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THE FAILURE OF EVEN CONJUGATE CHARACTERIZATIONS OF H^1 ON LOCAL FIELDS

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If K is a local field, the Hardy space $H^1(K)$ is defined as follows: If f is a distribution on K let $f(x, k)$ (defined on $K \times Z$) be its regularization. Let $f^*(x) = \sup_k |f(x, k)|$. Then $f \in H^1$ iff the maximal function f^* is integrable. Chao has given the following conjugate function characterization of H^1. Let π be a multiplicative character on K that is homogeneous of degree zero, ramified of degree 1, and is odd. Then $f \in L^1$ is in H^1 iff $(\pi\hat{f})^\vee \in L^1$. He also shows that if μ is a finite (Borel) measure then μ is absolutely continuous whenever $(\hat{\mu}\pi)^\vee$ is also a finite measure. In this paper proofs are given that these results fail if π is not odd.

It is shown that if π is even (but otherwise satisfies the conditions above) then there is a singular measure μ and an integrable function f. $f \notin H^1$ such that $\pi\hat{\mu} = \hat{\mu}$ and $\pi\hat{f} = \hat{f}$. These results were announced earlier [Gandulfo, Garcia-Cuerva, and Taibleson, Bull. Amer. Math. Soc., 82 (1976), 83-85].

A basic reference for this paper is [4]; in particular, Chapters I, II, and IV. Regularizations are discussed in detail in IV §1. The results proven here are [3; Thm. 1 and Lemma 1]. The theorem of Chao can be found in [4; IV § 3] or in [1]. Other characterizations of H^1 can be found in [2].

A local field is a locally compact field that is not connected and not discrete. A complete list of such fields is: the p-adic number fields and finite algebraic extensions of p-adic fields (these are of characteristic zero), and fields of formal Laurent series over a finite field, $GF(p^n)$, the so-called p^n-series fields (these are of characteristic p). We note that there is a "natural" ring multiplication for the dyadic group, 2^ω, so that the field of quotients of 2^ω is the 2-series field.

There is a norm, $|\cdot|$, on K that is ultrametric ($|x+y| \leqq \max[|x|, |y|]$ and so if $|x| \neq |y|$, $|x+y| = \max(|x|, |y|)$. If $x \in K$, $x \neq 0$, then $|x| = q^k$ for some $k \in Z$. The fractional ideals $\{\mathfrak{P}^k\}$ are the balls: $\mathfrak{P}^k = \{|x| \leqq q^{-k}\}$. We fix a character χ on the additive group of K such that χ is trivial (identically 1) on $\mathfrak{D} = \mathfrak{P}^0$ (the ring of integers in K) and is nontrivial on $\mathfrak{P}^{-1}$. We choose $\mathfrak{p}$ to be a generator of the prime ideal $\mathfrak{P} = \mathfrak{P}^1$ (in $\mathfrak{D}$). $|\mathfrak{p}| = q^{-1}$, and $\mathfrak{D}/\mathfrak{P} \cong GF(q)$ (the local class field of K) where $q = p^n$, p a prime. The measure of a set E is denoted

$|E|$. $|\mathfrak{P}^k| = q^{-k}$, so $|\mathfrak{D}| = 1$. For $u \in K$, we set $\chi_u(x) = \chi(ux)$, $\tau_u f(x) = f(x-u)$. Φ_k denotes the characteristic function of $\mathfrak{P}^k$.

DEFINITION. If K is of finite characteristic let $h_k = \chi_{\mathfrak{p}^{-k}}\Phi_0$. If K is of characteristic zero let $h_k = \sum_{l=1}^{q^{k-1}} \tau_{c_l^{k-1}}(\chi_{\mathfrak{p}^{-k}}\Phi_{k-1})$ where $\{c_l^k\}$ is a complete set of coset representatives of $\mathfrak{P}^k$ in $\mathfrak{D}$.

Note. (1) If K is of finite characteristic the two definitions essentially agree. (2) If $q = 2$, $\{h_k\}$ is the sequence of Rademacher functions.

LEMMA 1. *$\{h_k\}_{k\geq 1}$ is a sequence of independent, identically distributed random variables on $\mathfrak{D}$.*

Proof. Each h_k is supported on $\mathfrak{D}$ and we identify h_k with its restriction to $\mathfrak{D}$. The values of h_k are pth roots of unity. h_k is constant on the $q^k = p^{nk}$ cosets of $\mathfrak{P}^k$ in $\mathfrak{D}$. On each of the $q = p^n$ cosets of $\mathfrak{P}^{k-1}$ in cosets of $\mathfrak{P}^k \subset \mathfrak{D}$ it takes on each of its p possible values exactly p^{n-1} times. Thus, if ε is a pth root of unity $|\{h_k = \varepsilon\}| = p^{-1}$. We see that the h_k are identically distributed. To show independence we need to observe that if $\{k_j\}_{j=1}^t$ is a finite collection of distinct positive integers and $\{\varepsilon_j\}$ a set of pth roots of unity then $|\{h_{k_j} = \varepsilon_j,\ j = 1, \cdots, t\}| = p^t$. Using the facts above we get this result by systematically counting. This completes the proof.

