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THE SIMPLEST CLOSED 3-MANIFOLDS

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Every closed orientable 3-manifold has a Heegaard diagram and corresponding group presentations. We shall show here how to give a complete analysis of all closed orientable 3-manifolds that have genus two Heegaard diagrams having a corresponding presentation one of whose relators contains no more than 4 syllables. This is equivalent to saying that one of the determining simple closed curves in the diagram crosses a "waist band" of the diagram no more than 4 times. In the appendix we have given a catalog of manifolds with two generator presentations and not more than 20 syllables.

The catalog was produced with the aid of a computer using the techniques developed in [6] and [7]. The analysis comes to a grinding halt when "generalized knot spaces" are encountered because the author has not been able to show nontriviality of the groups encountered nor has he been able to decide which of these spaces are sufficiently large. The techniques used here have been sufficient for establishing homeomorphism between any pair of orientable 3-manifolds known to be homeomorphic. The author wishes to thank H. Zieschang for helpful conversations.

1. Preliminary definitions and theorems.

Let M denote a closed 3-manifold. A 2-complex K is called a spine of M if $M - B$ collapses to K, B denotes a (polyhedral) ball in M. In our discussion all spines will be assumed to have a cell decomposition with only one 0-cell (vertex). It is a simple matter to modify any spine by shrinking a maximal tree in the 1-skelton so that it has only one o-cell. It is easy to see how one obtains a group presentation from such a cell complex. The generators are in 1-1 correspondence with the 1-cells and the relators are read off from the formula by which the 2-cells attach to the 1-skelton. This group presentation will be called a *presentation of the spine.* Unfortunately the spine (and hence its presentation) does not always uniquely determine the manifold. For instance $\langle a | a^7 \rangle$ is a presentation of a spine of the lens spaces L_{7P} for $P = 1, 2, 3$. These spaces are not homeomorphic [11]. However for spines whose presentations have two generators and relators all of whose exponents are not ± 1 and ± 2, the spines uniquely determine the manifolds [5].

There are 2 quite common ways of building 3-manifolds-Heegaard diagrams and handle decompositions. A Heegaard diagram (H_1, H_2, h)

for a 3-manifold is a pair of handle bodies H_1 and H_2 of the same genus together with a homeomorphism $h: \partial H_1 \to \partial H_2$. It is not difficult to see that every closed orientable 3-manifold has a (in fact many) Heegaard diagram. Two Heegaard diagrams (H_1, H_2, h) and (H_1, H_2, h') are to be considered the same if h and h' are isotopic. Notice that this definition gives us no specific vehicle for saying how h is defined. Thus although this definition does not require it we have not really determined a manifold until some way of specifying h is given. The most common way of specifying h is by choosing meridian disks $D_1, \cdots, D_n$ for H_2 and specifying the simple closed curves $h^{-1}(\partial D_i)$ in ∂H_1. Note that there are infinitely many (non-isotopic) ways of choosing a system of meridian disks for H_2 and each of these (given h) determine the same Heegaard diagram. If one chooses a system $E_1, \cdots, E_n$ of meridian disks for H_1 one can now determine a Heegaard diagram by specifying only the simple closed curves $\{\partial E_i\}$ and $\{h^{-1}(\partial D_j)\}$ in ∂H_1. If we have (H_1, H_2, h) and $\{E_i\}$ and $\{D_i\}$ chosen we have also determined a handle decomposition as follows. Denote by $N(X)$ a regular neighborhood of X. The 0-handle will be $H_1 - \bigcup_{i=1}^n N(E_i)$; the 1-handles are $\{N(E_i)\}$, the 2-handles are $\{N(D_j)\}$ attached along $h^{-1}(N(D_j) \cap \partial H_2)$. The 3-handle is $H_2 \sim \bigcup_{j=1}^n N(D_j)$. Given any handle decomposition of a 3-manifold with one 3-handle and one 0-handle one obtains a spine by simply collapsing the 2-handles down to their central disks; collapsing the 1-handles down to the fins obtained by joining their centers with the central disks of the 2-handles; then collapsing the 0-handle down to the cone over the center of the intersection of the 2-complex so far obtained with the boundary of the 0-handle. In [14] the technique of getting a presentation of spine directly from ∂H_1, $\{\partial E_i\}$, $\{h^{-1}(\partial D_j)\}$ is given.

Now let M be a compact-orientable 3-manifold with nonempty boundary and suppose we are given a handle decomposition of M. From this we obtain the presentation $\varphi = \langle x_1, \cdots, x_n \mid R_1, \cdots, R_k \rangle$. Let α be any simple closed curve in ∂M. We may move α isotopically so that it lies entirely on the 0 and 1-handles of M. Again by an isotopic adjustment we may assume that the intersection of α with each of the 1-handles consists of parallel arcs. If we orient α and assign a direction to each 1-handle we can associate a (cyclic) word in the free semigroup on $x_1, \cdots, x_n$ with α by traveling around α and when α passes over the ith 1-handle writing x_i if the direction of travel agrees with the chosen direction and writing x_i^{-1} otherwise. (If α lies entirely in the 0-handle the corresponding word is 1.) This cyclic word $W(\alpha)$ will be called a *word corresponding to α or a presentation of α*. *Note*: two isotopic curves on ∂M may have different presentations; however, both words must represent the same element of $\Pi_1(M)$ as presented by φ.

Now suppose H is a genus 2 handlebobody. Let D be properly embedded disk that splits H into two genus 1 handlebodies, D will be called a *waist-cut* of H. If meridian disks for H have been chosen we always assume that the waist-cut is disjoint from the meridian disks. If α is a simple closed curve on ∂H then the minimum number of points of intersection of α with D will be the number of syllables of the word $w(\alpha)$ corresponding to α.

