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REDUCTIVITY IN C^*-ALGEBRAS AND ESSENTIALLY REDUCTIVE OPERATORS

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Reductivity is defined in the context of C^*-algebras other than the algebra of bounded operators on Hilbert space, and several examples are given. Essentially reductive operators are defined as operators whose images are reductive in the Calkin algebra. It is shown that every essentially reductive operator must be a compact perturbation of a reductive normal operator.

The concept of reductivity for operators on Hilbert space has generated a rich literature with a large number of interesting results. In this paper we will begin the investigation of reductivity in C^*-algebras other than $B(\mathscr{H})$ with special emphasis on the Calkin algebra. In what follows, Hilbert spaces are separable, operators are bounded, and subspaces are closed.

Reductive elements of C^*-algebras. If P is a projection and if A is an operator, a necessary and sufficient condition that the range of P be invariant under A is that $AP - PAP = 0$. Similarly, a necessary and sufficient condition that ran P reduce A is that $AP - PA = 0$. It thus makes good sense to speak of the operator A as having *invariant* and *reducing projections*. It is possible to recast the entire lexicon of invariant subspace theory in purely algebraic terms. We will say that an element p of a C^*-algebra $\mathscr{A}$ is a *projection* if $p^2 = p^* = p$. The projection p is *invariant* under an element a if $ap - pap = 0$; p *reduces* a if $ap - pa = 0$, or, equivalently, if both $ap - pap = 0$ and $a^*p - pa^*p = 0$. Eigenvalues make sense in this context also: p is an *eigenprojection* of a with eigenvalue λ if $(a - \lambda)p = 0$. The definition of reductivity is the same as for operators. An element a of $\mathscr{A}$ is *reductive* if every invariant projection reduces. An element a is *transitive* if the only invariant projections for a are 0 and 1; otherwise, a is *intransitive*.

EXAMPLE 1. The set $\mathscr{A}_K = \{\lambda + K : \lambda \in \boldsymbol{C}, K \text{ compact}\}$ is easily shown to be a C^*-algebra, once it is known that the set is closed; the inequality $\|\lambda + K\| \geqq |\lambda|$ is of use in proving this fact. The same inequality insures that if $\lambda_1 + K_1 = \lambda_2 + K_2$, then $\lambda_1 = \lambda_2$ and $K_1 = K_2$. Thus, if $\lambda + K$ is idempotent, we have $(\lambda + K)^2 = \lambda + K$, so that $\lambda^2 = \lambda$ and $2\lambda K + K^2 = K$. Thus λ is either 0 or 1. If in addition we require that $\lambda + K$ be self-adjoint, then either K or $-K$

must be self-adjoint and idempotent. The only compact projections are those of finite rank, so we have shown that the projections in $\mathscr{A}_K$ are those of finite rank (in case $\lambda = 0$) or finite co-rank (in case $\lambda = 1$). It is now easy to see that $\mathscr{A}_K$ contains transitive elements. Indeed, if an operator has an invariant subspace of finite dimension, the operator must have an eigenvalue. Hence if K_0 is a compact operator that has no eigenvalues, and whose adjoint has no eigenvalues, K_0 lies in $\mathscr{A}_K$ but no nontrivial projection in $\mathscr{A}_K$ can be invariant under K_0. K_0 is therefore trivially reductive in $\mathscr{A}_K$, so $\mathscr{A}_K$ contains nonnormal reductive elements as well.

This example is unsatisfying because K_0 is reductive for too simple-minded a reason: the only invariant projections are 0 and 1. A better example would have many invariant projections, preferably a whole chain of them. This suggests the idea of "complete reducibility" for an element a of a C^*-algebra $\mathscr{A}$.

We would like to say that a is completely reducible if, whenever p is a nonzero invariant projection for a then there is a nonzero invariant projection q that is a subprojection of p (i.e., $pq = q$). The difficulty is that for an arbitrary algebra $\mathscr{A}$, the projection p may have no nonzero subprojections; the corresponding difficulty for the algebra $B(\mathscr{H})$ is the case rank $p = 1$. We avoid this problem in the same way as for $B(\mathscr{H})$: the unpleasant p's are eliminated from consideration. Thus we say that an element a of $\mathscr{A}$ is completely reducible if the following condition holds: If p is a projection with $(1 - p)ap = 0$, and if there is a proper nonzero subprojection of p in $\mathscr{A}$, then there is a proper nonzero subprojection q for which $(1 - q)aq = 0$.

