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INVOLUTIONS OF SEIFERT FIBER SPACES

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A Seifert fiber space M is a compact 3-manifold which decomposes into a collection $\mathscr{F}$ of disjoint simple closed curves, called fibers, such that each fiber has a tubular neighborhood which consists of fibers and is a "standard fibered solid torus." We consider the question, given a PL involution h of M, can the fiber structure $\mathscr{F}$ be chosen in such a way that h will be fiber-preserving? We give an affirmative answer for the case when M is orientable, irreducible, and either $\partial M \neq \emptyset$ or M contains an incompressible fibered torus.

THEOREM. *Let h be a PL involution of the orientable, irreducible Seifert fiber space M. If the orbit-surface (Zerlegungsfläche) of the fiber structure is a 2-sphere, assume in addition that there exist at least four exceptional fibers. Then there exists a Seifert fiber structure on M with respect to which h is fiber-preserving.*

This theorem touches on two earier results. In [6] Montesinos considers the following problem. Given any orientable Seifert fiber space M, determine whether M is homeomorphic to a 2-fold cyclic covering of S^3 branched over a link. He shows that all orientable Seifert fiber spaces with orbit-surface either a 2-sphere or a non-orientable surface are such 3-manifolds. For those with an orientable orbit-surface of positive genus, he compiles a list of all those which are 2-fold branched cyclic covering spaces of S^3 with fiber-preserving covering transformations. We can now conclude that this list is complete since it follows from our theorem that all the PL involutions involved as covering transformations can be viewed as fiber-preserving involutions.

In [1] it is shown that if an irreducible, orientable, sufficiently large 3-manifold M is covered by a compact Seifert fiber space then M is either a Seifert fiber space or the union of two twisted line bundles over a closed nonorientable surface. It is not clear whether the union of these two twisted line bundles admits a Seifert fiber structure, but there exists a two-sheeted covering space M' of the union which is a Seifert fiber space. Thus, if one could show that M' always contains an incompressible fibered torus then it would follow that M is a Seifert fiber space.

Let us describe how one goes about constructing a fiber structure which is preserved by an involution. Let h be a PL involution of the Seifert fiber space M. If M is closed we construct a fibered torus T such that $h|T$ is fiber-preserving and such that M can be split along $T \cup h(T)$ to obtain a bounded 3-manifold with involution which is fiber-preserving along the boundary. If M has a boundary already (we may assume that h is fiber-preserving on the boundary) we construct a sequence of fibered annuli and tori on which h is fiber-preserving and which defines an equivariant hierarchy reducing M down to a union of fibered solid tori with the induced involution preserving the fibers along their boundaries. The fibering of these solid tori can then be deformed by an isotopy constant on their boundaries to make the involution fiber-preserving. This process can always be carried out on orientable, sufficiently large Seifert fiber spaces except for those with orbit-surface a 2-sphere and exactly three exceptional fibers. These exceptions are precisely the closed, orientable, sufficiently large Seifert fiber spaces not containing any fibered incompressible tori.

2. Notation and preliminary lemmas. We will work in the piece-wise linear (PL) category exclusively throughout this paper. Let M be a 3-manifold and let F be a surface embedded in M. The surface F is *two-sided* if there is an embedding $c: F \times [-1, 1] \to M$ such that $c(F \times [-1, 1])$ is a neighborhood of F and $c(x, 0) = x$ for all $x \in F$. Thus F is *properly embedded* in M; that is, $F \cap \partial M = \partial F$. We say that a two-sided, connected surface F is *incompressible* in M if $\pi_1(F) \to \pi_1(M)$ is a monomorphism. (We shall omit the base-points and all such unlabeled homomorphisms are assumed to be induced by the inclusion map.) We say that the 3-manifold M' is obtained from M by splitting M along the two-sided surface F if there is a local homeomorphism $p: M \to M'$ such that $p|p^{-1}(M - F)$ is a homeomorphism and $p^{-1}(F) \subset \partial M'$ consists of two copies of F, each mapped homeomorphically by p onto F. If g is a homeomorphism of M such that $g(F) = F$, then there exists a unique homeomorphism g' of M' lifting g.