We are now ready to give the technical tools of this section.

THEOREM 1.1. *Suppose* $\varphi_i = \langle x_{i,1}, x_{i,2}, \cdots, x_{i,m_i} | R_{i,1}, \cdots, R_{i,k_i}\rangle$, $i = 1, 2$; *present the spines* K_i *of* M_i. *Suppose further that the curves* $\alpha_1, \cdots, \alpha_n$ *cut* ∂M_1 *into an open disk and that* $g: \partial M_1 \to \partial M_2$ *is a homeomorphism. Let* M *be the manifold* $M_1 \bigcup_g M_2$. *Then* $\varphi = \langle\{x_{ij}; i=1,2; j=1,2,\cdots,m_i\} | \{R_{ij}; i=1,2; j=1,\cdots,k_i\} \cup \{W_1(\alpha_j) W_2^{-1}(g(\alpha_j)); j = 1, 2, \cdots, n\}\rangle$ *presents a spine of* M. *Here* $W_1(\alpha_j)$ *respectively* $W_2(\alpha_j)$ *denote the words in* $x_{1,1}, \cdots, x_{1,m_1}$ *respectively* $x_{2,1}, \cdots, x_{2,m_2}$ *corresponding to* α_j *respectively* $g^{-1}(\alpha_j)$.

A very closely related theorem proved by the same techniques is

THEOREM 1.2. *Suppose* φ_i, K_i *and* M_i *are as above and that* $\alpha_1, \cdots, \alpha_n$ *is a collection of disjoint simple closed curves in* ∂M_1 *and that* $N(\alpha_i)$ *is a small regular neighborhood of* α_i *in* ∂M_1. *Let* $g: \bigcup_{i=1}^n N(\alpha_i) \to \partial M_2$ *be a homeomorphism and let* $M = M_1 \bigcup_g M_2$. *Then* $\varphi = \{x_{ij}\} | \{R_{ij}\} \cup \{W_1(\alpha_i) W_2^{-1}(\alpha_j)\}$ *presents a spine of* M.

Proof. We assume without loss of generality that $\cup \alpha_i$ and $\cup g(\alpha_i)$ lies in the 0 and 1-handles of a handle decomposition of M_1 respectively M_2 determined by the spines K_1 and K_2 respectively. It is not difficult to see that we can enlarge K_1 by "joining" $\bigcup_{i=1}^n \alpha_i$ with the 1 complex that is the spine of the handlebody that is the union of the 0 and 1-handles. After enlarging the spine of M_2 in a similar way we have 2 complexes K_1' and K_2' of M_1 and M_2 respectively with the property M_i collapses to K_i and $K_1' \cap \partial M_1 = \bigcup_{i=1}^n \alpha_1$ while $K_2' \cap \partial M_2 = \bigcup_{i=1}^n g(\alpha_i)$. It is easy to see that (in both theorems) $K_1' \bigcup_g K_2'$ is a spine of M. Now choose an arc connecting the 0-cell of K_1' with the 0-cell of K_2' by crossing α_1 exactly once and crossing no other α_i, $i = 2, \cdots, n$. Shrink this arc to a point and read off the correspoding presentation to get the desired result.

Our next theorem enables us to show that none of the entires in our appendix contains a fake 3 ball unless it is simply connected. Further that all manifolds listed are irreducible if their fundamental group is not cyclic or is not a free product of two cyclic groups.

THEOREM 1.3. *Let φ be a presentation of a spine of the closed 3-manifold M. Denote by $g(\varphi)$ the number of generators in the presentation ϕ and let $r(\phi)$ by the rank (minimum number of generators) of the group presented by ϕ. If $g(\phi) \leqq r(\phi) + 1$ then M contains no fake 3-balls. If $g(\varphi) = r(\varphi)$ then M is reducible if and only if φ presents a nontrivial free product.*

Proof. Denote by $g(M)$ the genus of a minimal genus Heegaard diagram for M. Assume that $M = M_1 \# M_2$ where $\#$ denotes the connected sum along a 2-sphere S^2. In [1] Haken shows we may assume that S^2 intersects the Heegaard surface of a minimal Heegaard diagram in a simple closed curve. It follows that $g(M) = g(M_1) + g(M_2)$. Both results follow from this formula and the fact that no genus 1 counterexample for the Poincaré conjecture exists.

2. Sums of lens spaces. We consider first spaces having a spine with a presentation of the form $\langle a, b \mid a^m, R_2 \rangle$ where R_2 is some word in a and b.

THEOREM 2.1. *A closed 3-manifold with a presentation of the form $\langle a, b \mid a^m, R_2 \rangle$ is a spine of the connected sum of two lens spaces and of no other 3-manifolds. Furthermore one of the summands must be $L_{m,p}$ for some choice of p and, if there are two b-syllables with different exponents n and q then the second summand is uniquely determined.*

Proof. We use the $R-R$ system of φ as developed in [5]. In [5] we showed that every presentation of a spine can be derived from some $R-R$ system containing no free cancellations. The company corresponding to a^m is pictured in Figure 1 below. It is

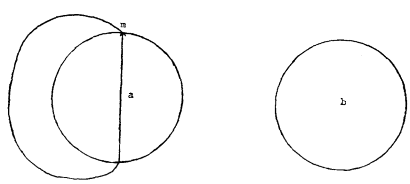


FIGURE 1

easy to see that every relator R_2 which fits in the above picture must have the properties that all exponents in a-syllables must be $\pm m$. Using Theorem 2.6 of [5] we eliminate a^m from R_2, thus producing a presentation of the form $\langle a, b \mid a^m, b^n \rangle$. This presents only spines of connected sums of 2 lens spaces. Note that if two different b exponents appear then the "gap" as discussed in [4] is determined so that the second summand is uniquely determined. The first summand corresponding to a^m is not uniquely determined.