A slight bit of trickery allows us to exhibit a completely reducible reductive element of $\mathscr{A}_K$ that fails to be normal. For this purpose let $\mathscr{H}$ be a Hilbert space, let K_0 be as above, and consider the operator $K_1 = 0 \oplus K_0$ on $\mathscr{H} \oplus \mathscr{H}$. The operator K_1 is compact, so it lies in $\mathscr{A}_K$. Suppose that K_1 has a finite-dimensional invariant subspace $\mathscr{M}$. Then K_1 has an eigenvector e_1 lying in $\mathscr{M}$. Since K_0 has no eigenvectors, e_1 must also lie in the subspace $\mathscr{H} \oplus 0$, and must in fact reduce K_1. Thus the relative orthogonal complement of e_1 in $\mathscr{M}$ is also a finite-dimensional invariant subspace of K_1, so K_1 has an eigenvector in $\mathscr{M} \ominus \{e_1\}$. Proceeding in this way, we can eventually show that $\mathscr{M}$ is a subspace of $\mathscr{H} \oplus 0$. Thus if $\mathscr{M}$ has dimension greater than one, it properly contains a nonzero invariant subspace of K_1.

On the other hand, if K_1 has an invariant subspace $\mathscr{N}$ of finite co-dimension, then K_1^* has one $(\mathscr{N}^\perp)$ of finite dimension and the same kind of analysis shows that $\mathscr{N}^\perp$ lies within $\mathscr{H} \oplus 0$. Hence $\mathscr{N}$ reduces K_1 and contains a nonzero invariant subspace of finite co-dimension. We have shown that K_1 is a reductive and completely reducible element of $\mathscr{A}_K$; on the other hand, K_1 is obviously not normal.

EXAMPLE 2. It is proved in [5, p. 184] that the smallest C^*-algebra containing the unilateral shift is the algebra $\mathscr{A}_U = \{T_\phi + K$: T_ϕ is a Toeplitz operator with continuous symbol and K is compact$\}$. Since the only compact Toeplitz operator is 0 [2], if $T_\phi + K = T_{\phi'} + K'$, then $T_\phi = T_{\phi'}$ and $K = K'$. In order that $T_\phi + K$ be idemptotent it is necessary that $(T_\phi)^2 + (KT_\phi + T_\phi K + K^2) = T_\phi + K$. By another theorem in [5, p. 184] for continuous ϕ the difference $(T_\phi)^2 - T_{\phi^2}$ is compact, say $(T_\phi)^2 - T_{\phi^2} = K'$. Thus for $T_\phi + K$ to be idempotent requires $T_{\phi^2} + (KT_\phi + T_\phi K + K^2 + K') = T_\phi + K$; by the above observation this equation shows that $T_{\phi^2} - T_\phi = 0$, and hence $\phi^2 - \phi = 0$. Since ϕ is continuous, we must have $\phi = 0$ or $\phi = 1$. Then $K' = 0$ and the computation reduces to the one in the previous example. Thus the projections in $\mathscr{A}_U$ are exactly the same as the ones in $\mathscr{A}_K$, namely those of finite rank or finite co-rank. This shows that a C^*-algebra can be made considerably larger without affecting the set of projections in the algebra. $\mathscr{A}_U$ of course contains at least as many transitive elements and nonnormal reductive elements as does $\mathscr{A}_K$.

One's feeling is that the above examples work because the class of projections in $\mathscr{A}_K$ and $\mathscr{A}_U$ is not rich—a paucity of projections makes it easier for an element either to be transitive or to be reductive, since with fewer projections there is less chance of finding one that is invariant but not reducing. Do transitive elements and nonnormal reductive elements always occur together? that is, if an algebra contains one, will it contain the other? The well known theorem of Dyer, Pedersen and Porcelli [6] says that this is the case for the algebra $B(\mathscr{H})$. The following example shows that arbitrary C^*-algebras may behave differently.

EXAMPLE 3. Let Λ be a connected subset of the complex plane, and let $C(\Lambda)$ be the C^*-algebra of continuous functions on Λ. Since Λ is connected, the only idempotent continuous functions on Λ are the constant functions $\phi_0 = 0$ and $\phi_1 = 1$. Hence every element is both reductive and transitive. On the other hand, every element is also normal, so this algebra contains transitive elements but no nonnormal reductive ones.

Normal operators on Hilbert spaces of dimension greater than one always have nontrivial invariant subspaces: the range of any spectral projection reduces the operator. As the above example indicates, it may be that none of the spectral projections (except 0 and 1) appear in the C^*-algebra generated by the normal operator. These spectral projections are, however, contained in the weakly closed C^*-algebra generated by the normal operator. In other words, any transitive element of a von Neumann algebra must be nonnormal (and, of course, reductive). We thus have the following question, corresponding to Dyer, Pedersen, and Porcelli's result:

QUESTION. If a von Neumann algebra contains transitive elements must it contain other nonnormal reductive elements? Conversely?

