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ON CHARACTERISTIC HYPERSURFACES OF SUBMANIFOLDS IN EUCLIDEAN SPACE

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The main purpose of this paper is to prove that $M^n \subset E^N$, where $N = n(n+1)/2$, the characteristic $(n-1)$-dimensional submanifolds of M^n are the asymptotic hypersurfaces.

1. Introduction. The concept of a characteristic submanifold of a given solution for a differential system, was introduced by E. Cartan in his theory of partial differential equations ([2], p. 79). Its importance appears in the treatment of the Cauchy problem.

Given an n-dimensional submanifold M^n of the Euclidean space E^N, we can define geometrically the notion of asymptotic submanifolds of M^n. The asymptotic lines have been used extensively for the study of the geometry of a surface in E^3. For higher dimension and codimension some results have been obtained, using the generalized concept [3], [4], [9], [10]. It is well known, that the characteristic curves of a surface in E^3 are the asymptotic lines ([2], p. 143).

In §2 we start with a brief introduction to the Cartan-Kähler theory of differential equations. Then given a Riemannian manifold M^n, we consider the differential ideal, whose integral submanifolds determine local isometries of M^n into E^N, $N = n(n+1)/2$. Next assuming $M^n \subset E^N$, we characterize the $(n-1)$-dimensional characteristic submanifolds of M^n.

In §3, we define the concept of asymptotic submanifolds of $M^n \subset E^N$, prove the main result and obtain a first order partial differential equation whose solutions are the characteristic hypersurfaces of M.

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2. Characteristic submanifold. Let M be an n-dimensional differentiable manifold. We denote by $\Lambda_k(M)$ the vector space of differential k-forms on M and $\Lambda(M) = \sum_{k=0}^{n} \Lambda_k(M)$. A *differential ideal* is an ideal U in $\Lambda(M)$ which is finitely generated, homogeneous (i.e., $U = \sum_{k=0}^{n} U_k$ where $U_k = U \cap \Lambda_k(M)$) are closed under exterior differentiation. We assume that U is a differential ideal which does not contain functions i.e., $U_0 = 0$. A p-dimensional submanifold S of M is said to be an (p-dimensional) *integral submanifold* for U, if $i^*(U) = 0$ i.e., $i^*(U_p) = 0$ where $i: S \to M$ is the inclusion map.

We denote by T_xM the tangent space to M at $x \in M$; $G_x^p(M)$ denotes the Grassman manifold of p-dimensional subspaces of T_xM

and $G^p(M) = \bigcup_{x\in M} G^p_x(M)$ is given the usual manifold structure. An element $E^p_x \in G^p_x(M)$ is said to be an *integral element* for U, if all the differential forms of U vanish when restricted to the elements of E^p_x.

Let $I^p_x(U)$ denote the set of p-dimensional integral elements for U at x, and let $I^p(U) = \bigcup_{x\in M} I^p_x(M)$ be given the topology as a subspace of $G^p(M)$. If E^p_x is an integral element for U generated by $\{v_1, \cdots, v_p\}$, we define the *polar space* $H(E^p_x)$ by

$$H(E^p_x) = \{v \in T_xM; \phi(v, v_1, v_2, \cdots, v_p) = 0, \forall\phi \in U_{p+1}\} .$$

An integral element E^p_x, $p \geqq 1$ is said to be *ordinary* if there exist integral elements $E^0_x, E^1_x, \cdots, E^{p-1}_x$ with $E^0_x \subset E^1_x \subset \cdots E^{p-1}_x \subset E^p_x$ such that $\dim H(E^i_x)$ is constant on a neighborhood of E^i_x in $I^i(U)$ for $i = 0, 1, \cdots, p-1$. A zero-dimensional integral element E^0_x is said to be *regular* if $\dim H(E^0_x)$ is constant on a neighborhood of E^0_x in $I^0(U)$. A p-dimensional integral element E^p_x, $p \geqq 1$ is said to be *regular* if it is ordinary and $\dim H(E^p_x)$ is constant on a neighborhood of E^p_x in $I^p(U)$. We remark that when M is connected, this definition of regularity is equivalent to Cartan's ([2], pp. 61-67) according to which, an integral element E^p_x is regular if it is ordinary and $\dim H(^p_x)$ is equal to the dimension of a generic p-dimensional ordinary integral element.