Note. It could be that one or both of the summands is S^3 (if for instance $m = 1$).

3. Seifert fiber spaces. We now investigate spaces with presentations of the form $\langle a, b \mid a^m b^n, R_2 \rangle$.

THEOREM 3.1. *A closed 3-manifold with a presentation of the form $\langle a, b \mid a^m b^n, R_2 \rangle$ is a Seifert fiber space over S^3 with $\leqq 3$ exceptional fibers.*

Note. If $n = 0$ we have the situation of 2, already discussed. It is known [10] that Seifert fiber spaces over S^3 with exactly 3 exceptional fibers are not sufficiently large unless their first homology is infinite.

Proof. All of these manifolds are obtained by sewing a solid torus to a manifold with spine presented by $\langle a, b \mid a^m b^n \rangle$. Since $\langle a, b \mid a^m b^n \rangle$ has center when $m, n \neq 0$ and, since the corresponding 3-manifold is irreducible (by Theorem 1.3) and has torus boundary, it is a Seifert fiber space over a disk with $\leqq 2$ exceptional fibers [10]. Attaching a solid torus gives a Seifert fiber space over S^2 with $\leqq 3$ exceptional fibers. If m or $n = 0$ we are again in the case discussed in § 2.

Note. For any of the presentations of this form listed in the appendix it is not difficult to identify which Seifert fiber spaces they present (in terms given by Seifert in [9]. For instance

$$\langle a, b \mid a^p b^n, (a^m b^{n+q})^k a^m b^q \rangle$$

presents a Seifert fiber space over S^3 with 3 exceptional fibers assuming $|p|$ and $|n| > 1$). The exceptional fibers are specified by the pairs of integers $p, m; n, q; k + 1, 1$. A rigiorus proof can be obtained using Theorems 1.1 and 1.2 and the construction of the specified Seifert fiber space as given by Seifert.

4. The 4 torus spaces.

THEOREM 4.1. *Let M be a closed 3-manifold presented by*

$$\langle a, b \mid (a^m b^q)^r b^n, a^m(a^p b^n)^s \rangle$$

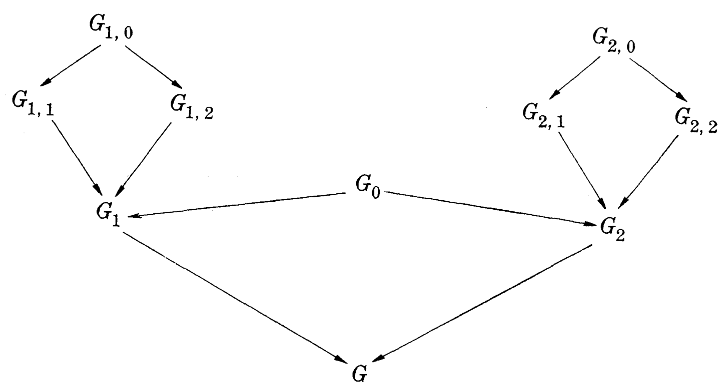
for some choice of m, n, p, q, r, s *with* $(m, p) = (n, q) = (r, s) = 1$. *Then* M *is the union of* 4 *solid tori, each pair of which meet in an annulus or a disk on their boundaries. Further, if* $|m|$ *and* $|n| > 1$ *then* M *is sufficiently large (hence uniquely determined among irreducible closed 3-manifolds by its fundamental group* [12]). *If* $|m|$ *or* $|n| = 1$ *then* M *is a Seifert fiber space over* S^2 *with* $\leqq 3$ *exceptional fibers.*

Proof. If m or $n = 0$ we have a case already investigated in §2. If $m = 1$ then setting $c = ab^q$ transforms our presentation into $\langle a, c \mid c^r b^n, R_2 \rangle$. Now such a transformation gives a new spine for M [5] also [13] so that we have the case treated in §3. If $|r|$ or $|s| \leqq 1$ we are again in the case of §3.

We assume now that $|m|, |n|, |r|, |s| \geqq 2$. We first note that $(a^m b^q)^r b^n = 1$ implies that a^m and b^n commute (write $a^m \rightleftarrows b^n$). To see this observe that $(a^m b^q) b^n (a^m b^q)^{r-1} (a^m b^q)^{-r} b^{-n} = 1$ but this is $a^m b^n a^{-m} b^{-n} = 1$. A similar argument shows that $a^m(a^p b^n)^s = 1$ implies $a^m \rightleftarrows b^n$. We now consider subgroups of our groups G determined by the generators listed below.

Subgroup	Generators
G_0	a^m, b^n
G_1	a^m, b
G_2	a, b^n
$G_{1,0}$	$(a^m b^q)^r$
$G_{1,1}$	$(a^m b^q)$
$G_{1,2}$	b
$G_{2,0}$	$(a^p b^n)^s$
$G_{2,1}$	$(a^p b^n)$
$G_{2,2}$	a

We have the following subgroup diagram. All maps are the inclusions.