The Fourier transform of a distribution f is denoted $\hat{f}$ and for $f \in L^1$, $\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx$. If μ is a finite Borel measure, $\hat{\mu}(\xi) = \int_K \overline{\chi_\xi(x)}d\mu(x)$. We note that $\overline{\chi_u} = \chi_{-u}$, $(\chi_u f)\hat{} = \tau_u \hat{f}$, $(\tau_u f)\hat{} = \overline{\chi_u} f$, and $\hat{\Phi}_k = q^{-k}\Phi_{-k}$.

LEMMA 2. *Let $g_k = Re\, h_k$ and*

$$\mu(x, k) = \begin{cases} g_1(x) \prod_{l=2}^{-k} (1 - g_l(x)), & x \in \mathfrak{D},\ k \leqq -1 \\ 0 & ,\ \text{otherwise.} \end{cases}$$

Then $\mu(x, k)$ is the regularization on K of a nontrivial, real-valued, finite Borel measure μ, that is singular, supported on $\mathfrak{D}$, $|\mu| < 1$, $\mu(\mathfrak{D}) = 0$, and $\hat{\mu}$ is supported on $C = \bigcup_{k\geq 1} \{(\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}) \cup (-\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1})\}$. If $q = 2$, μ is supported on a two point set. If $q > 2$, μ is continuous.

Proof. From Lemma 1 we see that $\{g_k\}$ is a sequence of independent, identically distributed random variables on $\mathfrak{D}$ (are i.i.d. on $\mathfrak{D}$). Observe that if J is a coset of $\mathfrak{P}^l$, $l < k$, then $\int_J g_k = 0$. We break the proof into smaller steps.

I. $\mu(x, k)$ is regular. Since g_k is constant on cosets of $\mathfrak{P}^k$, $\mu(x, k)$ is constant on cosets of $\mathfrak{P}^{-k}$. Next we see that $\int_{\mathfrak{D}} \mu(x, -1) = \int g_1 = 0 = \mu(x, 0)$. Finally we need to show that if $J = y + \mathfrak{P}^{-(k+1)} \subset \mathfrak{D}$, $k < -1$, then $\int_J \mu(x, k) = \int_J \mu(x, k+1) = q^{k+1}\mu(y, k+1)$. But, $\mu(x, k) = \mu(x, k+1)(1 - g_{-k})$, so $\int_J \mu(x, k) = \mu(y, k+1)\int_J (1-g_{-k}) = \mu(y, k+1)|J| = q^{k+1}\mu(y, k+1)$.

II. $|\mu(x, k)|$ is regular on the domain, $\mathfrak{D} \times \{k \leqq -1\}$. The proof for I works since $(1 - g_l(x)) \geqq 0$ for all x.

III. $\mu(x, k)$ is regularization of a nontrivial, real-valued, finite Borel measure, that is supported on $\mathfrak{D}$, and $\mu(\mathfrak{D}) = 0$. Using [4; IV (1.8)(e) and (1.9)(b)] we only need observe that $\mu(x, k)$ is real-valued; $\mu(x, k) = 0$ if $x \notin \mathfrak{D}$; and show that $\int \mu(x, k)dx \equiv 0$, $k \in \boldsymbol{Z}$; and $\int |\mu(x, k)|dx = \int |g_1| > 0$, $k \geqq -1$. $\int \mu(x, k)dx \equiv 0$ follows I. For $k \geqq 0$ it is trivial, for $k \leqq -1$, $\mu(x, k)$ is regular so

$$\int_K \mu(x, k)dx = \int_{\mathfrak{D}} \mu(x, k)dx = \int_{\mathfrak{D}} \mu(x, 0)dx = 0 .$$

That $\int \mu(x, k) = \int |g_1|$ follows from II. $|\mu(x, k)|$ is regular for $k \leqq -1$, so if $k \leqq -1$,

$$\int_K |\mu(x, k)|dx = \int_{\mathfrak{D}} |\mu(x, k)|dx = \int_{\mathfrak{D}} |\mu(x, -1)|dx = \int_K |g_1| > 0 .$$