It follows from Cartan-Kähler theorem ([2, pp. 68-74], [7, p. 26]) under the assumption that the manifold M and the differential forms are analytic, that given a q-dimensional ordinary integral element E^q_x, then there exists a q-dimensional integral submanifold S, which contains x ond satisfies the requirement $T_xS = E^q_x$.

An integral submanifold S for U is said to be *singular* if $\forall x \in S$, the integral element T_xS is not ordinary. We remark, that an integral submanifold S may be singular because none of its points is regular, or none of its tangential subspaces of dimension one, or two, $\cdots$, etc., or $p-1$ is regular, where p is the dimension of S. Hence one may have different classes of singular integral submanifolds, whose degree of singularity decreases in a certain sense when one goes from one class to the next one.

Let S be a p-dimensional nonsingular integral submanifold for U, a submanifold $\bar{S} \subset S$ of dimension $q < p$ is called *characteristic* if $\forall x \in \bar{S}$, the integral element $T_x\bar{S}$ is not regular.

The concepts introduced above, can be found with more details in [2] and [7]. The Cartan-Janet theorem [1], [6] asserts that any real analytic, n-dimensional, Riemannian manifold can be locally mapped by a real analytic isometric embedding, into a Euclidean space E^N of dimension $N = n(n+1)/2$. In what follows we consider the differential ideal, whose integral submanifolds give local isometries of M into E^N. Next assuming $M \subset E^N$, we characterize the $(n-1)$-dimensional characteristic submanifolds of M. We adopt the following indices convention

$$1 \leqq i, j, k, l \leqq n; \qquad n+1 \leqq \lambda, \mu, \alpha \leqq N;$$
$$1 \leqq I, J, K \leqq N; \qquad N = n(n+1)/2$$

and the summation convention with regard to repeated indices.

Let M be an n-dimensional Riemannian manifold with metric g. Let $F(M)$ denote the bundle of orthonormal frames over M, with the usual manifold structure. Under the action of the orthogonal group $O(n)$, $F(M)$ is a principal fiber bundle over M, with structural group $O(n)$. Let $\pi: F(M) \to M$ be the usual projection. We define the canonical forms $\omega^1, \cdots, \omega^n$ on $F(M)$ by

$\pi_{*z}(v) = \omega^i(v)e_i$ where $z = (x, e_1, \cdots, e_n) \in F(M)$ and $v \in T_z(F(M))$, hence $\pi^* g = \sum_i \omega^i \otimes \omega^i$. The connection forms ω_i^j on $F(M)$ are uniquely defined by

$$d\omega^i = \omega^j \wedge \omega_j^i , \qquad \omega_i^j + \omega_j^i = 0 .$$

Finally, if we consider

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

then there exist functions R_{ijkl}, the components of the Riemann curvature tensor, defined on $F(M)$ such that

$$\Omega_i^j = -\frac{1}{2} R_{ijkl} \omega^k \wedge \omega^l , \qquad R_{ijkl} = -R_{ijlk} .$$

Similarly for E^N, we denote by $F(E^N)$ the bundle of orthonormal frames over E^N, $\bar{\pi}: F(E^N) \to E^N$ the projection, $\bar{\omega}^I$ the canonical forms on $F(E^N)$, $\bar{\omega}_I^J$ the connection forms on $F(E^N)$.

We consider the product manifold $B = F(M) \times F(E^N)$, and define the differential ideal on B. Let $\rho: B \to F(M)$ and $\bar{\rho}: B \to F(E^N)$ be the usual projections. Using ρ and $\bar{\rho}$ we can pull the differential forms ω^i, ω_i^j, $\bar{\omega}^I$, $\bar{\omega}_I^J$ back to B, we will denote the pulled-back forms by the same symbols. Let U be the differential ideal on B generated by

$$(*) \qquad \begin{aligned} &\bar{\omega}^i - \omega^i \\ &\bar{\omega}^\lambda \\ &\bar{\omega}_i^j - \omega_i^j \\ &\omega^i \wedge \bar{\omega}_i^\lambda \\ &\bar{\omega}_i^\lambda \wedge \bar{\omega}_\lambda^j + \frac{1}{2} R_{ijlk} \omega^l \wedge \omega^k . \end{aligned}$$