We shall show that the above diagram gives a decomposition of G into the 4 infinite cyclic subgroups $G_{i,j}$ so that G is the tree product (see [2] for the definition of tree product) of the $G_{i,j}$. More simply stated G is the tree product of G_1 and G_2 with G_0 amalgamated, and G_i is the free product of $G_{i,1}$ and $G_{i,2}$ with $G_{i,0}$ amalgamated. In order to do this we build another diagram and show isomorphic equivalence. Let $G'_{i,j} = Z$ (the group of integers) for $i = 1, 2$ and $j = 0, 1, 2$. Define injections $\psi_{i,j}: G'_{i,0} \to G'_{i,j}$ by $\psi_{1,1}(1) = r$, $\psi_{1,2}(1) = -n$, $\psi_{2,1}(1) = s$, $\psi_{2,2}(1) = -m$. Now define $G'_i = G'_{i,1}{}_{\psi^*_{i,1}=\psi_{1,2}}G'_{i,2}$. Clearly G'_1 and G'_2 are presented by $\varphi_1 = \langle z_1, y_1 | z_1^r y_1^n \rangle$ and $\varphi_2 = \langle z_2, y_2 | z_2^s y_2^m \rangle$ respectively. Setting $z_1 = x_1 y_1^q$ and $z_2 = x_2^p y_2$ we get presentations

$$\varphi'_1 = \langle x_1, y_1 | (x_1 y_1^q)^r y_1^n \rangle$$

and q and n

$$\varphi'_2 = \langle x_2, y_2 | (x_2^p y_2)^s x_2^m \rangle .$$

Now let $G'_0 = Z \oplus Z$ and define homomorphisms $\psi_i: G'_0 \to G'_i$ by $\psi_1(1, 0) = X_1$, $\psi_1(0, 1) = y_1^n$, $\psi_2(1, 0) = x_2^m$, $\psi_2(0, 1) = y_2$. It must be that ψ_1 and ψ_2 are homomorphisms because $x_1 \rightleftarrows y_1^n$ in G'_1 and $x_2^m \rightleftarrows y_2$ in G'_2. We now show that ψ_1 and ψ_2 are monomorphisms. Suppose $\psi_1(u, v) = 1 \in G'_1$. Then $x_1^u y_1^{vn} = 1$. Note that $x_1^u y_1^{vu} = (z_1 y_1^{-q})^u y_1^{vn} = y_1^{vn}(z_1 y_1^{-q})^u$. But $z_1 \in G'_{1,1} - \psi_{1,1}(G'_{1,0})$ and $y_1^{-q} \in G'_{1,2} - \psi_{1,2}(G'_{1,0})$ because $|n| > 1$ and n and q are relatively prime. Furthermore $y_1^{vn} \in \psi_{1,2}(G_{1,0})$. Thus the length of $y_1^{vn}(z_1 y_1^{-q})^u$ is greater than 0. (See [3] for the definition of length in a free product with amalgamation.) It follows that $x_1^u y_1^{vn} \neq 1$ for $u \neq 0$. Suppose now that $u = 0$. Then our expression is y_1^{vn}. But clearly y_1 has infinite order so $y_1^{vn} = 1$ if and only if $v = 0$. We see then that ψ_1 (and by a similar argument ψ_2) is a monomorphism. We now form the free product with amalgamation $G' = G'_1{}_{\psi_1 \overset{*}{=} \psi_2} G'_2$. Clearly

$$\varphi = \langle x_1, x_2, y_1, y_2 | (x_1 y_1^q)^r y_1^n, (x_2^p y_2)^s x_2^m, x_1 = x_2^m, y_1^m = y_2 \rangle$$

presents G'. Eliminating x_1 and y_2 we get

$$\varphi' = \langle x_2, y_1 | (x_2^m y_1^q)^r y_1^n, (x_2^p y_1^n)^s x_2^m \rangle$$

which is our original presentation. Now we map $a \to x_2$ and $b \to y_1$ and lift subgroups to get the original decomposition for G.

Since we have $\Pi_1(M)$ is a nontrivial free product with amalgamation when $|m|, |n|, |r|, |s| > 1$, M is sufficiently large [10]. It follows from [12] that $\Pi_1(M)$ uniquely determines M among irreducible closed 3-manifolds. The 4 tori T_{ij} required in the theorem correspond to the 4 groups G'_{ij}; $i, j = 1, 2$. We now give a construction which yields our manifolds from 4 tori meeting as specified.

Let α_{11} be a simple closed curve in ∂T_{11} running longitudinally around ∂T_{11} r times while running meridianally around ∂T_{11} once and let α_{11} be a simple closed curve in ∂T_{12} running longitudinally around ∂T_{12} n times while running meridianally around ∂T_{12} p times. Now sew T_{11} to T_{12} along regular neighborhoods in ∂T_{11} and ∂T_{12} of α_{11} and α_{12} respectively. We have a new space T_1 which according to Theorem 1.2 has a spine presented by ϕ_1. A similar construction yields T_2 with a spine presented by ϕ_2. The presentations ϕ_1 and ϕ_2 define a handle decomposition for T_1 and T_2. If we change meridian disks in each of these handle decompositions by setting $z_1 = x_1 y_1^q$ and $z_2 = x_2^p y_2$ we get new spines with presentations ϕ_1' and ϕ_2'. Now sew ∂T_1 to ∂T_2 so that the simple closed curve presented by y_1^n in ∂T_1 is sewed to the curve presented by y_2 in ∂T_2 and the curve presented by x_1 in ∂T_1 is sewed to the curve presented by x_2^m in ∂T_2. By Theorem 1.1 the resulting manifold has a spine presented by φ. The eliminations of x_1 and y_2 can be done [5, Theorem 8.1] yielding a new spine presented by φ.

Our entire construction is summarized in Figure (2a) and (2b). Figure (2a) shows a R-R system [5] for T_1 while Figure (2b) shows a R-R system for T_2. These diagrams are divided into an upper and lower part representing regions on the boundaries of T_{11} or T_{21} and T_{12} or T_{22}, respectively. In order to construct our manifold M the dotted curves are to be sewed together as are the dashed curves. This diagram makes it easy to check that T_{ij} intersect in annuli or disks.

