

SIMPLIFYING SPINES OF 3-MANIFOLDS

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It is well known that every compact 3-manifold has a spine that is a 2-dimensional cell complex with just one vertex. Such a cell complex determines a group presentation in a natural way. It seems natural to call K a simpler spine than K' if the presentation corresponding to K is shorter than that corresponding to K' . In this paper we give an algebraic condition which is sufficient to guarantee the existence of a simpler spine.

Of course, identifying the simplest spine of a 3-manifold would allow one to solve the homeomorphism problem for 3-manifolds. From one point of view the difficulty with identifying the simplest spines arises from the lack of correspondence between algebraic operations on presentations and geometric alteration of spines. For example $\langle a, b \mid a^2b^3, a^3b^4 \rangle$ corresponds to a spine of S^3 but $\langle a, b \mid a^2b^3a^3b^4, a^3b^4 \rangle$ does *not* correspond to the spine of any 3-manifold. (See [7] for verification of this fact.) To state our result we need some definitions. Let $\phi = \langle X \mid \mathcal{R} \rangle$ be a group presentation, $X = \{x_1, x_2, \dots, x_n\}$ and $\mathcal{R} = \{R_1, R_2, \dots, R_k\}$ \mathcal{R} being a set of words in the free semi-group on $X \cup X^{-1}$. In what follows we will not distinguish between a relator R_i and any cyclic conjugate of it or its inverse. This convention is adopted because the complexes determined are the same. Let $\phi' = \langle X \mid R_1, R_2, \dots, R_{i-1}, R_iR_j, R_{i+1}, \dots, R_k \rangle$ where $j \neq i$. We shall say that ϕ' was obtained from ϕ by *multiplication of R_i and R_j* . If \mathcal{A} is an automorphism of $F(X)$ (the free group on X) we denote by $\mathcal{A}(R_i)$ the image of R_i under \mathcal{A} . We denote by $\mathcal{A}(\phi)$ the presentation $\langle X \mid \mathcal{A}(R_1), \dots, \mathcal{A}(R_k) \rangle$. The *length* of a presentation ϕ is the sum of the lengths of the freely reduced relators of ϕ .

THEOREM 1. *If K_ϕ is the spine of a 3-manifold M with corresponding presentation ϕ , $\bar{\phi}$ is obtained from ϕ by automorphism or multiplication and the length of $\bar{\phi}$ is less than the length of ϕ then M has a simpler spine than K_ϕ . This spine has a presentation ϕ' that can be obtained from ϕ by automorphism and or by multiplication. One can assume that ϕ' is at least as short as $\bar{\phi}$ if $\bar{\phi}$ was obtained from ϕ by automorphism or if ϕ could not be reduced in length by an automorphism.*

A proof of this theorem appears in §3. Actually a somewhat stronger theorem is proved, as will be pointed out. We now give some examples to illustrate Theorem 1.

EXAMPLE 1. $\phi = \langle a, b | a^2b^2, a^3b^3a^3b^5 \rangle$ corresponds to a spine of a closed 3-manifold M [7]. Furthermore ϕ cannot be reduced by an automorphism [9]. If we multiply a conjugate of the second relator by the inverse of the first relator and freely reduce we get $\bar{\phi} = \langle a, b | a^2b^2, aba^3b^5 \rangle$.

Theorem 1 guarantees that M has a simpler spine than that corresponding to ϕ . While $\bar{\phi}$ is not a presentation of such a spine we know that there is a spine of M whose presentation is at least as short as $\bar{\phi}$. (Incidentally, ϕ presents the quaternions and uniquely determines the known closed manifold with this fundamental group [4].)

