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BOUNDED MONOIDS

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Monoids whose left S-sets X always satisfy $sl(X) \leqq h(X)$ are characterized in terms of chain conditions on principal left ideals.

For S a monoid, a left S-set (S-operand, S-system) is a set X on which S operates from the left and such that $1x = x$ for all $x \in X$ where $1 \in S$ is the identity. For any $x \in X$, an S-subset of X of the form Sx is called an orbit of X. It is well-known that every left S-set is a union of orbits and that, up to isomorphism, orbits are characterized by left congruences on S (see [1], Chapt. 11).

In order to study the way orbits fit together in an S-set X the author has in [2] and [3] constructed two chains of S-subsets (to be defined more fully below) of an S-set X, each having the property that the subquotients are essentially 0-disjoint unions of orbits. The lengths of those two chains are denoted by $h(X)$ and $sl(X)$. In [3] it is shown that when $h(X)$ is finite, then $sl(X) \leqq h(X)$.

Let us call a monoid *bounded* if for every S-set X, one has $sl(X) \leqq h(X)$. Then a main goal of this paper is to show that a monoid is bounded if and only if there is a positive integer n such that the monoid contains no proper chain of principal left ideals of length exceeding n.

1. Preliminaries. Let X be a left S-set. An S-subset Y of X is a (possibly empty) subset Y of X such that $sy \in Y$ for all $s \in S$ and $y \in Y$. If X and Y are both orbits, we may say that Y is a suborbit of X. If Z is an S-set, a homomorphism $\phi: X \to Z$ is a function such that $\phi(sx) = s\phi(x)$ for all $x \in X$ and $s \in S$. A congruence $\sim$ on an S-set X is an equivalence relation such that $x \sim y$ implies $sx \sim sy$ for $x, y \in X$ and $s \in S$. Denoting the set of congruence classes by $X/\sim$ one finds that $X/\sim$ is a left S-set under the induced action and is a homomorphic image of X under the natural map $X \to X/\sim$.

If $Y \subset X$ is an S-subset, define a congruence $\sim_Y$ on X by setting $x \sim_Y x'$ if and only if $x = x'$ or $x, x' \in Y$. Let us denote $X/\sim_Y$ simply by X/Y. If $Y \neq \varnothing$, the class of Y in X/Y is denoted by 0.

If X is a left S-set, call $x, y \in X$ *separated in* X if there is no $z \in X$ such that $x, y \in Sz$. Then let us define a descending chain of S-subsets of X by setting $X_0 = X$, $X_{i+1} = \cup\{Sx \cap Sy: x, y \in X_i$ are separated in $X_i\}$ for $i > 0$, and $X_\sigma = \cap\{X_\tau: \tau < \sigma\}$ for σ a limit ordinal. Then $sl(X)$, the *saturation length* of X, is the first ordinal

α such that $X_\alpha = X_{\alpha+1}$. In general it need not be that $X_\alpha = \varnothing$, but whenever S satisfies the ascending chain condition on orbits, it must be that $X_\alpha = \varnothing$ (see [2] and [3]).

Alternately, if one has an S-subset $Y \subset X$, then an element $x \in X$ is called *Y-distinguished* if for every $z \in X$ either $x \in Sz$ or $z \in Sx$ or $Sx \cap Sz \subset Y$. The *orbit* $Sx \subset X$ is called *Y-distinguished* if x' is Y-distinguished for every $x' \in Sx$. Then one can define an ascending chain of S-subsets of X by letting $X^0 = \varnothing$, $X^{i+1} = \cup\{Sx: Sx$ is an X^i-distinguished orbit of $X\} \cup X^i$ for $i > 0$, and $X^\sigma = \cup\{X^\tau : \tau < \sigma\}$ if σ is a limit ordinal. Let $h(X)$, the *height* of X, be the first ordinal β such that $X^\beta = X^{\beta+1}$. (Note: The definition of Y-distinguished orbit here is more restrictive than that used in [3] and corrects the definition given there in the sense that some (correctable) gaps in proofs these are trivially closed by the altered definition, the current definition being what the author had in mind in [3]. The principal difference is that in [3] $x^1 = x$, whenever x is an orbit, while here that need no longer be so. Here $h(X)$ is at least as in [3].) In general $X^\beta \neq X$, but if S satisfies the descending chain condition on principal left ideals, then one always has $X^\beta = X$ (see [3]).

Throughout the rest of this paper we shall assume that S satisfies the descending chain condition on principal left ideals. This assumption is justified by the following proposition.

PROPOSITION 1.1. *If S does not satisfy the descending chain condition on principal left ideals, then there is a left S-set X such that $h(X) = 1$, but such that $sl(X) = 2$.*

Proof. Suppose there is an infinite proper descending chain of principal left ideals in S. Denote it by $S \supsetneqq Sa_1 \supsetneqq Sa_2 \supsetneqq \cdots$, and set $a_0 = 1$. Let $I = \{x \in S$: for all i, $a_i \notin Sx\}$. Then if $I \neq \varnothing$, I is a left ideal of S which is maximal with respect to the property that for all $i = 1, 2, \cdots$ one has $a_i \notin I$. Form the left S-set $S \times N$, where $N = \{0, 1, \cdots\}$, by letting $s(t, n) = (st, n)$ for all $s, t \in S$ and $n \in N$.