A *Seifert fiber space* is a compact 3-manifold M which can be decomposed into the union of disjoint simple closed curves (fibers) such that each fiber has a neighborhood, consisting of fibers, that can be mapped by a fiber-preserving homeomorphism onto a fibered solid torus. A *fibered solid torus* of type (μ, ν) is formed when one identifies the ends of the cyclinder $D^2 \times I$ by a homeomorphism of the disk D^2 which is a rotation through an angle of $2\pi\nu/\mu$ degrees, where μ and ν are relatively prime integers $\mu > \nu > 0$. The fibers arise from the joining of the arcs $\{x\} \times I$. Thus the core, arising from $\{0\} \times I$, meets $D^2 \times 0$ once and every other fiber meets $D^2 \times 0$ exactly μ times. If $\mu > 1$, the core (or any fiber of M corresponding to the core) is called an *exceptional fiber* of order μ. The quotient space of M by the fibers is a 2-manifold which we call the *orbit-surface*. A map between two Seifert spaces is said to be *fiber-preserving* if it maps fibers into fibers.

If M is an orientable Seifert fiber space in which $\pi_1(M)$ is infinite then $\pi_1(M)$ has a nontrivial cyclic normal subgroup N generated by the class of an ordinary fiber. In fact, with only a few exceptions, N is the unique maximal cyclic normal subgroup of $\pi_1(M)$.

LEMMA 1 [1]. *Let M be an orientable, irreducible Seifert fiber space with $\partial M \neq \emptyset$. Then the subgroup N generated by the class of a fiber in ∂M is the unique maximal cyclic normal subgroup of $\pi_1(M)$, unless M is $S^1 \times D^2$, $S^1 \times S^1 \times I$, or the orientable S^1-bundle over the Möbius band.*

We shall need to know under what circumstances we can split a Seifert fiber space along a surface F in such a way that the resulting 3-manifold M' is a Seifert fiber space with the projection fiber-preserving. For this we use the following lemma.

LEMMA 2 [1]. *Let M be an orientable, irreducible Seifert fiber space and let F be a two-sided incomplessible surface in M. Let N denote the normal subgroup generated by the class of an ordinary fiber.* (a) *If $N \not\subset \pi_1(F)$, then N is the center of $\pi_1(M)$ and M fibers over S^1 with F as a fiber.* (b) *If $N \subset \pi_1(F)$, then there exists a Seifert fiber structure of M in which F is the union of ordinary fibers.*

With only a few exceptions, the Seifert fiber structure of a given 3-manifold is unique up to isotopy. This follows from the next result due to Waldhausen [11].

LEMMA 3. *Let $f: M \rightarrow N$ be a homeomorphism of the orientable, irreducible, sufficiently large Seifert fiber spaces M and N, which we assume are not one of the following: $S^1 \times S^1 \times S^1$, $S^1 \times S^1 \times I$, an S^1-bundle over the Möbius band, an S^1-bundle over the Klein bottle. Then f is isotopic to a fiber-preserving homeomorphism.*

3. Equivariant pairs of surfaces. Consider a 3-manifold M with an involution h. Given a surface F properly embedded in M, we say that the pair $\{F, h(F)\}$ is *equivariant* if either (i) $h(F) \cap F = \emptyset$ or (ii) $h(F) = F$ and F is in general position with respect to $\mathrm{Fix}(h)$. In a Seifert fiber space M we will want to find equivariant pairs of annuli and tori which are unions of the fibers. The techniques we use are developed in detail in [3] and [5], although some modifications will be necessary.

Assume that the involution h is simplical with respect to some triangulation of M in which the properly embedded surface F can be viewed as a subcomplex. Then F can be moved into an equivariant general position, which we call *h-general position*, by the following procedure. First move F into general position with respect to $\operatorname{Fix}(h)$ and then, using only isotopies constant on $\operatorname{Fix}(h)$, move $F - \operatorname{Fix}(h)$ into general position with respect to $h(F) - \operatorname{Fix}(h)$. (Of course changing F also changes $h(F)$ so one must watch F and $h(F)$ simultaneously during the isotopy.) Now $F \cap h(F)$ is a graph in F with the vertices of the graph contained in $\operatorname{Fix}(h)$. If we start with a surface F which has some components B of ∂F invariant under h and transverse to $\operatorname{Fix}(h)$, we sometimes find it useful to move F into *h-general position modulo B*. We proceed as before except all isotopies will be constant on $B \cup \operatorname{Fix}(h)$. Thus $F - (\operatorname{Fix}(h) \cup B)$ will be in general position with respect to $h(F) - [\operatorname{Fix}(h) \cup B]$ and the graph $F \cap h(F)$ will include B.