EXAMPLE 2.¹ $\phi = \langle a, b, c | ac^{-1}ba^{-1}c, aba^{-1}b^{-1}c, ab^{-1}c^{-1}bc \rangle$ corresponds to a spine of a closed 3-manifold M . (Whose R - R system is shown in Figure 1.) There is an automorphism \mathcal{A} on $F(a, b, c)$ that

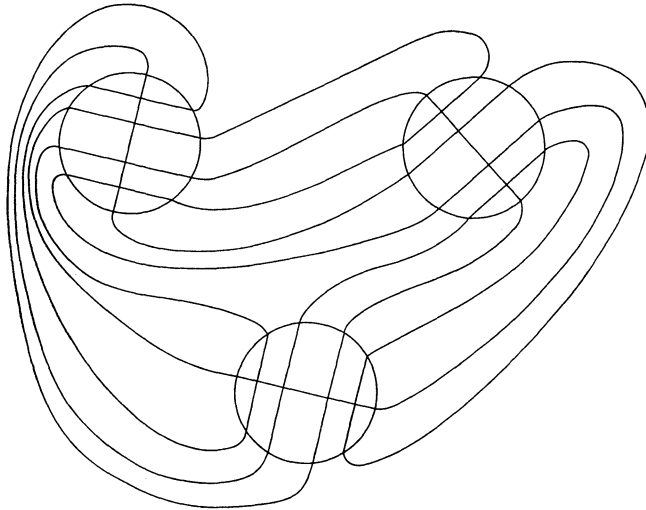


FIGURE 1

reduces the length of ϕ , namely that defined by $a \rightarrow a, b \rightarrow ba, c \rightarrow c$. Now $\mathcal{A}(\phi) = \langle a, b, c | ac^{-1}b^{-1}c, bc, aba^{-1}b^{-1}cbac \rangle$. The length of $\mathcal{A}(\phi)$ is 14, less than 15 which is the length of ϕ . It follows that M has a simpler spine than that corresponding to ϕ . This presentation is interesting because the group presented has a trivial abelianization. One can easily do an elimination and get a 2-generator presentation $\langle b, c | c^{-2}bcb^{-1}cb, b^{-2}ebc^{-1}bc \rangle$. It is not known to this author if this group is trivial or not. This is the simplest possible spine for a counterexample to the Poincaré conjecture. One can show that $b^5 = (c^{-1}b)^5$ and lies in the center of this group. Thus one can investigate

¹ Now known to be the binary dodecahedral space.

$\langle b, c | c^{-2}bc b^{-1}cb, b^{-2}cbc^{-1}bc, b^5, (c^{-1}b)^5 \rangle$. This group is trivial if and only if the original group is trivial. If this group is indeed trivial then we have a very exciting example to investigate because the manifold M survives every known method for showing that it is S^3 .

EXAMPLE 3. $\phi = \langle a, b | a^2b^3a^3b^3, a^2b^3a^2b^2 \rangle$ is the spine of a closed 3-manifold. Multiplying the first relator by the inverse of the second gives $\bar{\phi} = \langle a, b | ab, a^2b^3a^2b^2 \rangle$. By Theorem 1 there is a geometrically allowable multiplication that results in at least as short a presentation as $\bar{\phi}$. This implies that $\bar{\phi}$ is actually a spine of M since every multiplication gives this presentation or a longer one. Successive multiplications of the second relator by $(ab)^{-1}$ yield b . Again, if performed in the right order Theorem 1 implies these successive presentations must correspond to spines of M . We see then that M is S^3 . This proof is purely algebraic in nature. Once a presentation of this spine is obtained one need never consult the geometry again to show that we have a spine of S^3 .

2. Some powerful conjectures. In this section we present two group theoretic conjectures about presentations of spines which have very powerful consequences.

DEFINITION 2.1. The presentation ϕ is said to *reduce monotonically* to ϕ^* if there is a sequence of automorphisms and multiplications which monotonically reduce ϕ in length to ϕ^* . ϕ has the *monotone reduction property* if every presentation ϕ' to which ϕ reduces monotonically, itself reduces monotonically to the obviously trivial presentation $\langle x_1, x_2, \dots, x_n | x_1, x_2, \dots, x_n, 1, 1, \dots, 1 \rangle$.