Now let $X = (S \times N)/\sim$ where $\sim$ is the congruence on $S \times N$ given by $(s, m) \sim (t, n)$ if and only if either (i) $s, t \in I$ or (ii) $s = t \in Sa_k$ for some k and $m, n \leqq k$. (In (ii) one wants to use the largest possible k.)

Let $[y, n] \in X$ denote the class of $(y, n) \in S \times N$. If $z \in I$, then since $[z, m] = [z, n] = [z', n]$ for all $m, n \in N$ and all $z' \in I$ and since $s[z, m] = [sz, m] = [z, m]$, we can denote the class $[z, m]$ simply by 0. (If $I = \varnothing$, all comments about 0 below are vacuous.) The S-set X can be visualized schematically as in Figure 1, in which the action by S moves things downward along the indicated edges.

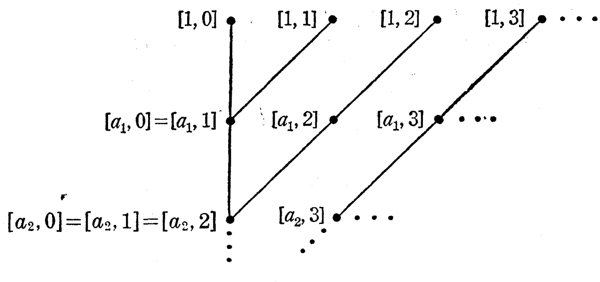

FIGURE 1

If $0 \neq [y, n] \in X$, then there is some m such that $y \in Sa_m$ but $y \notin Sa_{m+1} \cup I$. By construction of I, there must be some q such that $a_q \in Sy$. Without loss of generality we may assume that $q > m$ and $q > n$. If $[y, n] = [y, q]$, we must have $n = q$ or $n, q \leqq m$, both of which are impossible by choice of q; hence $[y, n] \neq [y, q]$. However, since $n < q$, we find that $0 \neq [a_q, q] = [a_q, n] \in S[y, n] \cap S[y, q]$. Hence for every nonzero orbit $S[y, n]$ of X, there is an orbit incomparable to it (under inclusion) with which it has nonzero intersection. This says $X^1 = \{0\}$ (or $X^1 = \varnothing$ if $I = \varnothing$) and that no $[y, n] \neq 0$ is X^1-distinguished. Hence $X^2 = X^1$, and $h(X) \leqq 1$.

On the other hand one easily sees that the elements $[1, n] \in X$ are all distinct for $n \in N$ and that these are the generators of the maximal orbits of X. Moreover, it is easy to see that for $n \neq m$ one has $S[1, n] \cap S[1, m] \subset S[a_1, 0] \cup \{0\}$. While $[a_1, 0] \in S[1, 0] \cap S[1, 1]$. Hence $X_1 = S[a_1, 0] \cup \{0\}$, which is either an orbit or the disjoint union of two orbits (depending on whether or not $0 \in S[a_1, 0]$). Thus $X_2 = \varnothing$, which gives $sl(X) = 2$.

Hence we will assume in what follows that S satisfies the descending chain condition on principal left ideals.

For any monoid S we now define two functions from S to $N \cup \{\infty\}$. Let $h = h_S: S \to N \cup \{\infty\}$ be given by $h(x) = \sup\{n$: there exists $a_1, a_2, \cdots, a_n \in S$ such that $Sx = Sa_1 \supsetneqq Sa_2 \supsetneqq \cdots \supsetneqq Sa_n\}$ for $x \in S$, and let $d = d_S: S \to N \cup \{\infty\}$ be given by $d(x) = \sup\{n$: there exist $a_1, a_2, \cdots, a_n \in S$ such that $Sx = Sa_1 \subsetneqq Sa_2 \subsetneqq \cdots \subsetneqq Sa_n\}$ for $x \in S$. We can call $h(x)$ the *height of x in S* and $d(x)$ the *depth of x in S*.

2. Bounded monoids.

We wish to characterize monoids S such that $sl(X) \leqq h(X)$ for every S-set X. The following theorem justifies calling such monoids *bounded*, and we may take any one of its equivalent conditions as a definition of bounded monoid.

THEOREM 2.1. *For a monoid S the following are equivalent.*
(i) *$sl(X) \leqq h(X)$ for every S-set X.*

(ii) *There exists an integer n such that S contains no proper chain of principal left ideals of length exceeding n.*

(iii) *S satisfies the descending chain condition on principal left ideals and $h(X) < \infty$ for every S-set X.*

(iv) *$h(a) < \infty$ for every $a \in S$.*

(v) *S satisfies the descending chain condition on principal left ideals and $d_T(a) < \infty$ for every $a \in T = S/I$ where $I = \{x \in S\colon Sx$ is a minimal left ideal$\}$.*

Proof. One sees immediately that (ii) implies (iv) and (ii) implies (v).

That (iii) implies (i) follows from Theorem 5 of [3] provided one observes that the descending chain condition on principal left ideals of S implies that every left S-set satisfies the descending chain condition on orbits (the hypothesis used in [3]). This follows since any chain $Sx_1 \supset Sx_2 \supset \cdots$ of orbits in an S-set X gives rise to a chain of principal left ideals $S = Sa_1 \supset Sa_2 \supset \cdots$ in S where $a_1 = 1$ and $a_{i+1} \in Sa_i$ is such that $a_{i+1}x_1 = x_{i+1}$ for $i \geqq 1$. Thus if $\phi\colon S \to X$ is the homomorphism of S-sets given by $\phi(s) = sx_1$ for $s \in S$, then $\phi(a_i) = x_i$ for all $i = 1, 2, \cdots$. Since the chain of left ideals $Sa_1 \supset Sa_2 \supset \cdots$ must terminate, so must the chain of orbits $Sx_1 \supset Sx_2 \supset \cdots$.

