generated by the $c^{(i)}$. Let $H_k'^{(i)} = K_k'^{(i)} \cap Gl_{l_i}(F''^{(i)})$. Let $Y(\psi'^{(i)})$ be the representations from which the $W(\psi'^{(i)})$ are induced.

By Lemma 13, if g_0 intertwines the $Y(\psi'^{(i)})$, we can take g_0 so that $\operatorname{Ad} g_0(c^1) = c^{(2)}$. If this is so, then by the same reasoning as in the construction of the $V(\psi')$, g_0 must intertwine the restrictions of the $Y(\psi'^{(i)})$ to the $H_1'^{(i)}$. From this, by induction, we conclude that if g_0 is to intertwine the $V(\psi')$ then necessarily $\operatorname{Ad} g_0(F'''^{(1)}) = F'''^{(2)}$, where $F'''^{(i)}$ is the subfield of $F''^{(i)}$ such that, on $U_1'^{(i)}$, $\psi'^{(i)} = \psi'''^{(i)} \circ N(F''^{(i)}/F'''^{(i)})$, and $\psi'''^{(i)}$ is nondegenerate on $U_1'''^{(i)}$. But then we see $F''^{(1)}$ and $F''^{(2)}$ must be conjugate, since they are determined by $F'''^{(1)}$ and $F'''^{(2)}$ respectively. Also, we see we may as well take $K'^{(1)} = K'^{(2)} = K'$, and then we can choose by Lemma 1, $g_1 \in K'$ such that $\operatorname{Ad} g_1(F''^{(1)}) = F''^{(2)}$. Then we are reduced to showing that two nonconjugate characters of $F''^{(1)} = F''$ do not yield the same representation, and this proceeds precisely as for the construction of the $V(\psi')$. This finishes Theorem 2.

(Strictly speaking, we should verify that the $V(\psi')$, which are obviously representations with compactly supported matrix coefficients, are in fact cuspidal. This could be done. (In fact, the $W(\psi')$ are already cuspidal on K'.) However, we prefer to cite a result of Jacquet ([5]), which says an irreducible representation of Gl_n with compactly supported matrix coefficients is automatically cuspidal. (This has been generalized by Harish-Chandra (see [2]) to general p-adic groups.)

CONCLUDING REMARKS. (a) The case of the basic inductive step of Theorem 2 when j is even contrasts sharply with the intricacy of our arguments to accomplish the same step when j is odd. Thus one may hope that Theorem 2 has a proof considerably simpler than the one we give.

(b) It seems likely that much of the construction given here for Gl_n can be carried over to other p-adic groups of classical or Chevalley type. This would require either a case-by-case analysis, or some general structure theorems involving considerably more detail then those now in the literature. The complete construction for Gl_n, however, hinges on the knowledge of the cuspidal representations of Gl_n over a finite field. Thus until the representation theory of other finite algebraic groups is better known, the full construction given here is limited to Gl_n.

(c) It follows from remarks of R.P. Langlands that Theorem 2 allows one to attach a supercuspidal representation of Gl_n to each irreducible representation of degree n of the Weil group of F, (for n prime to p). It should of course be checked that this correspondence has the proper L-function theoretic properties.

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