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AN INEQUALITY INVOLVING THE LENGTH, CURVATURE, AND TORSIONS OF A CURVE IN EUCLIDEAN n-SPACE

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Let x be a closed nondegenerate C^n curve in E^n parametrized by arc length s. We prove an inequality for such x which lie in a ball of radius R. For nonplanar curves in E^3 the inequality is

$$L \leqq R^2 \frac{\int_0^L \kappa^2 ds \int_0^L \tau^2 ds - \left(\int_0^L \kappa\tau ds\right)^2}{\int_0^L \tau^2 ds}$$

where L is the length of x, and κ and τ are the curvature and torsion of x, respectively. Equality holds only if x is a great circle on a sphere of radius R. We also obtain from the general inequality necessary conditions on the length, curvature, and torsions of x in order that x be a closed curve or a closed curve with at most one corner.

1. Definitions. We say a C^n curve x in E^n is *nondegenerate* if it has a Frenet framing. That is, there exists an orthonormal set of vector fields $e_1, e_2, \cdots, e_n$ along x such that

$$(1)\qquad \begin{aligned} x' &= e_1 \\ e_1' &= \kappa e_2 \\ e_2' &= -\kappa e_1 + \tau_1 e_3 \\ e_3' &= -\tau_1 e_2 + \tau_2 e_4 \\ &\vdots \\ e_n' &= -\tau_{n-2} e_{n-1}, \end{aligned}$$

where the prime denotes differentiation with respect to arc length, κ is the curvature, and $\tau_1, \tau_2, \cdots, \tau_{n-2}$ are the torsions of x. For the remainder of this paper, we assume that x is nondegenerate and $\tau_i \neq 0$, for $i = 1, 2, \cdots, n-2$. In what follows we also let $\tau_0 = \kappa$ and $\tau_{n-1} = 0$.

We say $x: [0, L] \to E^n$ is closed if it induces a C^n mapping $x: S^1 \to E^n$, where S^1 is the circle. To say $x: [0, L] \to E^n$ is closed with at most one corner means that $x(0) = x(L)$ but $x'(0)$ need not equal $x'(L)$.

Define $x_i = (x, e_i)$, for $i = 1, 2, \cdots, n$, where $(,)$ denotes the inner product in E^n. Then from (1) we obtain

$$
(2)\qquad
\begin{aligned}
x_1' &= 1 \qquad +\kappa x_2 \\
x_2' &= \qquad -\kappa x_2 \qquad +\tau_1 x_3 \\
x_3' &= \qquad\qquad -\tau_1 x_2 \qquad +\tau_2 x_4 \\
&\vdots \\
x_n' &= \qquad\qquad\qquad\qquad -\tau_{n-2} x_{n-1} .
\end{aligned}
$$

2. The inequality. Now suppose that x is closed with at most one corner; if x is not closed let $x(0) = x(L) =$ origin in E^n.

THEOREM. *Let $|x| \leqq R$. Then*

$$
L \leqq R^2\left[\sum_{j=1}^{q}\left|\prod_{k=1}^{j-1}\mu_k\right|\left[\frac{\int\tau_{2j-2}^2\int\tau_{2j-1}^2 - \left(\int\tau_{2j-2}\tau_{2j-1}\right)^2}{\int\tau_{2j-1}^2}\right]^{1/2} + \left|\prod_{k=1}^{q}\mu_k\right|\left[\int\tau_{2q}^2\right]^{1/2}\right]^2,
$$

where $q = [(n - 1/2)]$, $\mu_k = \int\tau_{2k-?}\tau_{2k-1}\Big/\int\tau_{2k-1}^2$, and all the integrals are taken with respect to s over $[0, L]$. Equality holds only if $x([0, L])$ is a circle of radius R in E^2. (Note that for n odd $\tau_{2q} = \tau_{n-1} = 0$ so that the last term in the sum is 0.)