To see that (iv) implies (ii) one need only observe that since $S = S1$ is a principal left ideal containing every other principal left ideal, one can have no proper chain of principal left ideals of length exceeding $h(1)$.

To show that (v) implies (iv) it suffices to show that $h(1) < \infty$. Now $I = \{x \in S\colon Sx$ is a minimal left ideal in $S\}$ is a two-sided ideal in S. In $T = S/I$, denote the image of $x \in S$ by $\bar{x} \in T$. Let $a_1 = 1$, $a_2, \cdots, a_n \in S$ be such that $Sa_1 \supsetneqq Sa_2 \supsetneqq \cdots \supsetneqq Sa_n$. Since S satisfies the descending chain condition on principal left ideals we may assume that Sa_n is minimal, i.e., that $a_n \in I$. Then in T one has $T\bar{a}_1 \supsetneqq T\bar{a}_2 \supsetneqq \cdots \supsetneqq T\bar{a}_{n-1} \supsetneqq \{0\}$. This says that $n \leqq d_T(0)$. Hence, in turn one can conclude that $h(1) = \sup\{n\colon S = Sa_1 \supsetneqq \cdots \supsetneqq Sa_n\} \leqq d_T(0) < \infty$, which establishes (iv).

To show that (iv) implies (iii) we observe that since (iv) implies (ii), S must satisfy the descending chain condition on principal left ideals. We now show that for every S-set X one has $\mathrm{h}(X) \leqq h(1) = n$. To do this let X be an S-set and for $x \in X$, let $h_X(x) = \sup\{n\colon Sx = Sx_1 \supsetneqq Sx_2 \supsetneqq \cdots \supsetneqq Sx_n$ for $x_1, \cdots, x_n \in X\}$. If $\psi\colon S \to Sx$ is the homomorphism of S-sets given by $\psi(s) = sx$ for $s \in S$ and $x \in X$, and if $Sx = Sx_1 \supsetneqq Sx_2 \supsetneqq \cdots \supsetneqq Sx_m$ is a proper descending chain of suborbits of Sx, then there are $a_1, \cdots, a_m \in S$ such that $\psi(a_i) = x_i$ for $i = 1, \cdots, m$ and such that $S = Sa_1 \supsetneqq Sa_2 \supsetneqq \cdots \supsetneqq Sa_m$. Hence, since we may take $a_1 = 1$, this shows that $m \leqq h(1) = n$. Therefore for every $x \in X$, one finds that $h_X(x) \leqq n$. Let $X(k) = \{x \in X: h_X(x) \leqq k\}$ for $k = 1, \cdots, n$. Hence $X(1) \subset X(2) \subset \cdots \subset X(n) = X$.

To complete the proof that $h(X) \leqq n$, we now show by induction that $X(k) \subset X^k$ for $k = 1, \cdots, n$. We observe first that $h_X(sx) \leqq h_X(x)$ for all $x \in X$ and all $s \in S$. Hence $X(k)$ is an S-subset of X for all $k = 1, \cdots, n$. Since $X(1)$ must be the union of the minimal orbits of X, one finds that $X(1) \subset X^1$. Now if $X(k) \subset X^k$ and if $x \in X(k+1)$ with $x \notin X^k$, then sx must be an $X(k)$-distinguished orbit (for every proper suborbit of Sx must lie in $X(k)$). Hence Sx must be an X^k-distinguished orbit, which shows that $x \in X^{k+1}$, and hence that $X(k+1) \subset X^{k+1}$ as desired. But thus $X^n \supset X(n) = X$, which says that $k(X) \leqq n$.

Finally we show that (i) implies (ii). By virtue of Proposition 1.1, we know that S must satisfy the descending chain condition on principal left ideals. We now complete the proof that (i) implies (ii) in two steps. First we show that if there is a proper chain of principal left ideals of S of length n, then there is a left S-set V_n such that $sl(V_n) = h(V_n) = n$. Second we show that if such V_n can be constructed for all $n \geqq 1$, then we can construct a left S-set X such that $h(X) = \omega$ while $sl(X) = \omega + 1$, where ω in the first infinite ordinal. Since this contradicts (i), it must be that there is a positive integer n such that S contains no proper chain of principal left ideals of length exceeding n, which is the statement of (ii). We now proceed with the construction.

Suppose that $Sa'_1 \subsetneqq Sa'_2 \subsetneqq Sa'_3 \subsetneqq \cdots \subsetneqq Sa'_n$ is a proper chain of principal left ideals. Let $U'_1 = \{x \in S: Sx \text{ is a minimal left ideal}\}$, and for $1 \leqq i \leqq n-1$ let $U'_{i+1} = U'_i \cup \{x \in S: Sx \text{ is minimal such that } x \notin U'_i\}$. If $U'_i \neq S$, then $U'_i \subsetneqq U'_{i+1}$ since S satisfies the descending chain condition on principal left ideals. However, we observe that $a'_2 \notin U'_1$, and inductively, that if $a'_k \notin U'_{k-1}$, then $a'_{k+1} \notin U'_k$ for $2 \leqq k \leqq n-1$. Hence we have a proper chain of left ideals $U'_1 \subsetneqq \cdots \subsetneqq U'_n$.