LEMMA 4. *Let M be an orientable, irreducible Seifert fiber space with a fiber structure $\mathscr{F}$. Assume that $\partial M \neq \varnothing$ and that M is not $S^1 \times D^2$ or the S^1-bundle over the Möbius band. If h is an involution of M such that $h|\partial M$ is fiber-preserving, then there exists a fiber structure $\mathscr{F}'$ of M and a surface S (either an annulus or a torus) properly embedded in M such that* (a) $\mathscr{F}|\partial M = \mathscr{F}'|\partial M$, (b) $\{S, h(S)\}$ *is equivariant,* (c) $S \cup h(S)$ *is a union of fibers from $\mathscr{F}'$ which are preserved by h,* (d) *if S separates M, then each component of M-S is neither $S \times I$ nor a trivially fibered torus.*

Proof. Procure an annulus S which is a union of fibers and satisfies (d) by lifting an appropriate arc from the orbit surface of $\mathscr{F}$. Such an annulus is always incompressible since it is two-sided and $\pi_1(S)$ is mapped injectively into $\pi_1(M)$.

Case 1. Every fiber in ∂M is invariant under h. Thus each component of ∂S is invariant under h. We use an isotopy constant on ∂M to move S into h-general position modulo ∂S and deform $\mathscr{F}$ at the same time. The desired surface will be constructed from this S.

Suppose there exists a disk E in $h(S)$ such that $E \cap S$ is a simple closed curve J. Let E_1 denote the disk in S bounded by J and let E_2 denote the closure of the other component of S-J. Consider the annulus $S' = E \cup E_2$. Since the 2-sphere $E \cup E_1$ bounds a 3-cell, we can isotope S to S' by an isotopy constant on ∂M. We now need a second isotopy to regain h-general position. Let U denote a small regular neighborhood of E in M such that $U \cap S$ is a regular neighborhood of J in S. Choose a new disk E' close to E in U such that (i) $E' \cap S = \partial E'$; (ii) $E' \cap h(S) = J \cap (\mathrm{Fix}\,(h) \cup \partial S)$; (iii) in E_2, $J \cup \partial E'$ bounds an annulus A pinched along $J \cap (\mathrm{Fix}\,(h) \cup \partial S)$ (that is, A is homeomorphic to the quotient space of $J \times I$ obtained by identifying $y \times I$ to a point for each $y \in ((\mathrm{Fix}\,(h) \cup \partial S) \cap J)$; (iv) the interior of the 3-cell in U bounded by $E \cup A \cup E'$ is disjoint from $S \cap h(S)$. Define S'' to be the annulus $E' \cup (S' - (A \cup E))$. If S'' fails to meet $\mathrm{Fix}\,(h)$ transversally along $J \cap \mathrm{Fix}\,(h)$ we can equivariantly push S'' away from $h(S'')$ at these bad points.

Thus there exists an isotopy constant on ∂M moving S first to S' and then to S''. Observe that the graph $S \cap h(S)$ will have been reduced at least to the extent that $J - (\mathrm{Fix}\,(h) \cup \partial S)$ has been removed.