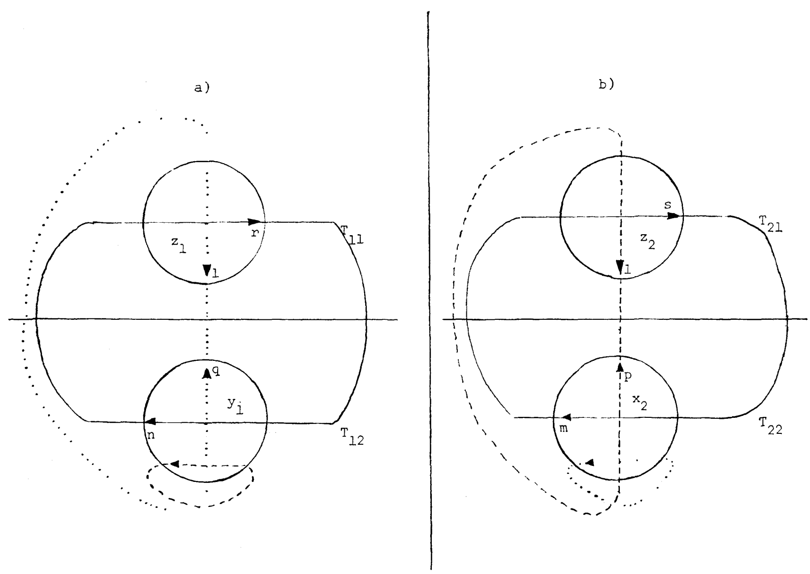


FIGURE 2

Note that the foregoing construction does not depend on the values of m, n, r, s being greater than 1.

There is another type of presentation encountered in the catalog in which one relator has 4-syllables. The simplest of these is the 19th entry. This space is again the result of sewing 4 tori together. But the sewing is not so simple as that given in Theorem 4.1. In particular, one shows that in

$$\langle a, b \mid a^{m+p}(a^{-p}b^{-n})^2, (a^{m+p}b^{n+q})^3 b^{-n}(a^{m+p}b^{n+q})^2 b^{-n} \rangle \qquad a^{m+p}$$

and b^n commute. This gives a decomposition for this group into a tree product. However, the presentations ϕ_1' and ϕ_2' in the proof of Theorem 4.1 now look like $\langle z_1, y_1 \mid z_1^3 y_1^{-m-p} \rangle$, and $\langle z_2, x_2 \mid z_2^3 x_2^{-n} z^2 x_2^{-n} \rangle$ respectively. The remaining steps of the analysis are the same except that the figure analogous to Figure 2 shows that some of the tori intersect in two of disks instead of one disk.

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APPENDIX

A CATALOG OF SPINES OF CLOSED 3-MANIFOLDS WITH GENUS 2 HEEGAARD DIAGRAMS

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We present here a complete list of the simplest closed orientable 3-manifolds that have genus 2 Heegaard diagrams. These manifolds are given by specifying their spines and by specifying the way in which these spines meet the boundary of a regular neighborhood of the single vertex in a cell decomposition of the manifold. The spines are uniquely determined by a group presentation. The spine is constructed by attaching a pair of disks to a figure 8 according to the formulae given by the relators of the group presentation. The beginning point for the construction of this catalog was a listing of the 598 distinct R-R systems whose presentations have two generators and no more than 20 syllables in their relators (see [5] for the definition of R-R system and how it determines the manifold). This listing was given by R. Stevens with help from the computer at the University of Michigan. The program has been rewritten and checked at Colorado State University. The techniques needed for writing this program were developed in [5] and [7]. Many of the presentations corresponding to a connected sum of lens spaces were not included in this listing. This listing contained a large number of redundancies in that it listed R-R systems that determined exactly the same set of manifolds. At this point the 2nd author with the help of R. Memmel undertook the writing of a computer program which grouped these R-R systems into 137 equivalence classes, the shortest member of which is listed in the catalog. The listing is complete in the sense that every orientable closed 3-manifold that has a spine whose presentation has two generators and no more than 20 syllables appears at least once.

Suppose M is a manifold obtained by assigning integer values to the exponents of some entry in the catalog. It is highly unlikely that M can also be obtained from a different entry. The likelihood of this type of coincidence rapidly decreases as the length of the presentation increases. For an example of such a coincidence consider the first and second listed classes whose presentations are

$$\langle a, b \,|\, a^m b^q, a^p b^n \rangle$$

and

$$\langle a, b \,|\, a^m b^{n+q} a^m b^q, a^p b^n \rangle \ .$$

If for the first presentation we choose $m = 2$, $q = 3$, $p = 3$, and $n = 4$, we get a spine of S^3 (see [7] for verification of this fact). If in the second presentation we choose $m = -1$, $q = 5$, $n = -3$, and $p = 1$ we also get a spine of S^3.