Observe now that if $x \in U'_k$ and $x \notin U'_{k-1}$ for $2 \leqq k \leqq n$, then there is some $y \in U'_{k-1} \cap Sx$ such that $y \notin U'_{k-2}$. (Here and below we let $U'_0 = \varnothing$.) This is so because $x \notin U'_{k-1}$ implies that there is an element $y \in Sx$ such that $Sy \subsetneqq Sx$ and such that $y \notin U'_{k-2}$. Choosing y so that Sy is minimal with respect to this property implies that $y \in U'_{k-1}$.

Let $a_n \in U'_n$ such that $a_n \notin U'_{n-1}$. Using the above observation inductively, one finds $a_i \in U'_i$ for $i = 1, \cdots, n$, such that $a_i \notin U'_{i-1}$ and such that $Sa_1 \subsetneqq Sa_2 \subsetneqq \cdots \subsetneqq Sa_n$.

Let us now consider the left S-set $U_n = U'_n/U'_1$. Denote the image of U'_i in U_n under the natural homomorphism by U_i for $i = 1, \cdots, n$, and denote the class of U'_1 by 0. Hence we have a proper chain of S-subsets of U_n given by $\{0\} = U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_n$. We may continue to write nonzero elements of U_n the same as elements of U'_n; thus we write $a_i \in U_i$ for $i = 2, \cdots, n$.

Finally we form the left S-set $U_n \times \{1, \cdots, 2^{n-1}\}$ by letting $s(x, \alpha) = (sx, \alpha)$ for $s \in S$, $x \in U_n$ and $1 \leqq \alpha \leqq 2^{n-1}$, and then we set $V_n = (U_n \times \{1, \cdots, 2^{n-1}\})/\sim$ where $\sim$ is the congruence relation given by $(x, \alpha) \sim (y, \beta)$ if and only if $x = y \in U^k$ for some k and $2^{n-k}p < \alpha, \beta \leqq 2^{n-k}(p+1)$ for some p with $0 \leqq p < 2^{k-1}$. (One wishes to use the smallest possible k here.)

Observe that this implies that if $x \in U_n$ and $x \notin U_{n-1}$, then $(x, \alpha) \sim (x, \beta)$ only when $\alpha = \beta$, while for $x = 0$, i.e., $x \in U_1$, $(x, \alpha) \sim (x, \beta)$ for all $1 \leqq \alpha, \beta \leqq 2^{n-1}$. We can picture V_n as a binary tree such that every path from a leaf to the root looks like a copy of U_n. See Figure 2. Denote the class of (x, α) in V_n by $[x, \alpha]$. We have a natural epimorphism of S-sets $\pi\colon V_n \twoheadrightarrow U_n$ given by $\pi([x, \alpha]) = x$.

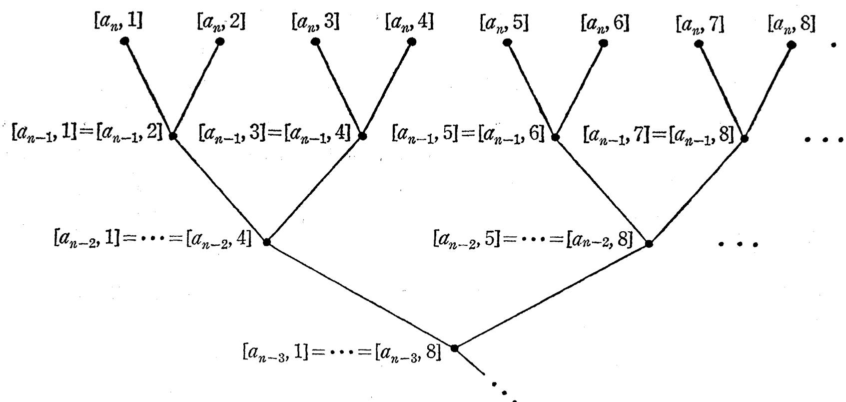

FIGURE 2

Let us now record properties of V_n in the following lemmas. By convention $U_0 = \varnothing$.

LEMMA 2.1.1. *Let $[x, \alpha] \in \pi^{-1}(U_k)$ with $[x, \alpha] \notin \pi^{-1}(U_{k-1})$ for some $1 \leqq k \leqq n$.*

(i) *For every $[y, \beta] \in V_n$ either $[x, \alpha] \in S[y, \beta]$ or $S[x, \alpha] \cap S[y, \beta] \subset \pi^{-1}(U_{k-1})$. Thus $[x, \alpha]$ is $\pi^{-1}(U_{k-1})$-distinguished.*

(ii) *$[y, \beta] \in \pi^{-1}(U_k)$ with $[y, \beta] \notin \pi^{-1}(U_{k-1})$, then either $S[x, \alpha] = S[y, \beta]$ or $[x, \alpha]$ and $[y, \beta]$ are separated in $\pi^{-1}(U_k)$ and $S[x, \alpha] \cap S[y, \beta] \subset \pi^{-1}(U_{k-1})$.*

Proof. If $[z, \gamma] \in S[x, \alpha] \cap S[y, \beta]$, then by construction of V_n, $[z, \gamma] = [z, \alpha] = [z, \beta]$ and $z \in Sx \cap Sy$ in U_n. But $x \in U_k$ implies that $z \in U_\kappa$. If $z \notin U_{\kappa-1}$, then by construction of U_κ, $Sx = Sz$. Thus

$[x, \alpha] \in S[x, \alpha] = S[z, \alpha] \subset S[y, \beta]$. Hence if $[x, \alpha] \notin S[y, \beta]$, then $z \in U_{k-1}$, which says $S[x, \alpha] \cap S[y, \beta] \subset \pi^{-1}(U_{k-1})$. This gives (i).