Now suppose that a disk such as E above does not occur. It follows that the graph $S \cap h(S)$ has no vertices on ∂S (it is not hard to rule out the case in which there is a single vertex on each component of ∂S). Suppose there exists an annulus $E \subset h(S)$ such that $E \cap S = \partial E$ and ∂E has a component K in common with $\partial h(S)$. It follows from Lemma 2 that we may assume $S \cup E$ is a union of fibers. For if we split M along S, we obtain a fibered space M' in which ∂E consists of two fibers in $\partial M'$. By Lemma 2 there is a fibering of M' in which E is the union of fibers. This fibering of M' can be deformed near $\partial M'$ to agree with the original fibering along $\partial M'$ while not changing it along E. This will define the desired fibering of M in which $S \cup E$ is a union of fibers. Let $J = \partial E - K$ and let D denote the annulus in S bounded by $K \cup J$. Let S_1 denote the 2-manifold $U \cup D$ which is invariant under h. In the case when $D \neq S$ we also have another annulus $S_2 = E \cup (S - D)$. If S_2 satisfies property (d), we can recover h-general position for S_2 by methods similar to those used above. Otherwise we must use S_1 which is either a torus or a Klein bottle. First we push S_1 into the interior of M by deforming it slightly in a small neighborhood of K. This can easily be done keeping S_1 invariant under h since $h(K) = K$. Consider a fibered regular neighborhood of S_1 in M which is invariant under h. Either S_1' or S_2 has property (d) except when S_1 is parallel to a component T of ∂M. If S_1' is parallel to T then so is S_1 and we use Theorem 1 of [3] to find an invariant annulus spanning K and a fiber in T. The same argument is used when $D = S$ unless D is parallel to S and then we apply [3]. Thus we get either a simpler annulus or the desired torus.

Case 2. Some fibers in ∂M are interchanged by h. We may assume that S was chosen such that $h(\partial S) \cap (\partial S) = \varnothing$. We proceed exactly an in Case 1 to remove any contractible closed curves in

$S \cap h(S)$. Suppose after this there exists an annulus $E \subset h(S)$ such that E has one boundary component K in common with $\partial h(S)$ and $\partial E - K = E \cap S$. Let D denote an annulus in S having one boundary component J in ∂S and such that $D \cap h(S) = \partial D - J = \partial E - K$. Now consider the two annuli $S_1 = E \cup D$ and $S_2 = E \cup (S - D)$, which we may assume are unions of fibers. At least one of these annuli, say S_i, satisfies (d). Either $h(S_i) = S_i$ or we can regain h-general position for S_i (as before) so as to obtain a simpler graph $S_i \cap h(S_i)$.

After a finite number of such steps this construction will produce the desired surface and deformation of the fiber structure $\mathscr{F}$ to $\mathscr{F}'$.

LEMMA 5. *Let M be a closed orientable Seifert fiber space. When M has a 2-sphere for an orbit-surface assume additionally that there exist at least four exceptional fibers. Let h be an involution of M. Then there exists a torus T and a fiber structure $\mathscr{F}$ for M such that* (a) *$\{T, h(T)\}$ is equivariant, and* (b) *$T \cup h(T)$ is a union of fibers of $\mathscr{F}$ which are preserved by h.*

Proof. First suppose that there exists a nonseparating torus T in M. (Recall that a nonseparating torus in an orientable, irreducible 3-manifold is always incomplessible.)

By using the proof in [5] for Theorem B (substituting "nonseparating" for the phrase "H_1-invariant" throughout), one can construct a nonseparating torus T such that $\{T, h(T)\}$ is equivariant. It follows from Lemmas 1 and 2 that there exists a fiber structure $\mathscr{F}$ for M such that either $T \cup h(T)$ is a union of fibers of $\mathscr{F}$ or T is transverse to the Seifert fibers in $\mathscr{F}$ and M is a torus bundle over S^1 with T and $h(T)$ fibers. In the latter case, we can view M as $T \times [0, 1]/\phi$ with $T = T \times 0$ and where ϕ is a homeomorphism of T of finite order [8]. It is not hard to show (see [2]) that we can parametrize T such that ϕ is given by one of the following: (1) identity, (2) $(x, y) \to (\bar{x}, \bar{y})$, (3) $(x, y) \to (y, \bar{x}\bar{y})$, (4) $(x, y) \to (y, \bar{x})$, (5) $(x, y) \to (y, \bar{x}y)$. In the first two cases there is clearly a Seifert fiber structure in which $T \cup h(T)$ is a union of fibers. In each of the last three cases, the orbit-surface of the natural Seifert fiber structure is a 2-sphere with 3, 4, and 5 exceptional fivers, respectively. Thus the third case is excluded by hypothesis and the last two will be taken care of by the next argument.