We remark that there is a left action of $O(n)$ on B which preserves the differential ideal U. Namely if $A = (a_{ij}) \in O(n)$ we consider $L_A: B \to B$, which associates to

$$z = ((x, e_1, \cdots, e_n), (\bar{x}, \bar{e}_1, \cdots, \bar{e}_N)) \in B$$

the point

$$L_A(z) = \Big(\Big(x, \sum_i a_{1i}e_i, \cdots, \sum_i a_{ni}e_i\Big), \Big(\bar{x}, \sum_i a_{1i}\bar{e}_i, \cdots, \sum_i a_{ni}, \bar{e}_i, \bar{e}_{n+1}, \cdots, \bar{e}_N\Big)\Big).$$

It is not difficult to verify that $L_A^*(U \cap \Lambda_1(B)) \subset U \cap \Lambda_1(B)$ and hence $L_A^*(U) = U$.

Since we want to determine the $(n -)$-dimentional characteristic submanifolds of $M^n \subset E^N$, we start characterizing the nonregular $(n-1)$-dimensional integral elements E_z^{n-1} for U in B, whose projections $\pi_* \circ \rho_*(E_z^{n-1})$ are $(n-1)$-dimensional. This characterization is obtained in Lemma 1(c).

Let p be an integer $0 \leqq p < n$, we adopt the additional index conventions

$$1 \leqq a, b, c \leqq p\ ; \qquad p + 1 \leqq r, s, t \leqq n\ .$$

Suppose that E_z^p is a p-dimentional integral element for U, generated by vectors $e_1, \cdots, e_p$ such that

$$\omega^a(e_b) = \delta_b^a\ , \qquad \omega^r(e_b) = 0\ .$$

If we denote, $h_{ia}^\lambda = \bar{\omega}_i^\lambda(e_a)$ then it follows, from the fact that the generators of U vanish when restricted to E_z^p, that

$$h_{ab}^\lambda = h_{ba}^\lambda \tag{1}$$

$$\sum_\lambda (h_{ia}^\lambda h_{jb}^\lambda - h_{ib}^\lambda h_{ja}^\lambda) - R_{ijab} = 0\ . \tag{2}$$

Denote by

$$H_{ia} = (h_{ia}^{n+1}, \cdots, h_{ia}^N)$$

the vector in the $(N - n)$-dimentional Euclidean space.

Let J^p denote the set of p-dimensional integral elements E_z^p, which satisfy the following conditions:

1. $\omega^1 \wedge \cdots \wedge \omega^p \neq 0$ and $\omega^{p+1} = \cdots = \omega^n = 0$ when restricted to E_z^p.

2. the vectors $\{H_{ma}: 1 \leqq a \leqq p, a \leqq m \leqq n-1\}$ are linearly independent. Let $V^p = \{E_z^p \in I^p(U): L_{A^*}(E_z^p) \in J^p$ for some $A \in O(n)\}$.

Then V^p is an open subset of $I^p(U)$. Part of the next lemma is proved following ([5], with the obvious modifications).

LEMMA 1.
(a) *If* $0 \leqq p < n$, *then* $\dim H(E_z^p)$ *is constant on* V^p;
(b) *For* $0 \leqq p < n$, *if* $E_z^p \in V^p$, *then it is a regular element*;
(c) *If* $p = n-1$, *and* E_z^{n-1} *is an integral element such that* $\pi_* \circ \rho_*(E_z^{n-1})$ *is* $(n-1)$-*dimensional, then* E_z^{n-1} *is regular if and only if* $E_z^{n-1} \in V^{n-1}$.