For (ii) suppose $[x, \alpha], [y, \beta] \in S[w, \delta]$ for some $[w, \delta] \in \pi^{-1}(U_k)$. Then as in proof of (i) we show $S[x, \alpha] = S[y, \beta] = S[w, \delta]$. Hence if $[x, \alpha]$ and $[y, \beta]$ are not separated in $\pi^{-1}(U_\kappa)$, then $S[x, \alpha] = S[y, \beta]$. This gives (ii).

LEMMA 2.1.2. (i) *For* $1 \leqq k \leqq n-1$, $\{[a_{n-k}, 2^k p + 1]: 0 \leqq p < 2^{n-k-1}\} \subset (V_n)_k \subset \pi^{-1}(U_{n-k})$.

$$\text{(ii)} \quad sl(V_n) = n\,.$$

Proof. (i) implies (ii), since (i) for $k = n-1$ implies that $\phi \neq (V_n)_{n-1} \subset \pi^{-1}(U_1) = \{0\}$ which implies $(V_n)_n = \varnothing$. Thus $sl(V_n) = n$.

We establish (i) by induction on k. For $k = 1$, we see by Lemma 2.1.1 that the set $\{[x, \alpha]: [x, \alpha] \in \pi^{-1}(U_n) = V_n$ but $[x, \alpha] \notin \pi^{-1}(U_{n-1})\}$ generates maximal orbits in V_n and that the intersection of orbits generated by a separated pair of elements is contained in $\pi^{-1}(U_{n-1})$. Hence $(V_n)_1 \subset \pi^{-1}(U_{n-1})$.

On the other hand, $[a_n, 2p+1]$ and $[a_n, 2(p+1)]$ are separated in V_n for $0 \leqq p < 2^{n-2}$ by Lemma 2.1.1. Thus $[a_{n-1}, 2p+1] \in S[a_n, 2p+1] \cap S[a_n, 2(p+1)] \subset (V_n)_1$. This gives (i) for the case $k = 1$.

Suppose by induction that (i) is true for $1 < k < n-1$. We establish it for $k+1$.

Since $(V_n)_k \subset \pi^{-1}(U_{n-k})$, we see that if $[x, \alpha]$ and $[y, \beta] \in (V_n)_k$ are separated in $\pi^{-1}(U_{n-k})$, then they are separated in $(V_k)_\kappa$ and by Lemma 2.1.1 $S[x, \alpha] \cap S[y, \beta] \subset \pi^{-1}(U_{n-k-1})$. Thus $(V_n)_{k+1} \subset \pi^{-1}(U_{n-(k+1)})$.

Finally, if $\{[a_{n-k}, 2^k p + 1]: 0 \leqq p < 2^{n-k-1}\} \subset (V_n)_k$, we observe by Lemma 2.1.1 that $[a_{n-k}, 2^k(2p) + 1]$ and $[a_{n-k}, 2^k(2p+1) + 1]$ are separated in $(V_n)_k$ for $0 \leqq p < 2^{n-k-2}$. Thus $[a_{n-k-1}, 2^{k+1}p + 1] \in S[a_{n-k}, 2^k(2p) + 1] \cap S[a_{n-k}, 2^k(2p+1) + 1] \subset (V_n)_{k+1}$ for $0 \leqq p < 2^{n-k-2}$. This completes the lemma.

LEMMA 2.1.3. (i) *For* $1 \leqq k \leqq n-1$, $(V_n)^k \supset \pi^{-1}(U_k)$ *but* $[a_{k+1}, 1] \notin (V_n)^k$.

$$\text{(ii)} \quad h(V_n) = n\,.$$

Proof. To see that (ii) follows from (i) we see that when $k = n-1$ we have $V_n \supsetneqq (V_n)^{n-1} \supset \pi^{-1}(U_{n-1})$. But by Lemma 2.1.1 every remaining element in V_n will be $\pi^{-1}(U_{n-1})$-distinguished and hence $((V_n)^{n-1})$-distinguished. This gives $(V_n)^n = V_n$, which says $h(V_n) = n$.

We establish (i) by induction on k. For $k = 1$, $\pi^{-1}(U_1) = \{0\}$, a fixed point, so that $(V_n)^1 \supset \{0\} = \pi^{-1}(U_1)$. However, by Lemma 2.1.1, $[a_2, 1]$ and $[a_2, 2^{n-2}+1]$ generate orbits which are incomparable under inclusion. Yet $0 = [a_1, 1] \in S[a_2, 1] \cap S[a_2, 2^{n-2}+1]$. This assures us that $[a_2, 1] \notin (V_n)^1$.