Suppose that M does not admit a Seifert fiber structure with respect to which M contains a nonseparating fibered torus. It follows that the orbit-surface is a 2-sphere and, with only one exception, the fiber structure is unique. This lone exception is $\{0; (0_1, 0);$

(2, 1), (2, 1), (2, 1), (2, 1)}, which admits a second Seifert fiber structure $\{0, (n_2, 2)\}$ with respect to which there does exist a fibered nonseparating torus. Thus, all spaces M considered in the remainder of this proof have unique Seifert fiber structures and $\pi_1(M)$ contains a unique maximal cyclic normal subgroup.

Let T be a fibered torus in M such that each component of $M-T$ contains at least two exceptional fibers. Move T into h-general position by an isotopy, deforming the fiber structure at the same time so as to keep T fibered. The torus T can be modified to obtain the desired torus by the methods in [5]. We content ourselves here with merely observing that the constructions in [5] can always be done in such a manner to yield a fibered torus T satisfying (b).

We assume that $T \cap h(T)$ has the minimal complexity among all such fibered tori T. If $T \cap h(T) = \varnothing$ than all we have to is adjust the fiber structure, which we do for all the cases at the end.

Suppose $T \cap h(T) \neq \varnothing$. Let M_1 and M_2 denote the two components of the Seifert fiber space obtained by splitting M along T. Let $A \subset M_1$ denote an innermost component of "$h(T)$ split along $T \cap h(T)$." Since $T \cap h(T)$ is assumed to be minimal, it follows that Int(A) is not an open disk. Thus A is an incompressible annulus in M_1. Let B denote one of the annuli in T bounded by ∂A. Recall that the class of an ordinary fiber in T represents the cyclic normal subgroup $N \subset \pi_1(M_1) \subset \pi_1(M)$.

If $N \subset \pi_1(A)$ then there exists a Seifert fiber structure of M_1 (which can be extended to one on M) in which $A \cup T$ is fibered. It follows that $T' = A \cup B$ is a torus rather than a Klein bottle. (For if T' were a Klein bottle, then the orbit-surface would contain a nonseparating simple closed curve, contradicting the fact that it is a 2-sphere.) If this torus T' is incompressible, then we proceed exactly as in [5] to show that when $h(T') \neq T'$ we can move T' into h-general position with $T' \cap h(T')$ having a lower complexity than $T \cap h(T)$. Hence T' must be compressible. Let X denote the solid torus bounded by T'. We may assume that B is innermost in T, for if not there would be another such pair of annuli available to us inside X. By the minimality of $T \cap h(T)$, the two annuli A and B are not parallel. Thus X must be a fibered torus which contains an exceptional fiber. We can adjust ∂X by an isotopy to obtain an equivariant pair $\{\partial X, h(\partial X)\}$ by the usual technique. Observe that since $h|\partial A$ is already fiber-preserving, the fibering of M can be deformed near X to make $h|\partial X$ fiber-preserving.

If $N \subset \pi_1(A)$, then both M_1 and M_2 are annulus-bundles over S^1 and hence $M = \{b; (n_2, 2)\}$, where the orbit surface for this fiber structure is a Klein bottle. Only in the case when $b = 0$ does such a Seifert fiber space admit a second fiber structure with orbit-surface a 2-sphere and we have already treated this case.