Of course we mean the words "transitive" and "reductive" to be interpreted relative to the algebra; a transitive element of a von Neumann subalgebra of $B(\mathscr{H})$ need not be transitive in $B(\mathscr{H})$. Example 3 does not dispose of the following possibility either:

QUESTION. If a C^*-algebra contains nonnormal reductive elements, must it contain transitive ones?

Essentially reductive operators. Let $\mathscr{K}$ be the ideal of compact operators in $B(\mathscr{H})$ and let $\mathscr{C}$ denote the quotient space $\mathscr{C} = B(\mathscr{H})/\mathscr{K}$, that is, the Calkin algebra. From the point of view of the first question above, the Calkin algebra is quite interesting, since it is one of the few C^*-algebras for which the answer to the question "Does this algebra contain transitive elements?" is both known and nontrivial. Brown and Pearcy [3] have shown that the Calkin algebra contains no transitive elements; the statement for operators on $\mathscr{H}$ is that for any operator A, there is a projection P of infinite rank and co-rank such that $(1-P)AP$ is compact. An immediate question is whether $\mathscr{C}$ contains any nonnormal reductive elements; the answer is no (Thm. 2).

Let the canonical surjection from $B(\mathscr{H})$ to $\mathscr{C}$ be denoted by π. We will say that an element A in $B(\mathscr{H})$ is *essentially reductive* if its image $\pi(A)$ is a reductive element of the Calkin algebra. In this section we shall obtain several conditions equivalent to essential reductivity, and give a pertinent example. Calkin's original paper [4, p. 850] shows that any Hermitian idempotent element of $\mathscr{C}$ is the image of some projection in $B(\mathscr{H})$. Thus, a necessary and sufficient condition for a projection $p \in \mathscr{C}$ to be invariant for $\pi(A)$ is that there exist a projection $P \in B(\mathscr{H})$ such that $\pi(P) = p$ and $(1-P)AP \in \mathscr{K}$. Similarly, the condition that $AP - PA$ be compact is necessary and sufficient for $p = \pi(P)$ to reduce $\pi(A)$. The proof of the following lemma is straightforward.

LEMMA 1. *Let P be a projection and A an operator.* (i) *If $(1-P)AP$ is compact, then there is a compact operator K such that*

$(1-P)AP = (1-P)KP$. (ii) *If $AP - PA$ is compact, then there is a compact K' such that $AP - PA = K'P - PK'$.*

If P is a projection and if $(1-P)AP$ is compact, we shall say that the range of P is *essentially invariant* under A; if $AP - PA$ is compact, we will say that ran P is *essentially reducing* for A. The sets of essentially invariant and essentially reducing subspaces of A will be denoted by $\mathrm{Inv}_e A$ and $\mathrm{Red}_e A$ respectively and the sets of invariant and reducing subspaces by $\mathrm{Lat}\, A$ and $\mathrm{Red}\, A$. Since $(1-P)AP = (AP - PA)P$, $\mathrm{Red}_e A \subseteqq \mathrm{Inv}_e A$.

THEOREM 1. *The following statements are equivalent:*

(i) *A is essentially reductive.*

(ii) $\mathrm{Red}_e A = \mathrm{Inv}_e A$.

(iii) *If C is compact, and if $\mathscr{M} \in \mathrm{Lat}\,(A + C)$ then there is a compact K such that $\mathscr{M} \in \mathrm{Red}\,(A + K)$.*

(iv) *If $\mathscr{M} \in \mathrm{Inv}_e A$, then there is a compact K such that $\mathscr{M} \in$* $\mathrm{Red}\,(A + K)$.

Proof. (i) $\Leftrightarrow$ (ii) by the remark preceding Lemma 1 and the fact that $\mathrm{Red}_e A \subseteqq \mathrm{Inv}_e A$.

(ii) $\Rightarrow$ (iii): Let $\mathscr{M} \in \mathrm{Lat}\,(A + C)$ and let P be the projection onto $\mathscr{M}$; so $(1-P)(A+C)P = 0$. Then $(1-P)AP$ is compact, or $\mathscr{M} \in \mathrm{Inv}_e A$. By (ii), $\mathscr{M} \in \mathrm{Red}_e A$, so $AP - PA$ is compact. By Lemma 1 there is a compact operator K' such that $AP - PA = K'P - PK'$. Set $K = -K'$; then P commutes with $A + K$, so $\mathscr{M} \in \mathrm{Red}\,(A + K)$.