Conjecture 1. If ϕ corresponds to the spine of a 3-manifold and ϕ reduces monotonically to the obviously trivial presentation, then ϕ has the monotone reduction property.

Conjecture 2. Conjecture 1 holds if we add the assumption that ϕ corresponds to the spine of a *closed* manifold.

Intuitively, these conjectures say that one cannot do the wrong reduction and get stuck when simplifying the spine of a simply connected 3-manifold.

THEOREM 2.1. *If K_ϕ is a spine of M and ϕ has the monotone reduction property then M must be a cube with holes.*

Proof. Since ϕ can be reduced in length it follows from Theorem 1 that M has a simpler spine than K_ϕ . The resulting shorter

presentation can again be reduced. Induction yields finally the obviously trivial presentation which can only correspond to a spine of a cube with holes.

COROLLARY 2.2. *Conjecture 1 implies that the Poincaré conjecture is true.*

Proof. In [3] Haken proves that a homotopy 3-sphere has a cell decomposition with one vertex whose 2-skeleton K_ϕ has a presentation ϕ which can be monotonically reduced to the obviously trivial presentation.

If Conjecture 1 holds then Theorem 2.1 implies that a regular neighborhood of K_ϕ is a cube with holes. This in turn implies that the original manifold was S^3 .

It is tempting to try to give a completely algebraic proof of Conjecture 1, that is try to prove all group presentations that can be monotonically reduced to the obviously trivial presentation must have the monotone reduction property. That this cannot be done is shown by the presentation $\langle a, b | a^{-1}bab^{-1}ab, a^2b^{-1}aba^{-1}b^{-1}a^{-1}b \rangle$. If we multiply the second relator by a conjugate of the inverse of the first relator we get $\langle a, b | a^{-1}bab^{-1}ab, b^{-2}a^{-1}ba \rangle$. Now multiplying the first relator by the second we get $\langle a, b | a^{-1}ba^2b^{-1}, b^{-2}a^{-1}ba \rangle$. Now we multiply the first relator by a conjugate of the inverse of the second relator to get $\langle a, b | ab, b^{-2}a^{-1}ba \rangle$. This presentation easily reduces monotonically to $\langle a, b | a, b \rangle$. We now return to the original presentation and do another series of multiplications and automorphisms. First multiply as shown:

$$(ab^{-1}aba^{-1}b)(b^{-1}a^{-1}ba^2b^{-1}aba^{-1}) = ab^{-1}aba^{-2}ba^2b^{-1}aba^{-1}$$

which cyclically reduces to $a^2ba^{-2}ba^2b^{-1}$. Our presentation is now

$$\langle a, b | a^{-1}bab^{-1}ab, a^2ba^{-2}ba^2b^{-1} \rangle .$$

Now we do the automorphism defined by $a \rightarrow a$ and $b \rightarrow ab^{-1}$ to get $\langle a, b | abab^{-2}, a^2b^{-1}a^{-1}b^{-1}a^2b \rangle$. We now multiply the second relator by the first to get $\langle a, b | abab^{-2}, ab^2ab^{-1}a^{-1}b^{-1} \rangle$. Now we do the automorphism defined by $a \rightarrow ab^{-1}$ and $b \rightarrow b$. We get $\langle a, b | a^2b^{-3}, abab^{-1}a^{-1}b^{-1} \rangle$. It is easy to check that this presentation cannot be reduced further by automorphism or multiplication. The above presentation gives a counterexample for Conjectures 2 and 4 of [1]. None of the above presentations correspond to a spine of a 3-manifold, hence do not provide counterexamples for our Conjectures 1 or 2.

THEOREM 2.3. *Conjecture 2 implies that monotone reduction of the presentation of a spine gives an algorithmic method for identify-*

ing S^3 among compact 3-complexes. The proof of this theorem is long and somewhat involved and will be presented in a subsequent paper.