Thus in every case we have shown that there exists a fibered torus T such that the pair $\{T, h(T)\}$ is equivariant. If T happens to be compressible, then $h(T) = T$ and we have already made $h|T$ fiber-preserving. When T is incompressible, it remains to adjust the fiber structure on $T \cup h(T)$ to make $h|T$ fibe-preserving. Let us first consider the case when $\pi_1(M)$ has a unique maximal cyclic normal subgroup N. We may assume $T \cup h(T)$ is a union of fibers in view of Lemma 2. Then h carries a fiber in T onto a simple closed curve in $h(T)$ which is homotopic to a fiber in $h(T)$. Hence the fibering can be deformed so that $h|T$ is fiber-preserving, even when $h(T) = T$. The only two cases when N is not unique are $S^1 \times S^1 \times S^1$ and $\{0; (n_2, 2)\}$. These spaces can be viewed as the torus-bundles $T \times [0, 1]/\phi$, where $T = T \times \{0\}$ and ϕ induces $\phi_* = \pm I$ on $H_1(T; Z)$. According to [10] we can view h as $h([x, t]) = [\beta(x), \lambda(x)]$, for a suitable parametrization where β is an involution of T and $\lambda(t) = t$, $1 - t$, or $t + 1/2$. There exists a simple closed curve $J \subset T$ such that $\beta(J) = J$ or $\beta(J) \cap J = \emptyset$. Since $\phi(J)$ is homotopic to J we can define a Seifert fiber structure of $T \times [0, 1]/\phi$ in which $J \times \{0\}$ and $h(J \times \{0\})$ are fibers and $T \cup h(T)$ is a union of fibers. It follow that the fiber structure can be further adjusted to make $h|T$ fiber-preserving. This completes the proof of the lemma.

4. Fiber-preserving involutions. Given an involution h of a Seifert fiber space M, we are ready to construct a fiber structure on M with respect to which h is fiber-preserving. The first two lemmas deal with two basic Seifert fiber spaces, the fibered solid torus and $S^1 \times S^1 \times I$. In the proof of the main theorem we essentially reduce the consideration of M down to these two spaces, each of which admit infinitely many nonisotopic fiberings.

LEMMA 6. *Let $M = S^1 \times D^2$ and let $\mathscr{F}$ be a Seifert fiber structure for M. Let h be an involution of M such that $h|\partial M$ is fiber-preserving. Then there exists a fiber structure $\mathscr{F}'$ for M such that h preserves the fibers of $\mathscr{F}'$ and $\mathscr{F}'$ agrees with $\mathscr{F}$ on ∂M.*

Proof. We may assume that M is parametrized by $\{(rx, y) | x, y \in S^1$ and $0 \leqq r \leqq 1\}$ in such a way that $h(rx, y) = (r\alpha(x), \beta(y))$ [4]. Let $P_r \colon \partial M \to M$ be the map defined by $P_r(x, y) = (rx, y)$. Define $\mathscr{F}'$ to be the collection $\{P_r(J) | J \subset \partial M$ and $J \in \mathscr{F}\}$.

LEMMA 7. *Let $M = S^1 \times S^1 \times I$ and let $\mathscr{C}_0$ and $\mathscr{C}_1$ be fiberings of $S^1 \times S^1 \times \{0\}$ and $S^1 \times S^1 \times \{1\}$, respectively, by circles which are* *homotopic in M. Suppose that h is an involution of M such that h preserves the fibers of $\mathscr{C}_0$ and $\mathscr{C}_1$. Then there exists a Seifert fiber structure $\mathscr{F}$ of M extending $\mathscr{C}_0 \cup \mathscr{C}_1$ such that h preserves the fibers of $\mathscr{F}$.*

Proof. Extend $\mathscr{C}_0 \cup \mathscr{C}_1$ to a Seifert fiber structure of M. Then there exists a nonseparating fibered annulus A spanning the two components of ∂M. By Lemma 4, we may further assume that $\{A, h(A)\}$ is equivariant and $A \cup h(A)$ is a union of fibers preserved by h. (Recall that M has no exceptional fibers.) Split M along $A \cup h(A)$ to obtain M', a union of disjoint solid tori. If M' consists of two components which are interchanged by h' we may simply refiber one component by the images of the fibers in the other. If each component is invariant under h' then we apply Lemma 6 to M' and h' to complete the proof.

Proof of the main theorem. We first treat the case when M has a nonempty boundary. There exists a hierarchy for M, determined by a sequence of fibered annuli, reducing M down to a union of disjoint fibered solid tori. We define the complexity of M to be the sum of the number of exceptional fibers together with the minimal length of the sequence determining such a hierarchy.

Since we have already considered $S^1 \times D^2$ and $S^1 \times S^1 \times I$, let us assume that M is not homeomorphic to either of these. If M is the S^1-bundle over the Möbius band, observe that $\pi_1(M)$ contains only two maximal cyclic normal subgroups and both of these are characteristic subgroups. Otherwise, there exists a unique maximal cyclic normal subgroup N, which is generated by the class of a fiber in ∂M. It follows in either case that the fibers in ∂M are preserved up to homotopy by h. Hence we can deform the fiber structure of M near ∂M to make $h|\partial M$ fiber-preserving. We use an induction argument on the complexity of M to prove the theorem.