IV. μ is a singular measure. To see that μ is not absolutely continuous we use [4; (1.8)(d)]. This implies that the regularization of an absolutely continuous measure is Cauchy in L^1. We use the fact that $\{g_k\}$ is i.i.d. on $\mathfrak{D}$. Then for $k \geqq -1$,

$$\begin{aligned}\int_K |\mu(x, k) - \mu(x, k-1)| &= \int_{\mathfrak{D}} |g_1|(1 - g_2) \cdots (1 - g_{-k})|g_{-k+1}| \\ &= \int_{\mathfrak{D}} |g_1| \int_{\mathfrak{D}} (1 - g_2) \cdots \int_{\mathfrak{D}} (1 - g_{-k}) \int_{\mathfrak{D}} |g_{-k+1}| = \left[\int |g_1|\right]^2 > 0 .\end{aligned}$$

Note that $|\{(1 - g_k(x)) = 0\}| = p^{-1}$, so $|\{\mu(x, k)\} \neq 0\}| = (1 - p^{-1})^{-(k+1)}$ and so $\mu(x, k) \to 0$ a.e. From which it follows that $\mu^*(x) < \infty$ a.e. [4; V (2.3)]. Let $E_N = \{\mu^*(x) < N\}$. By the dominated convergence theorem $|\mu|(E_N) = 0$ (use II) and so $|\mu|(\bigcup_N E_N) = 0$, but $\bigcup_N E_N$ is a set of full measure, so μ is supported on a set of measure zero.

Actually we can do the whole thing in one simple step if we carefully analyse the set in $\mathfrak{D}$ on which $\mu(x, k) = 0$. That set, call it F_k, is a union of cosets of $\mathfrak{P}^{-k}$, $|F_k| \to 1$, $\{F_k\}$ is increasing. Thus μ is supported on the set $\sim(\bigcup_k F_k)$ which is a closed set of measure zero.

V. If $q = 2$ μ is a 2-point measure. If $q > 2$, μ is continuous. For $q = 2$ a little computation shows that there are decreasing sequences of cosets $\{I_k^i\}$, $i = 1, 2$, such that I_k^i is a coset of $\mathfrak{P}^{-k}$ and

$$\mu(x, k) = \begin{cases} 2^{-k-1}, & x \in I_k^1 \\ -2^{-k-1}, & x \in I_k^1 \\ 0, & \text{otherwise.} \end{cases}$$

Since $|I_k^i| = 2^k$, we see that $\mu(\cdot, k)$ converges W^* to a 2-point measure with mass 1/2 at one point and mass $-1/2$ at the other. More generally we note that $|\mu(x, k)| \leqq 2^{-k-1}$ for all x, so that if I_k is a coset of $\mathfrak{P}^{-k}$, then

$$\begin{aligned} |\mu|(I_k) &= \lim_{l \to -\infty} \int_{I_k} |\mu(x, l)|\, dx \\ &= \int_{I_k} |\mu(x, k)| \leqq |I_k| 2^{-k-1} = (1/2)(q/2)^{-k} \longrightarrow 0 \end{aligned}$$

as $k \to -\infty$ if $q > 2$. Thus, if $\{I_k\}$ is a decreasing sequence of cosets, $|\mu|(I_k) \to 0$ and so μ has no atomic component.

VI. $\hat{\mu}$ is supported on C. It will suffice to show that each $\hat{\mu}(\cdot, k)$ is supported on C. Note also that for $q = 2$, this is an uninteresting statement since $C = K \sim \mathfrak{D}$. To show that $\mu(\cdot, k)$ is supported on C it will be sufficient to show that if $\{k_j\}$ is a finite set of distinct positive integers with $k_s = \max_j k_j$ then $(g_{k_1} \cdots g_{k_s})\hat{}$ is supported on

$$\{(\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}) \cup (-\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1})\}\,.$$

We consider two cases. If K is of finite characteristic,

$$\begin{aligned} g_{k_1} \cdots g_{k_s} &= 2^{-s}(\chi_{\mathfrak{p}^{-k_1}} + \chi_{-\mathfrak{p}^{-k_1}}) \cdots (\chi_{\mathfrak{p}^{-k_s}} + \chi_{-\mathfrak{p}^{-k_s}})\Phi_0 \\ &= 2^{-s} \sum \chi_{\pm\mathfrak{p}^{-k_1}} \cdots \chi_{\pm\mathfrak{p}^{-k_s}}\Phi_0 = 2^{-s} \sum \chi_{(\pm\mathfrak{p}^{-k_1} \pm \cdots \pm \mathfrak{p}^{-k_s})}\Phi_0\,. \end{aligned}$$

Thus,

$$(g_{k_1} \cdots g_{k_s})\hat{} = \sum \tau_{(\pm\mathfrak{p}^{-k_1} \pm \cdots \pm \mathfrak{p}^{-k_s})}\Phi_0\,.$$