Connections between transformations of group presentations and manifolds have a long history. Whitehead studied multiplications and automorphisms in [9]. In [2] Andrews and Curtis discuss changing presentations by what amounts to sequences of multiplications and their relations with 4-manifolds. In [5] the author relates these to the 3- and 4-dimensional Poincaré conjecture. Rapaport [8] shows that presentations of the trivial group which can be reduced to the obviously trivial presentation by automorphism and multiplication can be reduced by multiplication alone. This process does not preserve montone reduction, however.

3. Proof of Theorem 1. If ϕ can be reduced in length by an automorphism then Theorem 1 of [10] can be applied to show that there is a geometrically allowable transformation which yields at least as short a presentation. (See also [6] for the connection between Zieschang's theorem and presentations of spines.) We now assume that the manifold M has a spine K_ϕ whose corresponding presentation ϕ cannot be reduced by automorphism. Let us assume that $\phi = \langle x_1, \dots, x_n \mid R_1, \dots, R_k \rangle$ and that

$$\bar{\phi} = \langle x_1, \dots, x_n \mid R_1 R_2, R_2, \dots, R_k \rangle$$

is shorter than ϕ when the free cancellations in $R_1 R_2$ are done. Let us further suppose that $R_1 = U x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_r}^{n_r}$ and

$$R_2 = x_{i_r}^{-n_r} x_{i_{r-1}}^{-n_{r-1}} \dots x_{i_1}^{-n_1} V$$

where U and V are words on x_1, x_2, \dots, x_n and UV is a cyclically reduced word. Since $R_1 R_2$ is assumed to be shorter than R_1 it must be that the length of V is less than one half the length of R_2 .

Now we examine the (reduced) R - R system corresponding to

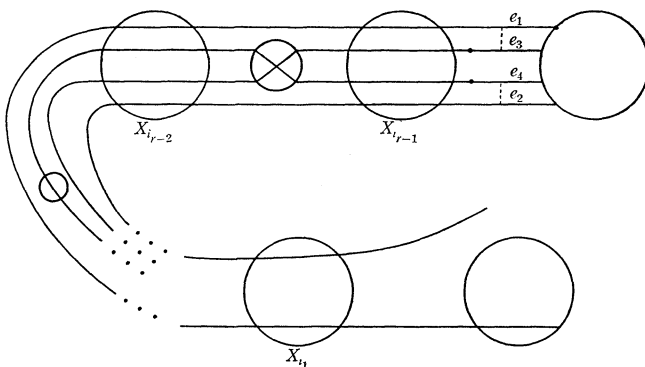


FIGURE 2

the faithfully embedded P -graph of K_ϕ . In what follows we shall assume that the reader is familiar with R - R systems and operations on them as developed in [6]. We locate on this R - R system the route e_1 of R_1 corresponding to the gap between $x_{i_r-1}^{n_r}$ and $x_{i_r}^{n_r}$ and also the route e_2 of R_2 corresponding to the gap between $x_{i_r-1}^{-n_r}$ and $x_{i_r}^{-n_r}$. (See Figure 2.) Now if we could always geometrically perform the multiplication between these two routes then the theorem would be proved. However we cannot always do this multiplication geometrically because there may be intervening routes. We shall find a different place to multiply geometrically that results in at least as much free cancellation as resulted between R_1 and R_2 . We assume without loss of generality that an end of e_2 lies clockwise from an end of e_1 in town x_{i_r-1} . Beginning at e_2 we proceed counter-clockwise until we encounter the first route not lying in R_2 . Call this route e_3 and the corresponding relator R_3 . (Note that $R_3 = R_2$ is possible.) Denote by e_4 the first clockwise route from e_3 . Of necessity e_4 is a route of R_2 . We multiply R_2 and R_3 between e_4 and e_3 . The idea of the proof that