Proof. We rewrite (2) by means of integral formulas. All the integrals are taken with respect to s over $[0, L]$. Since x is either closed or has its "corner" at the origin, we obtain

$$
(3.1)\qquad L = -\int\kappa x_2
$$

$$
(3.\mathrm{i})\qquad 0 = \int\tau_{i-2}x_{i-1} - \int\tau_{i-1}x_{i+1} .
$$

Here $i = 2, \cdots, n$. Let μ_j, $j = 1, \cdots, q$ be arbitrary real numbers. Then $(3\cdot 2j + 1)$, for $j = 0, 1, \cdots, q$ imply

$$
\begin{aligned}
L &= -\int\tau_0 x_2 + \sum_{j=1}^{q}\prod_{j=1}^{j}\mu_k\left[\int\tau_{2j-1}x_{2j} - \int\tau_{2j}x_{2j+2}\right] \\
&= \sum_{j=1}^{q}\prod_{k=1}^{j-1}\mu_k\left[\int\left(\mu_j\tau_{2j-1} - \tau_{2j-2}\right)x_{2j}\right] + \prod_{k=1}^{q}\mu_k\int\tau_{2q}x_{2q+2} .
\end{aligned}
$$

Taking absolute values of each term in the sum and applying the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
L \leqq \sum_{j=1}^{q}&\left|\prod_{k=1}^{j-1}\mu_k\right|\left(\int(\mu_j\tau_{2j-1} - \tau_{2j-2})^2\right)^{1/2}\left(\int x_{2j}^2\right)^{1/2} \\
&+ \left|\prod_{k=1}^{q}\mu_k\right|\left(\int\tau_{2q}^2\right)^{1/2}\left(\int x_{2q+2}^2\right)^{1/2} .
\end{aligned}
$$

But $|x_{2j}| \leqq R$, for $j = 1, 2, \cdots, q + 1$. Also letting

$$\mu_j = \int \tau_{2j-2}\tau_{2j-1} \Big/ \int \tau_{2j-1}^2 ,$$

which minimizes each of the integrals $\int (\mu_j \tau_{2j-1} - \tau_{2j-2})^2$, we establish our inequality.

It is easy to check that equality holds only if $x([0, L])$ is a circle of radius R in E^2. (Remember that we demand that $\tau_i \neq 0$, $i = 1, \cdots, n - 2$.)

REMARK. The inequality in the theorem is sometimes better and sometimes worse than the inequality $L \leqq R \int \kappa$. As an example of a curve for which our inequality is better consider the curve in E^3

$$x(t) = \left(\left(c + \frac{1}{n}\cos t\right)\cos\frac{1}{n^2}t, \left(c + \frac{1}{n}\cos t\right)\sin\frac{1}{n^2}t, \frac{1}{n}\sin t\right),$$

where $0 \leqq t \leqq 2\pi n^2$, $c + 1/n = 1$, and n is a positive integer. This is a curve that winds n^2 times around a torus of radii c and $1/n$. For this curve $R = 1$, $L = O(n)$, $\int \kappa = O(n^2)$, but

$$\frac{\int \kappa^2 \int \tau^2 - \left(\int \kappa\tau\right)^2}{\int \tau^2} = O(n)$$

as $n \to \infty$.

3. Some corollaries. By a theorem of Rutishauser and Samelson [1], we know that any closed curve in E^n of length L is contained inside a sphere of radius $L/4$. Hence we may replace R by $L/4$ in our inequality if x is closed and obtain an inequality involving only L, κ, and τ_i, $i = 1, \cdots, n - 2$. We state the result only for closed curves in E^3.

COROLLARY 1. *Let x be a closed curve in E^3. Then*

$$\frac{16}{L} < \frac{\int \kappa^2 \int \tau^2 - \left(\int \kappa\tau\right)^2}{\int \tau^2}.$$

A similar result holds if x has one corner.

COROLLARY 2. *Let x be a closed curve in E^n with at most one* *corner, where n is odd. It is not the case that $\tau_{2j-2}/\tau_{2j-1} = c_j$, a constant, for $j = 1, \cdots, (n-1)/2$.*

Proof. Since $|x| \leqq R$ for some R we may apply the theorem. If $\tau_{2j-2}/\tau_{2j-1} = c_j$, for $j = 1, 2, \cdots, (n-1)/2$, then $\int\tau_{2j-2}^2\int\tau_{2j-1}^2 - \left(\int\tau_{2j-2}\tau_{2j-1}\right)^2 = 0$, for $j = 1, \cdots, (n-1)/2$. This implies for n odd that $L = 0$, which is an obvious contradiction.

References

1. H. Rutishauser and H. Samelson, *Sur le rayon d'une sphere dont la surface contient une courbe fermée*, C. R. Acad. Sci. Paris, **227** (1948), 755-757.

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