Proof. (a) Since $L_A^*(U) = U$ it suffices to show that $\dim H(E_z^p)$ is constant on J^p. Assume that E_z^p is generated by $e_1, \cdots, e_p$ such that $\omega^a(e_b) = \delta_b^a$ and $\omega^r(e_b) = 0$. We consider the polar space

$$\begin{aligned} H(E_z^p) &= \{v \in T_zB;\, \phi(v, e_1, \cdots, e_p) = 0 \forall \phi \in U_n\} \\ &= \{v \in T_zB;\, \phi_1(v) = 0 \text{ and } \phi_2(v, e_a) = 0 \forall \phi_1 \in U_1,\ \phi_2 \in U_2\} \end{aligned}$$

where last equality follows from the fact that U is generated by (*). Hence $H(E_z^p)$ consists of vectors $v \in T_zB$ which satisfy the following system of equations:

$$\bar{\omega}^i(v) - \omega^i(v) = 0 \tag{3}$$

$$\bar{\omega}^\lambda(v) = 0 \tag{4}$$

$$\bar{\omega}_i^j(v) - \omega_i^j(v) = 0 \tag{5}$$

$$h_{ia}^\lambda \omega^i(v) - \bar{\omega}_a^\lambda(v) = 0 \tag{6}$$

$$\sum_\lambda h_{ja}^\lambda \bar{\omega}_i^\lambda(v) + \sum_\lambda h_{ia}^\lambda \bar{\omega}_\lambda^j(v) - R_{ijla}\omega^l(v) = 0\ , \qquad i < j\ . \tag{7}$$

If we specify $\omega^i(v)$, $\omega_i^j(v)$ then equations (3)-(6) will uniquely determine $\bar{\omega}^I(v)$, $\bar{\omega}_i^j(v)$ and $\bar{\omega}_a^\lambda(v)$. Moreover we remark that for $1 \leqq i, j \leqq p$, equation (7) is an immediate consequence of (1), (2) and (6). So we need only to consider (7) where $1 \leqq i \leqq p$, $p+1 \leqq j \leqq n$ and $p+1 \leqq i < j \leqq n$, i.e.,

$$\begin{aligned} &\sum_\lambda h_{sa}^\lambda \bar{\omega}_b^\lambda(v) + \sum_\lambda h_{ba}^\lambda \bar{\omega}_\lambda^s(v) - R_{bsla}\omega^l(v) = 0 \\ &\sum_\lambda h_{sa}^\lambda \bar{\omega}_t^\lambda(v) + \sum_\lambda h_{ta}^\lambda \bar{\omega}_\lambda^s(v) - R_{tsla}\omega^l(v) = 0\ . \end{aligned} \tag{8}$$

Since in (8), for $a \neq b$, interchanging a and b does not modify the equation, we need only to consider

$$\sum_\lambda h_{ba}^\lambda \bar{\omega}_s^\lambda(v) = \Big(\sum_\lambda h_{sa}^\lambda h_{ib}^\lambda - R_{bsia}\Big)\omega^i(v)\ , \qquad a \leqq b \tag{9}$$

$$\sum_\lambda h_{sa}^\lambda \bar{\omega}_t^\lambda(v) - \sum_\lambda h_{ta}^\lambda \bar{\omega}_s^\lambda(v) = R_{tsia}\omega^i(v)\ , \qquad s < t\ . \tag{10}$$

Denote the vectors

$$H_i(v) = (\bar{\omega}_i^{n+1}(v), \cdots, \bar{\omega}_i^N(v)) .$$

We determine the vectors $H_{p+1}(v), \cdots, H_n(v)$ so that they satisfy (9) and (10). The system (9) determines the dot product of $H_{p+1}(v)$ with the $p(p+1)/2$ linearly independent vectors H_{ba}, $a \leqq b$. Once we have chosen a particular $H_{p+1}(v)$ which satisfies this liner system of rank $p(p+1)/2$, the dot product of $H_{p+2}(v)$ with each of the $p(p+1)/2 + p$ linearly independent vectors $\{H_{ma}: 1 \leqq a \leqq p, a \leqq m \leqq p+1\}$ is completely determined by (9) and (10). We continue in this fashion. Finally we find that the dot product of $H_n(v)$ with each of the $p(p+1)/2 + p(n-p-1)$ linearly independent vectors $\{H_{ma}: 1 \leqq a \leqq p, a \leqq m \leqq n-1\}$ is completely determined. Hence we find that $\bar{\omega}_t^\lambda(v)$ must satisfy a consistent system of linear equations which has rank $np(n-p)/2$. The polar system of E_z^p consists of these equations together with (3)-(6). Hence $\dim H(E_z^p)$ depends only on n and p whenever $E_z^p \in J^p$.