$$\phi' = \langle x_1, \dots, x_n | R_1, R_2, R_3R_2, R_4, \dots, R_k \rangle$$

is as short as $\bar{\phi}$ is to argue that because e_3 and e_4 are “caught between” e_1 and e_2 the words R_2 and R_3 must be parallel (go through the same stations) at least as long as R_1 and R_2 . We refer to the region in which the edges of R_1 and R_2 pass through the same stations, where the syllables $x_{i_1}^{n_1}, \dots, x_{i_r}^{n_r}$ appear, as the free cancellation region. However, there are a few things that could go wrong with this argument. One of these is shown in Figure 2. As shown, there may be one or several towns in the free cancellation region so that the tracks of R_2 and R_3 do not remain parallel in this region. In this case we may remove all such towns from the free cancellation region by pulling them through the towns $x_{i_r-1}, x_{i_r-2}, \dots, x_{i_1}$ along tracks parallel to the tracks of R_1 and R_2 . This process amounts to an automorphism of ϕ . This automorphism does not change the length of the presentation. For if it did then pulling one of these

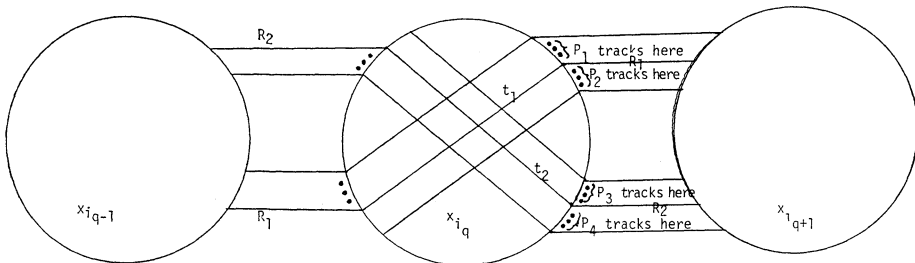


FIGURE 3

towns in the opposite direction would decrease the length by an automorphism, contradicting our hypothesis. There is one more type of difficulty which might arise. This is illustrated in Figure 3. This difficulty occurs when there is a cross of the supposedly parallel tracks of R_1 and R_2 somewhere in the free cancellation region. Since the corresponding syllables must be the same length it follows that they must both be length 1 or -1 since crossing syllables must have relatively prime length. Assume without loss of generality that both syllables have length 1. Let us assume that the cross appears in the town x_{i_q} with tracks t_1 and t_2 crossing, t_1 lying in R_1 , t_2 in R_2 . We shall further assume that there are: p_1 tracks in town x_{i_q} lying counterclockwise from t_1 in the same station as t_1 ; p_2 tracks in town x_{i_q} lying clockwise from t_1 in the same station as t_1 ; p_3 tracks lying counterclockwise from t_2 in the same station as t_2 ; and p_4 tracks in the same station as t_2 lying clockwise from t_2 . These assumptions imply that there are: $p_1 + p_4$ pairs of syllables of the form $(x_{i_{q-1}}^n x_{i_q})^{\pm 1}$; $p_2 + p_3$ pairs of syllables of the form $(x_{i_q} x_{i_{q+1}}^n)^{\pm 1}$ in ϕ ; and the two occurrences of syllables of the form, $(x_{i_{q-1}}^n x_{i_q} x_{i_{q+1}}^n)^{\pm 1}$ not counted above. All occurrences of x_{i_q} must be in one of the above forms or in the form $(x_{i_{q-1}}^n x_{i_q}^2 x_{i_{q+1}}^n)^{\pm 1}$. If $p_2 + p_3 + 2 > p_1 + p_4$ then the automorphism $x_{i_q} \rightarrow x_{i_q} x_{i_{q+1}}^{-1}$ and $x_j \rightarrow x_j$ for $j \neq i_{q+1}$ reduces the length of ϕ contrary to hypothesis. If $p_2 + p_3 \leq p_1 + p_4$ then $x_{i_q} \rightarrow x_{i_{q-1}}^{-1} x_{i_q}$ and $x_j \rightarrow x_j$ for $j \neq i_{q-1}$ defines an automorphism of ϕ that decreases its length. From the above argument we see that a cross cannot occur in the free cancellation region. A cross may occur at the ends of this region but this presents no problem since all free cancellations may be (geometrically) performed. This completes the proof.

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