1. Determining the embedding from the 7 numbers in the catalog. Denote these numbers by α, β, γ, α', β', γ', δ. See Figure 3 for an illustration of this construction for 3, 0, 2, 1, 1, 3; 1. We

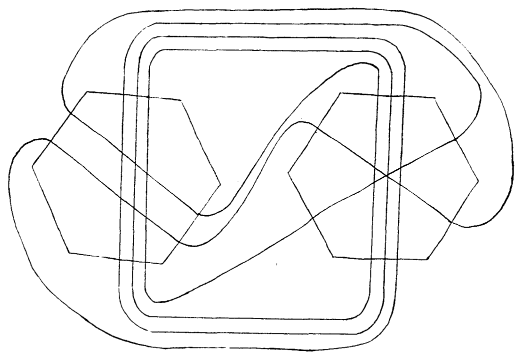

FIGURE 3

shall construct the R-R system of the presentations listed in the catalog. Let D_a and D_b be disjoint regular hexagons in the plane. In D_a lable successive sides by the integers m, $m + p$, and p. Now construct α oriented parallel line segments with one end (tail) on the side opposite the edge labelled m, the other end (head) on the side labelled m. Next construct β parallel line segments with heads on the side $m + p$ and tails on the opposite side. Similarly construct γ parallel line segments with heads on the side labelled p and tails on the opposite side. Construct systems of parallel line segments in D_b corresponding to the numbers α', β', and γ' in the same way. These line segments are called tracks. We next draw an arc connecting the last (clockwise) head of a track in D_a with the δth track in D_b counting clockwise from the first (clockwise) tail of a track in D_b. This first track should be counted as the 0th track. Now draw the remaining arcs joining the tracks in D_a and D_b so that these arcs are disjoint. The R-R system is now completed. To obtain the group presentation corresponding to this R-R system we proceed as follows. Trace out a closed curve by following a track through D_a then along arc connecting the end of the track to a track in D_b. Next follow the arc connecting the end of this track with the end of a track in D_a. We continue until we return to the starting point. This closed curve determines a relator in a presentation in the following way. As we travel along a closed curve, we record a syllable a^m if we travel from tail to head along a track whose head lies on the side labelled m. Next we follow the closed curve to a track in D_b. If we follow along this track from tail to head we record a syllable b^n (or b^{n+p} or b^p). If we travel along a track from head to tail we record the syllable with a minus sign preceding the exponent, e.g., a^{-m} or b^{-q}. We continue in this way until we return to the starting point. This gives us one of the relators listed in the catalog. The R-R systems obtained from the table have two relators, hence we will always have exactly two of these closed curves.

Table of Group Presentations

The letters that appear represent alternatively the exponents of the generators a and b. For example, the 4th entry denotes the presentation form $\langle a,b/a^{m}b^{-t}a^{-p}b^{-t},a^{s}b^{q}a^{s}b^{-n}\rangle$. Here s denotes $m+p$ and t denotes $n+q$. By choosing integer values for m, n, p, and q such that $(m,p)=1$, $(n,q)=1$ one gets a presentation of the spine of a closed 3-manifold.

1,0,1,1,0,1; 0	$mq,\ pn$
2,0,1,1,1,1; 0	$mtmq,\ pn$
3,0,1,1,2,1; 0	$mtmtmq,\ pn$
1,2,1,1,2,1; 2	$m-t-p-t,\ sqs-n$
4,0,1,1,3,1; 0	$mtmtmtmq,\ pn$
1,3,1,2,1,2; 0	$mqsnpnsq,\ st$
1,3,1,2,1,2; 1	$m-n-p-n,\ stsqsq$
5,0,1,5,0,1; 1	$mn-mnm-n,\ mqm-n-p-n$
5,0,1,1,4,1; 0	$mtmtmtmtmq,\ pn$
5,0,1,3,1,2; 3	$mqm-nmqm-nm-n,\ pt$
1,4,1,3,1,2; 0	$mqsnsq,\ snpnst$
1,4,1,3,1,2; 5	$m-q-p-q,\ s-ns-ns-ns-t$
3,1,2,3,1,2; 0	$mtmqmq,\ snpnpn$
3,1,2,3,2,1; 0	$mtmtmq,\ snpnpn$
3,1,2,3,1,2; 2	$mqsqm-n,\ m-n-p-t-p-n$
6,0,1,1,5,1; 0	$mtmtmtmtmtmq,\ pn$
6,0,1,1,5,1; 2	$mtm-t-m-q-m-t,\ mtptm-n$
1,5,1,4,1,2; 6	$m-q-p-q,\ s-ns-ns-ns-ns-t$
1,5,1,2,3,2; 1	$m-n-p-n,\ ststsqstsq$
1,5,1,2,3,2; 2	$m-nsts-n,\ stptsqsq$
4,1,2,3,1,3; 0	$mtmqmqmq,\ snpnpn$
4,1,2,3,1,3; 4	$m-nm-nm-nm-t,\ sqpqpq$
4,1,2,3,1,3; 5	$m-nm-q-p-q-p-q,\ m-ns-nm-t$
3,1,3,2,3,2; 3	$m-nm-nm-t-p-t,\ sqptpq$
7,0,1,1,6,1; 0	$mtmtmtmtmtmtmq,\ pn$
7,0,1,1,6,1; 2	$mtm-nmtm-t-m-t,\ mtptmq$
7,0,1,5,1,2; 3	$mn-m-q-m-q-mnm-n,\ mtm-n-p-n$
7,0,1,5,1,2; 5	$mqm-nm-nmqm-nm-nm-n,\ pt$
7,0,1,4,1,3; 4	$mqm-nmqm-nmqm-nm-n,\ pt$
1,6,1,1,6,1; 2	$m-t-s-t-p-t-s-t,\ stsqsts-n$
1,6,1,5,1,2; 7	$m-q-p-q,\ s-ns-ns-ns-ns-ns-t$
1,6,1,4,1,3; 0	$mqsnsqsnsq,\ snpnst$
1,6,1,3,2,3; 3	$m-nsqsqsqs-n,\ stpts-n$
5,0,3,4,1,3; 6	$m-nm-tm-nm-q-p-q,\ m-npqp-n$
5,1,2,5,1,2; 2	$mtm-n-p-n-p-n,\ mqmqm-n-s-n$
5,1,2,4,1,3; 3	$mqm-nmqsqm-n,\ m-n-p-t-p-n$
5,1,2,4,3,1; 4	$mqm-nm-nm-nm-n,\ stptpt$
5,1,2,4,1,3; 5	$m-nm-nm-nm-nm-t,\ sqpqpq$
5,1,2,3,2,3; 0	$mtmqmtmqmq,\ snpnpn$
5,1,2,3,2,3; 5	$m-nm-nm-tm-nm-t,\ sqpqpq$
5,1,2,3,2,3; 6	$m-nm-q-p-q-p-q,\ m-ns-nm-tm-t$
4,1,3,4,1,3; 0	$mtmqmqmq,\ snpnpnpn$
4,1,3,4,3,1; 0	$mtmtmtmq,\ snpnpnpn$
4,1,3,4,3,1; 1	$mtm-n-p-n-p-n-p-n,\ mtstmq$
4,1,3,4,1,3; 6	$m-ns-nm-q-p-q,\ m-npqp-nm-t$
4,1,3,3,2,3; 1	$mqmqmqm-n-p-n,\ stpnpt$