(b) Suppose that $E_z^p \in J^p$ is generated by $e_1, \cdots, e_p$, such that $\omega^a(e_b) = \delta_b^a$ and $\omega^r(e_b) = 0$. If $0 \leqq q \leqq p$, we let E_z^q be the q-dimensional integral element generated by $e_1, \cdots, e_q$. Then $E_z^q \in J^q$ and hence $\dim H(E_z^q)$ is constant in a neighborhood of E_z^q in $I^q(U)$. It follows that E_z^p is regular. Consequently if $E_z^p \in V^p$, then it is a regular integral element.

(c) From (b) we only need to prove that if E_z^{n-1} is a regular integral element then $E_z^{n-1} \in V^{n-1}$. Since $\pi_* \circ \rho_*(E_z^{n-1})$ is $(n-1)$-dimensional, we can find an element $A \in O(n)$ such that $\omega^n = 0$ on $L_{A*}(E_z^{n-1})$. Hence, we can assume that E_z^{n-1} is generated by $e_1, \cdots, e_{n-1}$, such that $\omega^a(e_b) = \delta_b^a$ and $\omega^n(e_b) = 0$, where $1 \leqq a, b \leqq n-1$. Since E_z^{n-1} is regular, it follows that $\dim H(E_z^{n-1})$ is constant in a neighborhood of E_z^{n-1} in $I^{n-1}(U)$. The polar system of E_z^{n-1} is given by (3)-(6) and (7) reduces to

$$\sum_\lambda h_{ba}^\lambda \bar{\omega}_n^\lambda(v) = \Big(\sum_\lambda h_{na}^\lambda h_{ib}^\lambda - R_{bnia}\Big)\omega^i(v) , \qquad a \leqq b . \tag{11}$$

As in (a) if we specify $\omega^i(v)$, $\omega_i^j(v)$ then $\bar{\omega}^I(v)$, $\bar{\omega}_i^j(v)$ and $\bar{\omega}_a^\lambda(v)$ will be uniquely determined by (3)-(6). Moreover the $n(n-1)/2$ components $\bar{\omega}_n^\lambda(v)$ must satisfy the linear system (11) which has exactly $n(n-1)/2$ equations. Hence, if $\dim H(E_z^{n-1})$ is constant in a neighborhood of E_z^{n-1}, then the determinant of the coefficient matrix in (11) is nonzero, i.e., the vectors $\{H_{ba}: 1 \leqq a \leqq b \leqq n-1\}$ are linearly independent, which implies $E_z^{n-1} \in J^{n-1}$.

Let M be an n-dimensional Riemannian manifold and $f: M \to E^N$ an isometric imbedding. If $x_0 \in M$, there exists a neighborhood V of

x_0 in M and a section $\bar{\sigma}$: $V \rightarrow F(E^N)$ such that if $\bar{\sigma}(x) = (f(x), \bar{e}_1(x), \cdots, \bar{e}_N(x))$, then $\bar{e}_1(x), \cdots, \bar{e}_n(x)$ are tangent to $f(M)$. We consider the section σ: $V \rightarrow F(M)$, defined by $\sigma(x) = (x, e_1(x), \cdots, e_n(x))$ where $f_*(e_i(x)) = \bar{e}_i(x)$. For simplicity, we denote by ω^i, ω_i^j the differential forms $\sigma^*\omega^i$, $\sigma^*\omega_i^j$ induced on V and similarly $\bar{\omega}^I$, $\bar{\omega}_I^J$ will denote the pulled-back forms $\bar{\sigma}^*\bar{\omega}^I$, $\bar{\sigma}^*\bar{\omega}_I^J$ on V. Consider the map Γ: $V \rightarrow B$ defined by $\Gamma(x) = (\sigma(x), \bar{\sigma}(x))$. Since f is an isometry, $\Gamma(V)$ is an integral submanifold for U in B. We say that a q-dimensional vector space $L^q \subset T_{x_0}M$, $0 \leqq q < n$ is regular if $\Gamma_*(L)$ is a regular integral element for U. Similarly, a q-dimensional submanifold S of V is said to be characteristic, if $\Gamma(S)$ is a characteristic submanifold of $\Gamma(V)$. The characteristic hypersurfaces of M have at each point a nonregular tangent space. Our next lemma characterizes the nonregular $(n-1)$-dimensional spaces tangent to M.