Suppose by induction that (i) is true for $1 < k < n-1$. Thus $(V_n)^k \supset \pi^{-1}(U_k)$. By Lemma 2.1.1 every element of $\pi^{-1}(U_{k+1})$ is either in $\pi^{-1}(U_k)$ or is $\pi^{-1}(U_k)$-distinguished. Hence if $[x, \alpha] \notin (V_n)^k$ but $[x, \alpha] \in \pi^{-1}(U_{k+1})$, then $S[x, \alpha]$ is a $\pi^{-1}(U_k)$-distinguished orbit, and hence a $(V_n)^k$-distinguished orbit. Thus $[x, \alpha] \in (V_n)^{k+1}$, which gives $\pi^{-1}(U_{k+1}) \subset (V_n)^{k+1}$.

On the other hand, if $[a_{k-1}, 1] \notin (V_n)^k$, then since $[a_{k+1}, 1] \in S[a_{k+2}, 1] \cap S[a_{k+2}, 2^{n-k-2}+1]$ and since $S[a_{k+2}, 1]$ and $S[a_{k+2}, 2^{n-k-2}+1]$ are incomparable under inclusion, we see that $[a_{k+2}, 1] \notin (V_n)^{k+1}$. This concludes the induction.

Finally, to complete the proof that (i) implies (ii) in Theorem 2.1, let us suppose that S contains chains of principal left ideals of all lengths. Thus for each $n \geqq 1$ we can construct an S-set V_n as above. Notice that each V_n has a fixed point $0_n \in V_n$. Let X be the 0-disjoint union of the $\{V_n : n \geqq 1\}$, that is, $X = (\bigcup_{n=1}^{\infty} V_n)/\approx$ where $\bigcup_{n=1}^{\infty} V_n$ is the disjoint union of the V_n and where $\approx$ is the congruence on this disjoint union given by $v \approx v'$ if and only if $v = v'$ or $v = 0_n$ and $v' = 0_m$ for some m and n.

To compute $h(X)$ one finds that $X^1 = \{0\}$ and that $X^k = (\bigcup_{n=1}^{\infty} (V_n)^k)/\approx$. Thus for every integer k one has $X^k \neq X$, but $\bigcup_{k=1}^{\infty} X^k = X$. This gives $h(X) = \omega$.

However one also sees for every integer k that $\{0\} \in X_k = (\bigcup_{n=1}^{\infty} (V_n)_k)/\approx$. Hence $\{0\} = \bigcap_{k=1}^{\infty} X_k = X_\omega$. But then $X_{\omega+1} = \varnothing$, which says that $sl(X) = \omega + 1 > h(X)$ contrary to (i). This completes the proof of Theorem 2.1.

REMARK. To see that the result fails if one simply demands that S satisfy both the ascending and descending chain condition on principal left ideals, or even both chain conditions on orbits, or if in (v) one only demands that $d(a) < \infty$ for all $a \in S$, we can consider the following example. Let $C_n = \{a_n, a_n^2, \cdots, a_n^{n-1}, a_n^n = 0_n\}$ be finite, cyclic, nil semigroups. Let $S = (\bigcup_{n=1}^{\infty} C_n) \cup \{1\}$ where 1 is a two-sided identity and where we set $a_n^k a_m^l = 0_m$ for $m \neq n$ and multiply as in C_n otherwise. Then S satisfies both chain conditions on principal left ideals and on orbits and every $a \in S$ is such that $d(a) < \infty$. But $h(1) = \infty$ and S chains of principal left ideals of arbitrary length.

3. Monoids bounded by n.

In order to analyze in more detail what can happen let us call a monoid S *bounded by* $n \geqq 0$ if for every S-set X we have $sl(X) \leqq h(X) \leqq sl(X) + n$. In [3] it was shown that for every given positive integer n there is a monoid S and an S-set X such that $h(X) = sl(X) + n$.

LEMMA 3.1. *If S is a bounded monoid and if S contains a proper chain of principal left ideals of length $n + 2$ for an integer $n \geqq 0$, then there exists a left S-set such that $h(X) = sl(X) + n$.*

Proof. Let us note first that the condition of the lemma is equivalent to the statement that $h(1) \geqq n + 2$ in S.

Since S is bounded, it satisfies both chain conditions on principal left ideals. We can use a construction similar to that used in Proposition 1.1 to create an S-set X such that $sl(X) = 2$ and $h(X) = n + 2$. Suppose that we have $Sa_0 \supsetneqq Sa_1 \supsetneqq \cdots \supsetneqq Sa_{n+1}$ a proper chain of principal left ideals. Since S is bounded, we may assume that the a_i are chosen so that $h(a_0) = n + 2$ in S. This guarantees that Sa_0 contains no longer proper chain of principal left ideals, that Sa_{n+1} is minimal, and that each Sa_i is minimal among principal left ideals properly containing Sa_{i+1} for $i = 0, \cdots, n$. Now as in Proposition 1.1 we set $X = (Sa_0 \times \{0, \cdots, n + 1\})/\sim$ where $(x, \alpha) \sim (x, \beta)$ if $x \in Sa_i$ and $\alpha, \beta \leqq i$ for $0 \leqq i \leqq n + 1$. We denote the class of (x, α) in X by $[x, \alpha]$.