3,2,3,3,2,3; 2	*mqsqsqm − n, m − n − p − t − p − t − p − n*
3,2,3,3,2,3; 4	*m − nm − t − p − q − p − t, m − nsqpqs − n*
8,0,1,1,7,1; 0	*mtmtmtmtmtmtmtmq, pn*
8,0,1,5,1,3; 3	*mn − m − q − m − q − m − q − mnm − n, mtm − n − p − n*
8,0,1,5,1,3; 5	*mqm − nmqm − nm − nmqm − nm − n, pt*
7,0,2,1,7,1; 2	*mt − mtm − t, mqm − t − p − tm − nm − t − p − t*
1,7,1,6,1,2; 8	*m − q − p − q, s − ns − ns − ns − ns − ns − ns − t*
1,7,1,2,5,2; 1	*m − n − p − n, ststststsqststsq*
1,7,1,2,5,2; 3	*m − t − s − q − s − q − s − t, sts − nstpts − n*
1,7,1,4,2,3; 0	*mqsnsqsnsq, snpnstst*
1,7,1,4,2,3; 5	*m − t − p − t, sqs − nsqs − nsqs − ns − n*
1,7,1,4,2,3; 6	*m − tsqs − t, sqpqs − ns − ns − ns − n*
6,1,2,5,1,3; 2	*mtm − n − p − n − p − n, mqmqmqm − n − s − n*
6,1,2,5,3,1; 5	*mqm − nm − nm − nm − nm − n, stptpt*
6,1,2,5,1,3; 6	*m − nm − nm − nm − nm − nm − t, sqpqpq*
6,1,2,3,3,3; 1	*mtm − n − p − n − p − n, mtstmqmqmq*
5,1,3,2,5,2; 0	*mtmqmtstmq, mtpnpnpt*
5,1,3,4,1,4; 0	*mtmqmqmqmq, snpnpnpn*
5,1,3,4,1,4; 4	*mqpqm − nm − nm − nm − n, sqptpq*
5,1,3,4,1,4; 5	*m−nm−nm−nm−nm−t, sqpqpqpq*
5,1,3,4,1,4; 8	*m − np − nm − qm − tm − q, m − q − pn − sn − p − q*
5,1,3,4,2,3; 0	*mtmqmtmqmq, snpnpnpn*
5,1,3,4,3,2; 0	*mtmtmqmtmq, snpnpnpn*
5,1,3,4,3,2; 1	*mtm − n − p − n − p − n − p − n, mtstmqmq*
5,1,3,4,2,3; 3	*mqm − nmqsqm − n, m − n − p − t − p − t − p − n*
5,1,3,4,3,2; 5	*m − nm − nm − nm − nm − t − p − t, sqptpq*
5,1,3,4,2,3; 7	*m − ns − nm − q − p − q, m − npqp − nm − tm − t*
4,2,3,3,3,3; 1	*mqmqmqm − n − p − n, ststpnpt*
9,0,1,9,0,1; 3	*−m − nmn − mnm − n − mn, −m − nmqm − n − mnpn*
9,0,1,9,0,1; 1	*mn − mn − mnm − nm − n, mqm − nm − n − p − nm − n*
9,0,1,1,8,1; 0	*mtmtmtmtmtmtmtmtmq, pn*
9,0,1,1,8,1; 3	*mtm − t − m − tmtm − t − m − q − m − t, mtptm − n*
9,0,1,7,1,2; 1	*mnmqmnmqmnm − n − m − n, mnpnmt*
9,0,1,7,1,2; 7	*mqm − nm − nm − nmqm − nm − nm − nm − n, pt*
9,0,1,5,1,4; 3	*−m − nmqmqmqmqm − n − mn, −m − t − mnpn*
9,0,1,5,1,4; 5	*mqm − nmqm − nmqm − nmqm − nm − n, pt*
1,8,1,7,1,2; 1	*m − n − s − n − p − n − s − n, snstsnsqsnsq*
1,8,1,7,1,2; 2	*−mnsqsnpnsqsn, −s − n − snstsn*
1,8,1,7,1,2; 6	*m − ns − n − p − ns − n, sts − nsqs − nsqs − n*
1,8,1,7,1,2; 9	*−mqpq, −sn − sn − sn − sn − sn − sn − sn − st*
1,8,1,6,1,3; 0	*mqsnstsnsq, snsnpnsnsq*
1,8,1,6,1,3; 2	*−mnsqstsqsn, −s − n − p − n − snsqsn*
1,8,1,5,1,4; 0	*mqsnsqsnsqsnsq, snpnst*
1,8,1,5,2,3; 4	*m − nsts − nsts − n, sqsqsqs − n − p − n*
1,8,1,5,2,3; 6	*m − t − p − t, sqs − nsqs − ns − nsqs − ns − n*
1,8,1,5,2,3; 7	*m − tsqs − t, sqpqs − ns − ns − ns − ns − n*
1,8,1,3,4,3; 3	*m − nsts − nsts − n, stptsqsqsq*
7,0,3,5,2,3; 8	*m − nm − tm − nm − tm − nm − q − p − q, m − npqp − n*
7,1,2,7,1,2; 0	*mnmqmnsnmq, mnpnmnpnmt*
7,1,2,6,3,1; 4	*mtm − nmtstm − n, mqm − n − p − nm − n − p − n*
7,1,2,6,3,1; 6	*mqm − nm − nm − nm − nm − nm − n, stptpt*
7,1,2,6,1,3; 7	*m − nm − nm − nm − nm − nm − nm − t, sqpqpq*
7,1,2,6,1,3; 8	*−mn − mn − mqpqpq, −mn − mn − sn − mn − mt*
7,1,2,5,1,4; 2	*mtm − n − p − n − p − n, mqmqmqm − n − s − n*