We denote the matrix $H^\lambda = (h_{ij}^\lambda)$ where $h_{ij}^\lambda = \bar{\omega}_i^\lambda(e_j)$. Moreover, given a matrix A, we denote by A_b the bth row of A and A_b^t denotes the transpose of A_b. Assume $\Gamma(V)$ is not a singular integral submanifold for U, then as an immediate concequence of Lemma 1(c), we obtain

LEMMA 2. *Let $u_i\omega^i = 0$ be an $(n-1)$-dimensional subspace of $T_{x_0}M$. We may assume that $\sum_{i=1}^n u_i^2 = 1$. Choose $A = (a_{ij}) \in O(n)$ such that $a_{ni} = u_i$. Then $u_i\omega^i = 0$ is nonregular if and only if the vectors*

$$(A_aH^{n+1}A_b^t, \cdots, A_aH^NA_b^t) , \qquad 1 \leqq a \leqq b \leqq n-1$$

are linearly dependent, as vectors in E^{N-n}.

We remark that this condition determines a first order partial differential equation, and the characteristic hypersurfaces of M are the solutions of this equation. In the next section as a consequence of Lemma 3, the partial differential equation will be given in another form, which will not involve the choice of matrix A.

3. Asymptotic submanifolds; proof of main result. Let M be an n-dimensional C^∞ submanifold of E^N, $N = n(n+1)/2$ with the induced metric and such that the inclusion i: $M \rightarrow E^N$ is nondegenerate. Let $x \in M$ and denote by s the second fundamental form. A q-dimensional $0 < q < n$ linear subspace L of the tangent space T_xM is called *asymptotic* if there exists a vector ξ normal to T_xM such that $\langle s(X, Y), \xi \rangle = 0$, $\forall X, Y \in L$ where $\langle\ ,\ \rangle$ denotes the Euclidean metric. If L is of codimension one, we have an *asymptotic hyperplane* at x. A q-dimensional submaniford V of M, $q < n$ is called *asymptotic at* $x \in V$ if T_xV is asymptotic and asymptotic if this is true for each $x \in V$. It is not difficult to see that V is an asymptotic hypersurface of M if and only if there exists a normal to the osculating space of V, which is also normal M. The notation of asymptotic submanifold in a more general context can be found in [4].

Let $e_1, \cdots, e_N$ be an orthonormal frame defined on a neighborhood of $x \in M$, such that $e_1, \cdots, e_n$ are tangent to M and $e_{n+1}, \cdots, e_N$ are normal to M. Let $\omega^1, \cdots, \omega^N$ be the dual frame. With the same indices convention as in §2, we denote by $h_{ij}^\lambda = \omega_i^\lambda(e_j)$ where ω_i^λ are the connection forms. It follows from the definition that a hyperplane $u_i\omega^i = 0$ is asymptotic if and only if the second fundamental forms $h_{ij}^\lambda\omega^i \otimes \omega^j$ are linearly dependent when restricted to $u_i\omega^i = 0$.

The following algebraic lemma shows that the condition obtained in Lemma 2 is eqivalent to saying that $u_i\omega^i = 0$ is asymptotic. As in §2 given a matrix A we denote by A_b the bth row of A and A_b^t denotes the transpose of A_b.

LEMMA 3. *Let φ^λ be $n \times n$ symmetric matrices $\lambda = n+1, \cdots, N$. $N = n(n+1)/2$ and let $A = (a_{ij}) \in O(n)$. Then the vectors*

$$(A_b\varphi^{n+1}A_c^t, \cdots, A_b\varphi^N A_c^t), \qquad 1 \leqq b \leqq c \leqq n-1$$

are linearly dependent, as vectors in E^{N-n}, if and only if the quadratic forms $\varphi_{ij}\omega^i \otimes \omega^j$ are linearly dependent when restricted to $a_{ni}\omega^i = 0$, where ω^i are n independent 1-forms.