In X the set $\{[a_0, \alpha]: 0 \leqq \alpha \leqq n - 1\}$ is a set of generators for all the distinct maximal orbits of X. As in Proposition 1.1 we see that $X_1 = S[a_1, 0]$ since $[x, \gamma] \in S[a_0, \alpha] \cap S[a_0, \beta]$ for $\alpha \neq \beta$ implies that $[x, \gamma] = [x, 0]$ and $x \in Sa_1$, and since $[a_1, 0] \in S[a_0, 0] \cap S[a_0, 1]$. Thus since X_1 is an orbit, $X_2 = \varnothing$, which gives $sl(X) = 2$.

In order to show that $h(X) = n + 2$, we can first observe that if we define a function $h_X: X \to N$ as in the proof that (iv) implies (iii) in Theorem 2.1, then $h_X([a_0, \alpha]) = h_S(a_0) = n + 2$. Thus as in the proof of Theorem 2.1, we can conclude that $h(X) \leqq n + 2$. It now suffices to show that for $k < n + 2$, we have $[a_{n-1-k}, 0] \notin X^k$. This we do by induction on k. For $k = 0$, there is nothing to prove since $X^0 = \varnothing$. Thus suppose that for $0 \leqq k < n + 1$ we have $[a_{n-1-k}, 0] \notin X^k$. By construction of X we see that $[a_{n-1-\kappa}, 0] = [a_{n-1-k}, n - 1 - k] \in S[a_0, n - 1 - k]$. However $[a_{n-1-k}, 0] \in S[a_{n-2-k}, 0]$, and $X, S[a_{n-2-k}, 0]$ and $S[a_0, n - 1 - k]$ are incomparable under inclusion. Thus since $[a_{n-1-k}, 0] \in S[a_{n-2-k}, 0] \cap S[a_0, n - 1 - k]$ and since $[a_{n-1-k}, 0] \notin X^k$, we find that $[a_{n-1-(k+1)}, 0] \notin X^{k+1}$, which completes the induction. Hence letting $k = n + 1$ we get $[a_0, 0] \notin X^{n+1}$ while $X = X^{n+2}$, so that $h(X) = n + 2$ as desired.

We can now characterize monoids bounded by n.

THEOREM 3.2. *Let $n \geqq 0$ be an integer. A monoid S is bounded by n if and only if S contains no proper chain of principal left* *ideals of length exceeding $n + 2$. Furthermore, if there is a chain of this length, then there is a left S-set X such that $h(X) = sl(X) + n$.*

Proof. By Theorem 2.1 we know that $sl(X) \leqq h(X)$ for every left S-set X if and only if the lengths of chains of principal left ideals in S are bounded. If S contains a chain of principal left ideals of length $n + 2$, then Lemma 3.1 produces a left S-set X such that $sl(X) \leqq h(X) = sl(X) + n$. Thus the inequality $h(X) \leqq sl(X) + n$ for every S-set X implies that S contains no proper chain of principal left ideals of length exceeding $n + 2$.

It remains to show that if every chain of principal left ideals in S is of length not exceeding $n + 2$, then for every left S-set X we have $h(X) \leqq sl(X) + n$. Now as in Lemma 3.1 we observe that our condition says that $h(1) \leqq n + 2$ in S and as in the proof of Lemma 3.1 and Theorem 2.1, this implies that $h(X) \leqq n + 2$ for every left S-set X. Since chains of principal left ideals are bounded in S, one finds that S satisfies both chain conditions on orbits. Thus $sl(X) = 0$ if and only if $X = \varnothing$, which is if and only if $h(X) = 0$. Hence we need only consider the case $sl(X) = 1$. But $sl(X) = 1$ implies that X is a disjoint union of orbits. Thus it suffices to suppose that X is a single orbit, say, $X = Sx_0$.

As in Lemma 3.1 and the proof that (iv) implies (iii) in Theorem 2.1 we can define $h_X: X \rightarrow N$ and find that $h_X(x_0) \leqq h_S(1) \leqq n + 2$ and that $X_k \supset \{x \in X: h_X(x) \leqq k\}$ for $k > 0$. Thus $X^n \supset \{x \in X: h_X(x) \leqq n\}$. Suppose now that $x \notin X^n$. Then $h_X(x) \geqq n + 1$. If $h_X(x) = n + 2$, then since $h_X(x) \leqq h_X(x_0) \leqq n + 2$, we must have $Sx = Sx_0 = X$. Thus $y \in Sx$ for all $y \notin X^n$, and x is X^n-distinguished. If $h_X(x) = n + 1$ and if $y \notin X^n$ is such that $x \notin Sy$, then we must have $h_X(y) = n + 1$. Thus either $y \in Sx$ or $h_x(z) \leqq n$ for all $z \in Sx \cap Sy$ since $Sx \cap Sy \subsetneqq Sx$ and $Sx \cap Sy \subsetneqq S_y$. But this says $Sx \cap Sy \subset X^n$. Hence every element of X is X^n-distinguished. Therefore $X = Sx_0$ is an X^n-distinguished orbit so that $X = X^{n+1}$. This shows that $h(X) \leqq n + 1 = n + sl(X)$ as desired.

4. Further properties. Let S be a bounded monoid and let $h(1) = n$, where h is the height function on elements of S. Let $I_k = \{x \in S: h(x) \leqq k\}$, $1 \leqq k \leqq n$. Then we can observe the following properties of S.