7,1,2,5,1,4; 4	$mqm - nmqm - nmqsqm - n,\ m - n - p - t - p - n$
7,1,2,5,2,3; 3	$mtstm - n,\ mqmqm - n - p - nmqm - n - p - n$
7,1,2,5,3,2; 5	$mqm - nm - nmqm - nm - nm - n,\ stptpt$
7,1,2,5,2,3; 7	$m - nm - nm - nm - tm - nm - nm - t,\ sqpqpq$
7,1,2,3,4,3; 0	$mtmtmqmtmqmtmq,\ snpnpn$
7,1,2,3,4,3; 7	$m - nm - tm - nm - tm - nm - tm - t,\ sqpqpq$
7,1,2,3,4,3; 8	$-mn - mqpqpq,\ -mn - sn - mt - mt - mt - mt$
6,1,3,6,1,3; 2	$mtm - n - p - n - p - n - p - n,\ mqmqmqm - n - s - n$
6,1,3,6,1,3; 4	$mqm - nmqsqm - n,\ m - n - p - nm - n - p - t - p - n$
6,1,3,6,1,3; 8	$m - nm - tm - nm - q - p - q,\ m - ns - nm - npqp - n$
6,1,3,5,4,1; 2	$mtm - n - p - n - p - nmtm - n - p - n,\ mtstmq$
6,1,3,5,1,4; 5	$mqpqm - nm - nm - nm - nm - n,\ sqptpq$
6,1,3,5,4,1; 5	$mqm - nm - nm - nm - nm - n,\ stptptpt$
6,1,3,5,1,4; 6	$m - nm - nm - nm - nm - nm - t,\ sqpqpqpq$
6,1,3,5,3,2; 6	$m - nm - nm - nm - nm - nm - t - p - t,\ sqptpq$
6,1,3,5,3,2; 7	$-mn - mn - sn - mn - mt,\ -mn - mtpqpqpt$
6,1,3,3,4,3; 1	$mtstm - n - p - n,\ mtpnptmqmqmq$
6,1,3,3,4,3; 4	$mqm - t - p - t - p - t - p - t,\ mqsqm - nm - nm - n$
5,0,5,5,0,5; 2	$mqm - n - p - q - m - q - p - n,\ mqpnpqm - n - p - n$
5,1,4,5,1,4; 0	$mtmqmqmqmq,\ snpnpnpnpn$
5,1,4,5,4,1; 0	$mtmtmtmtmq,\ snpnpnpnpn$
5,1,4,5,1,4; 4	$mqptpqm - nm - n,\ m - n - p - q - s - q - p - nm - n$
5,1,4,5,2,3; 0	$mtmqmtmqmq,\ snpnpnpnpn$
5,1,4,5,3,2; 0	$mtmtmqmtmq,\ snpnpnpnpn$
5,1,4,5,3,2; 1	$mtm - n - p - n - p - n - p - n - p - n,\ mtstmqmq$
5,1,4,5,2,3; 2	$mqmqm - n - p - nmqm - n - p - n,\ stpnpt$
5,1,4,5,2,3; 7	$m - nm - tm - ns - nm - t,\ m - npqpqpqp - n$
5,1,4,5,2,3; 9	$-mn - pn - sn - pn - mq,\ -mt - mt - mqp - npq$
5,1,4,3,4,3; 6	$-mn - mt - mn - sn - mt,\ -mtpqpqpqpt$
5,2,3,5,2,3; 0	$mtmqmtmqmq,\ snpnsnpnpn$
5,2,3,5,3,2; 0	$mtmtmqmtmq,\ snpnsnpnpn$
5,2,3,5,2,3; 4	$mqsqsqm - nm - n,\ m - n - p - t - p - t - p - nm - n$
5,2,3,5,2,3; 8	$m - ns - ns - nm - q - p - q,\ m - npqp - nm - tm - t$
5,2,3,3,4,3; 8	$m - tm - tm - tm - tm - q - p - q,\ s - ns - npqp - n$
5,2,3,3,4,3; 9	$-mt - mqp - npq - mt - mq,\ -mt - sn - pn - st$
3,4,3,3,4,3; 2	$mqstptsqm - n,\ m - n - p - t - s - q - s - t - p - n$
3,4,3,3,4,3; 4	$m - nm - t - p - t - p - t - p - t,\ m - nsqsqsqs - n$