Each term is the characteristic function of a coset of $\mathfrak{D}$ in one or the other of $\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}$ or $-\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}$. For K of finite characteristic we proceed more carefully.

$$g_{k_1} \cdots g_{k_s} = 2^{-s}(h_{k_1} + \overline{h_{k_1}}) \cdots (h_{k_s} + \overline{h_{k_s}}) .$$

$$\hat{h}_k = [q^{-k+1} \textstyle\sum_{l=1}^{q(k-1)} \overline{\chi}_{c_l^{k-1}}] \tau_{\mathfrak{p}^{-k}} \Phi_{-k+1} .$$

Since $c_l^{k-1} \in \mathfrak{D}$ it follows that the term in the "square" brackets is constant on cosets of $\mathfrak{D}$. $\tau_{\mathfrak{p}^{-k}}\Phi_{-k+1}$ is the characteristic function of $\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$ so $\hat{h}_k$ is a finite linear combination of characteristic functions of cosets of $\mathfrak{D}$ contained in $\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$. Thus h_k is a finite linear combination of terms of the form $\chi_u \Phi_0$, $u \in \mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$, $k > 0$. Similarly, $\bar{h}_k$ is a finite linear combination of such terms with $u \in -\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$. The proof now proceeds as in the finite characteristic case.

This completes the proof of Lemma 2.

Note. μ is defined as a local field version of a Riesz product. See [5; V § 7]. It should then come as no surprise that μ is a continuous singular measure when $q \geqq 3$. We also note that if $q = 3$, then μ (except for a trivial factor) is the Cantor-Lebesgue measure supported on the Cantor set, if one identifies $\mathfrak{D}$ with $[0, 1]$ in the usual way.

COROLLARY. *Let π be a multiplicative character on K that is ramified of degree 1, homogeneous of degree zero, and is even. Let μ be the real-valued, singular measure defined in Lemma 2. Then $\pi\hat{\mu} = \hat{\mu}$.*

Proof. We show that $\pi(x) \equiv 1$ on C. π is ramified of degree 1 so π is constant on each coset $\pm\mathfrak{p}^k + \mathfrak{P}^{k+1}$ so we only need to determine $\pi(\mathfrak{p}^k)$ and $\pi(-\mathfrak{p}^k)$. π is homogeneous of degree zero so we only need to determine $\pi(1)$ and $\pi(-1)$. π is even so $\pi(-1) = \pi(1)$. π is a multiplicative character so $\pi(1) = 1$. This completes the proof.

THEOREM. *Let μ be as above, and let $\{c_k\}$ be a collection of distinct coset representatives of $\mathfrak{D}$ in K. Then there is a sequence $\{a_k\}$ of real numbers such that if $f(x) = \sum_{k=1}^{\infty} a_k \tau_{c_k}\mu(x, -k)$, then $f \in L^1$, but $f \notin H^1$. Furthermore, $\hat{f}$ is supported on C.*

Proof. Let $f_k = \tau_{c_k}\mu(\cdot, -k)$. f_k is supported on $c_k + \mathfrak{D}$.

$$f_k(x, l) = \begin{cases} \mu(x - c_k, l) & , \quad l > -k \\ \mu(x - c_k, -k) , & l \leqq -k . \end{cases}$$

Thus $f_k(\cdot, l)$ is supported on $(c_k + \mathfrak{D}) \times \boldsymbol{Z}$. Consequently,

$$\int |f| = \sum |a_k| \int |f_k| = \int |g_1| \sum |a_k| , \quad \text{and} \quad \int f^* = \sum |a_k| \int (f_k)^* .$$

We claim that $\left\{\int (f_k)^*\right\}$ is unbounded. If this claim is valid we simply choose $\{a_k\}$ so $\sum |a_k| < \infty$ and $\sum a_k \int (f_k)^* = \infty$. To prove the claim suppose $\left\{\int (f_k)^*\right\}$ is bounded. We note that $(f_k)^*(x) = \sup_{l \geqq -k} |\mu(x - c_k, l)|$, so $\{(f_k)^*(x + c_k)\}$ is a nondecreasing sequence with limit μ^*. By the Lebesgue monotone convergence theorem $\mu^* \in L^1$. But $\mu(x, k)$ converges a.e. so by the Lebesgue dominated convergence theorem $\{\mu(\cdot, k)\}$ converges in L^1 and hence is Cauchy in L^1. But $\{\mu(\cdot, k)\}$ is not Cauchy in L^1, a contradiction.

We need to show that $\hat{f}$ is supported on C. But $\hat{f} = \sum a_k \chi_{c_k} \hat{\mu}(\cdot, k)$, and $\hat{\mu}(\cdot, k)$ is supported on C for all k, so $\hat{f}$ is also supported on C. This completes the proof of the theorem.

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