Proof. The vectors $(A_b\varphi^{n+1}A_c^t, \cdots, A_b\varphi^N A_c^t)$ are linearly dependent iff $\exists \alpha_\lambda \in \boldsymbol{R}$ not all zero, such that

$$A_b\Big(\sum_{\lambda=n+1}^{N} \alpha_\lambda\varphi^\lambda\Big)A_c^t = 0, \qquad \forall 1 \leqq b \leqq c \leqq n-1.$$

We denote by D the matrix $D = \sum_\lambda \alpha_\lambda\varphi^\lambda$ and $W = (\omega^1, \cdots, \omega^n)$. We will prove that $A_bDA_c^t = 0 \quad \forall 1 \leqq b \leqq c \leqq n-1$ if and only if $WDW^t = 0$ whenever $A_nW^t = 0$.

Consider

$$WDW^t = WA^t(ADA^t)AW^t. \tag{12}$$

Suppose $A_bDA_c^t = 0$, $\forall 1 \leqq b \leqq c \leqq n-1$, then since D is symmetric

$$WDW^t = [WA_1, \cdots, WA_{n-1}^t\, WA_n^t]\begin{bmatrix} & & A_1DA_n^t \\ & 0 & \vdots \\ & & A_{n-1}DA_n^t \\ A_nDA_1^t & \cdots & A_nDA_n^t \end{bmatrix}\begin{bmatrix} A_1W^t \\ \vdots \\ A_{n-1}W^t \\ A_nW^t \end{bmatrix}.$$

Hence if $A_n W^t = 0$ then $WDW^t = 0$, i.e., the quadratic forms $W\varphi^\lambda W^t$ are linearly dependent whenever $A_n W^t = 0$.

Conversely, suppose $WDW^t = 0$ when $A_n W^t = 0$, then it follows from (12) that

$$0 = \sum_{b=1}^{n-1} A_b D A_b^t \Big(\sum_{k=1}^{n} a_{bk}\omega^k\Big)^2 + 2 \sum_{\substack{b,c=1 \\ b<c}}^{n-1} A_b D A_c^t \Big(\sum_{k,l=1}^{n} a_{bk} a_{cl} \omega^k \otimes \omega^l\Big) . \tag{13}$$

Let e_i be the dual basis of ω^i, i.e., $\omega^i(e_j) = \delta^i_j$. If we evaluate (13) at the pair (e_k, e_k) we get

$$\sum_{b=1}^{n-1} A_b D A_b^t a_{bk}^2 + 2 \sum_{\substack{b,c=1 \\ b<c}}^{n-1} A_b D A_c^t a_{bk} a_{ck} = 0 , \qquad \forall k = 1, \cdots, n .$$

Adding over k, since $A \in O(n)$ we get

$$\sum_{b=1}^{n-1} A_b D A_b^t = 0 . \tag{14}$$

If we apply (13) to the pairs $(e_k, e_l)(e_l, e_k) l \neq k$ and subtract we get

$$\sum_{\substack{b,c=1 \\ b<c}}^{n-1} A_b D A_c^t (a_{bk} a_{cl} - a_{bl} a_{ck}) = 0 , \qquad \forall 1 \leqq k \leqq l \leqq n . \tag{15}$$

This is an homogeneous linear system of $n(n-1)/2$ equations with $(n-1)(n-2)/2$ unknowns $A_b D A_c^t$, $1 \leqq b < c \leqq n-1$. We claim that the rank of this system is $(n-1)(n-2)/2$. In fact, otherwise it follows from Sylvester-Franke theorem on determinants ([8], p. 94, take $m = 2$), that the cofactor of a_{ni} in A is zero, $\forall i = 1, \cdots, n$, which contradicts the fact that $\det A \neq 0$. Hence from (15) we have that

$$A_b D A_c^t = 0 , \qquad 1 \leqq b < c \leqq n-1 . \tag{16}$$

Now (13) reduces to

$$\sum_{b=1}^{n-1} A_b D A_b^t \Big(\sum_{k=1}^{n} a_{bk}\omega^k\Big)^2 = 0 \tag{17}$$