It follows from Lemmas 6 and 7 that the theorem holds for M with a complexity equal to one. Thus assume the induction hypothesis: Let h be an involution of M such that $h|\partial M$ is fiber-preserving. If the complexity of M is $\leqq n$, then the fibering of M can be redefined, leaving it unchanged on ∂M, such that h is fiber-preserving.

Suppose that M has a complexity equal to $n+1$. It follows from Lemma 4 (except when M is S^1-hundle over the Möbius band) that there exists a deformation of the fibering, constant on ∂M, and an equivariant pair of surfaces $\{A, h(A)\}$ such that $A \cup h(A)$ is fibered, $h|A \cup h(A)$ is fiber-preserving, and splitting M along $A \cup h(A)$ gives a Seifert fiber space M' for which each component has a complexity less than or equal to n. We apply the induction hypothesis to each invariant component of M'. It is easy to redefine the fiber structure on any components which are interchanged by h' to make h' fiber-preserving on these. Together, these fiber structures define a new fiber structure of M with respect to which h is fiber-preserving. In case M is the S^1-bundle over the Möbius band, we apply the method used in the proof of Lemma 4 to construct an equivalent pair $\{A, h(A)\}$ as above. If the construction is successful we can proceed as before. Notice however, that the construction fails only when we encounter an invariant Klein bottle with an invariant fibered neighborhood U such that $\overline{M - U} \approx \partial M \times I$. In this case we may deform the fibering on $\bar{U}$ such that $h | \bar{U}$ is fiber-preserving and then apply Lemma 7 to extend the fibering of $\partial M \cup \bar{U}$ over $\overline{M - U}$ in such a way that h will preserve the fibers on M. This completes the inductive step.

Now suppose that M is a closed 3-manifold. By hypothesis, M contains a fibered incompressible torus. Thus it follows from Lemma 5 that there exists a Seifert fiber structure for M and a fibered torus T such that $\{T, h(T)\}$ is equivariant and $h | T \cup h(T)$ is fiber-preserving. Split M along $T \cup h(T)$ to obtain the Seifert fiber space M' with involution h' which is fiber-preserving along $\partial M'$. It follows from the previous case that there is a Seifert fiber structure of M', agreeing with the original on $\partial M'$, with respect to which h' is fiber-preserving. This defines the desired fiber structure on M.

REFERENCES

1. C. McA. Gordon and W. Heil, *Cyclic normal subgroups of fundamental groups of 3-manifolds*, Topology, **14** (1975), 305–309.

2. J. Hempel, *Free cyclic actions on* $S^1 \times S^1 \times S^1$, Proc. Amer. Math. Soc., **48** (1975), 221–223.

3. P. Kim and J. Tollefson, *Pl involutions of fibered 3-manifolds*, Trans. Amer. Math. Soc., **232** (1977), 221–237.

4. ———, *Splitting the PL involutions on nonprime 3-manifolds*, Michigan Math. J., (to appear).

5. K. Kwun and J. Tollefson, *PL involutions of* $S^1 \times S^1 \times S^1$, Trans. Amer. Math. Soc., **203** (1975), 97–106.

6. J. Montesinos, *Variedades de Seifert que son recubridores ciclicos ramificados de dow hojas*, Bol. Soc. Math. Mexicana, **18** (1973), 1–32.

7. P. Orlik, *Seifert Manifolds*, Lecture Notes in Mathematics, Vol. **291**, Springer-Verlag, Berlin (1972).

8. P. Orlik and F. Raymond, *On 3-manifolds with local SO(2) action*, Quart. J. Math. Oxford (2), **20** (1969), 143–160.

9. H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math., **60** (1933), 147–238.

10. J. Tollefson, *Periodic homeomorphisms of 3-manifolds fibered over* S^1, Trans. Amer. Math. Soc., **223** (1976), 223–234.

11. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten II*, Invent. Math., **4** (1976), 87–117.

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