PROPOSITION 4.1. *Let S, h, n, and I_k be as above.*

(i) *For all $x, y \in S$, one has $h(xy) \leqq h(x)$ and $h(xy) \leqq h(y)$.*

(ii) *Each I_k is a two-sided ideal in S.*

(iii) *I_{n-1} consists of the nonunits of S.*

(iv) *$x \in S$ is a unit if and only if $h(s) = h(1) = n$.*
(v) *If $x \mathscr{J} y$ in S, then $h(x) = h(y)$.*

Proof. (i) Since $xy \in Sy$, it is clear that $h(xy) \leqq h(y)$. On the other hand, since one has an S-map $\phi\colon Sx \twoheadrightarrow Sxy$ given by $\phi(t) = ty$ for $t \in Sx$, the inverse image of any proper chain of principal left ideals contained in Sxy gives rise to such a chain contained in Sx. Hence $h(xy) \leqq h(x)$.

(ii) By (i) if $h(x) \leqq k$, then $h(ax)$, $h(xa) \leqq h(x) \leqq k$ for all $a \in S$. Thus each I_k is a two-sided ideal.

(iii) and (iv). For $x \in S$, $h(x) = h(1) = n$ if and only if $Sx = S$, i.e., if and only if x has a left inverse. But if $a \in S$ is a left inverse of x, then $ax = 1$ and by (i) $n = h(1) = h(ax) \leqq h(a) \leqq n$, so that a also has a left inverse. Thus $xa = 1$ and both are two-sided units.

(v) If $x \mathscr{J} y$ in S, then $SxS = SyS$ so that $x \in I_k$ if and only if $y \in I_k$ for $1 \leqq k \leqq n$. But since $x \in I_k$ for $k = h(x)$ and $y \in I_k$ for $k = h(y)$ we get $h(x) = h(y)$ as desired.

We should observe that S being bounded is not a strong enough condition for S to satisfy $\mathscr{J} = \mathscr{D}$. If $S = T \cup \{1\}$ where 1 is an identity and T is a simple, idempotent free semigroup which contains a minimal left ideal, then S is bounded (in fact, $h(1) = 2$), but $\mathscr{J} \neq \mathscr{D}$ (see [1], v. 2, Ex. 1, p. 93).

Suppose now that S is a finite monoid. Then certainly S is bounded, and the bounds expressed in Theorem 3.2 apply to any S-set. Suppose further that we consider S as a left S-set over itself. Then since S is a single orbit, we always have $sl(S) = 1$. However, the height of this S-set need not be 1, and in fact the difference $h(S) - sl(S)$ gives some measure of the complexity of the left ideal structure of S. If S is cyclic, then $h(S) = sl(S) = 1$. However, we show in conclusion that for some S, the difference $h(S) - sl(S)$ attains the limit imposed by Theorem 3.2.

PROPOSITION 4.2. *Let $n \geqq 0$ and let T_{n+2} be the monoid of all transformation on the set $\{1, \cdots, n + 2\}$ (with functions written on the left). Then letting T_{n+2} act on itself from the left we see that $sl(T_{n+2}) = 1$ and $h(T_{n+2}) = n + 1$.*

Proof. That $sl(T_{n+2}) = 1$ follows since T_{n+2} is a single orbit over itself. Also since T_{n+2} satisfies the hypotheses of Theorem 3.2 and is an orbit over itself, we may conclude as a corollary to the proof that $h(T_{n+2}) \leqq n+1$. Thus it suffices to show that $(T_{n+2})^n \neq T_{n+2}$.

Let $\alpha_1, \cdots, \alpha_{n+1}, \beta_2, \cdots, \beta_{n+1} \in T_{n+2}$ be given by

$$\alpha_k(x) = \begin{cases} x & 1 \leqq x \leqq k \\ k & k < x \end{cases} \quad \text{and} \quad \beta_k(x) = \begin{cases} x & 1 \leqq x \leqq k-1 \\ k-1 & x = k \\ k & k < x \end{cases}$$

for $1 \leqq x \leqq n+2$. One easily verifies that $\alpha_k\alpha_{k+1} = \alpha_k$ and that $\alpha_k\beta_{k+1} = \alpha_k$ for $1 \leqq k \leqq n$. Furthermore, one sees that all the principal left ideals $T_{n+2}\alpha_\kappa$, $1 \leqq k \leqq n+1$, and $T_{n+2}\beta_k$, $2 \leqq k \leqq n+1$ are distinct since two elements of T_{n+2} generate the same principal left ideal if and only if they induce the same partition on $\{1, \cdots, n+2\}$. (See [1], v. 1, §2.2.) Thus one has $T_{n+2}\alpha_{n+1} \supsetneqq \cdots \supsetneqq T_{n+2}\alpha_2 \supsetneqq T_{n+2}\alpha_1$, $T_{n+2}\beta_{k+1} \supsetneqq T_{n+2}\alpha_k$ for $k = 1, \cdots, n$, and $T_{n+2}\beta_{k+1}$ and $T_{n+2}\alpha_{k+1}$ are incomparable under inclusion for $k = 1, \cdots, n$. Thus since $\alpha_k \in T_{n+2}\alpha_{k+1} \cap T_{n+2}\beta_{k+1}$ for $k = 1, \cdots, n$, we can conclude by induction that $\alpha_k \notin (T_{n+2})^{k-1}$ for $k = 1, \cdots, n$. Hence $\alpha_{n+1} \notin (T_{n+2})^n$ so that $(T_{n+2})^n \neq (T_{n+2})$. Therefore $h(T_{n+2}) = n+1$ as desired.

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