and from (14) we have

$$A_{n-1} D A_{n-1}^t = - \sum_{b=1}^{n-2} A_b D A_b^t . \tag{18}$$

If we substitute (18) in (17) we get

$$\sum_{b=1}^{n-2} A_b D A_b^t \Big(\sum_{k=1}^{n} (a_{bk} - a_{n-1k})\omega^k\Big)\Big(\sum_{k=1}^{n} (a_{bk} + a_{n-1k})\omega^k\Big) = 0 .$$

Applying this equation to the pairs of vectors (e_k, e_l), (e_l, e_k), $l \neq k$ and subtracting we get

$$\sum_{b=1}^{n-2} A_b D A_b^t (a_{bk} a_{n-1l} - a_{n-1k} a_{bl}) = 0 , \qquad 1 \leqq k < l \leqq n .$$

This is a linear system of $n(n-1)/2$ equations with $n-2$ unknowns $A_b D A_b^t$, $1 \leqq b \leqq n-2$. The rank of this system is $n-2$. Otherwise, using Laplace's development of a determinant in the general version (i.e., the determinant is a linear function of the minors comprised in any number of lines) we get that the system (15) has rank lower than $(n-1)(n-2)/2$, which is a contradiction. Therefore $A_b D A_b^t = 0$ for $b = 1, \cdots, n-2$ and finally from (16) and (18) we conclude that $A_b D A_c^t = 0$ $\forall 1 \leqq b \leqq c \leqq n-1$.

Let $f: M \to E^N$ be an isometric embedding, with the same notation as in 2, we say that f is *singular* if $\forall x \in M$, $\Gamma_*(T_x M)$ is not an ordinary integral element for U in B. Then our main result follows immediately from Lemmas 2 and 3:

THEOREM. *Let $f: M \to E^N$ be a nonsingular isometric imbedding. An $(n-1)$-dimensional submanifold of M is characteristic if and only if it is asymptotic.*

We remark that f being nonsingular implies that f is nondegenerate, but for $n > 2$ it may exist a nondegenerate isometric imbedding which is singular; in this case all hypersurfaces would be asymptotic.

We observe that it is not difficult to prove that $u_i \omega^i = 0$ is asymptotic if and only if there exist real numbers a_λ, b_i not all zero, such that

$$a_\lambda h^\lambda_{ij} \omega^i \otimes \omega^j \equiv u_i \omega^i \otimes b_j \omega^j .$$

This reduces to a homogeneous equation in u_i of degree n, $P(u_1, u_2, \cdots, u_n) = 0$. In order to describe the polynomial P we consider the matrices

$$U_0 = \begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix} \qquad U_p = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ u_{p+1} & u_{p+2} & \cdots & u_n \\ u_p & & & 0 \\ & u_p & & \\ & & \ddots & \\ 0 & & & u_p \end{bmatrix}$$

where U_p has the first $(p-1)$ rows equal to zero, $1 \leqq p \leqq n-1$

$$A_0 = \begin{bmatrix} h_{11}^{n+1} & h_{22}^{n+1} & \cdots & h_{nn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{11}^{N} & h_{22}^{N} & \cdots & h_{nn}^{N} \end{bmatrix} \qquad A_p = 2\begin{bmatrix} h_{pp+1}^{n+1} & h_{pp+2}^{n+1} & \cdots & h_{pn}^{n+1} \\ \vdots & \vdots & & \vdots \\ h_{pp+1}^{N} & h_{pp+2}^{N} & \cdots & h_{pn}^{N} \end{bmatrix},$$

$$1 \leqq p \leqq n-1 .$$

Then

$$P(u_1, u_2, \cdots, u_n) = \det \begin{bmatrix} U_0 & U_1 & \cdots & U_{n-1} \\ A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix} = 0 .$$

Hence the characteristic hypersurfaces of M are the solutions of the first order partial differential equation defined by $P(u_1, \cdots, u_n) = 0$. For $n = 3$ this equation was obtained by Cartan ([2], p. 208).

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