(iii) $\Rightarrow$ (iv): Let $\mathrm{ran}\, P = \mathscr{M} \in \mathrm{Inv}_e A$. Then $(1-P)AP$ is compact, and by Lemma 1 there is a compact K_0 such that $(1-P)AP = (1-P)K_0P$. Thus $(1-P)(A - K_0)P = 0$, $\mathscr{M} \in \mathrm{Lat}\,(A - K_0)$, and the conclusion follows from (iii).

(iv) $\Rightarrow$ (i): If P is a projection for which $(1-P)AP$ is compact, by (iv) there is a compact K so that P commutes with $A + K$; but then $AP - PA$ is necessarily compact.

In several ways, $\mathrm{Inv}_e A$ and $\mathrm{Red}_e A$ behave like $\mathrm{Lat}\, A$ and $\mathrm{Red}\, A$; for instance, if $\mathscr{M} \in \mathrm{Inv}_e A$, then $\mathscr{M}^\perp \in \mathrm{Inv}_e A^*$. However, $\mathrm{Inv}_e A$ lacks what is perhaps the most important property of $\mathrm{Lat}\, A$: it can fail to be a lattice.

EXAMPLE 4. We consider the Hilbert space $\mathscr{H}^{(3)} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$, and the operator A defined by $A\langle f, g, h\rangle = \langle h, 0, 0\rangle$. Let T be a compact operator on $\mathscr{H}$ whose range is dense. Let $\mathscr{M}$ and $\mathscr{N}$ be subspaces of $\mathscr{H}^{(3)}$ defined as follows:

$$\mathscr{M} = \{\langle 0, g, 0\rangle : g \in \mathscr{H}\}$$
$$\mathscr{N} = \{\langle 0, g, Tg\rangle : g \in \mathscr{H}\} .$$

This is a standard example of subspaces whose algebraic sum is not closed. It is easy to check that $\mathscr{M}$ is invariant under A, and therefore essentially invariant. The projection Q with range $\mathscr{N}$ can be determined to be the following:

$$Q\langle f, g, h\rangle = \langle 0, S(g + T^*h), TS(g + T^*h)\rangle$$

where we have set $S = (1 + T^*T)^{-1}$. Thus

$$AQ\langle f, g, h\rangle = \langle TS(g + T^*h), 0, 0\rangle .$$

T is compact, so AQ is compact and therefore so is $AQ - QAQ$. Thus $\mathscr{M}$ and $\mathscr{N}$ both lie in $\operatorname{Inv}_e A$ (in fact, it is easy to check that both lie in $\operatorname{Red}_e A$). On the other hand, the span of $\mathscr{M}$ and $\mathscr{N}$ is the subspace $0 \oplus \mathscr{H} \oplus \mathscr{H}$. (Let x and y be any two vectors in $\mathscr{H}$. Find g so that Tg is "near" y (the range of T is dense); then the vector $\langle 0, x, Tg\rangle = \langle 0, x - g, 0\rangle + \langle 0, g, Tg\rangle$ lies in $\mathscr{M} + \mathscr{N}$ and is "near" $\langle 0, x, y\rangle$.) Let R be the projection having $0 \oplus \mathscr{H} \oplus \mathscr{H}$ for its range. Then

$$\begin{aligned}(1 - R)AR\langle f, g, h, \rangle &= (1 - R)A\langle 0, g, h\rangle \\ &= (1 - R)\langle h, 0, 0\rangle \\ &= \langle h, 0, 0\rangle .\end{aligned}$$

Clearly the range of $(1 - R)AR$ is closed and infinite-dimensional, and consequently $(1 - R)AR$ cannot be compact. Thus $\mathscr{M}$ and $\mathscr{N}$ lie in $\operatorname{Inv}_e A$, but their span does not.

We remark that the above example is possible because we chose subspaces $\mathscr{M}$ and $\mathscr{N}$ whose algebraic sum is not closed. In other words, it is possible to prove the following statement: If $\mathscr{M}, \mathscr{N} \in \operatorname{Inv}_e A$, and if $\mathscr{M} + \mathscr{N}$ is closed, then $\mathscr{M} + \mathscr{N}$ is in $\operatorname{Inv}_e A$.

Essential reductivity and essential normality. A recent powerful result of Voiculescu gives the following fact as an immediate corollary:

THEOREM 2. *Every essentially reductive operator is essentially normal.*

Proof. For any set $\mathscr{F}$ consisting of self-adjoint projections in $\mathscr{C}$, let $\operatorname{Alg}(\mathscr{F}) = \{a \in \mathscr{C} : (1 - p)ap = 0 \text{ for all } p \in \mathscr{F}\}$. For A in $\mathscr{B}(\mathscr{H})$, let $\mathscr{A}(\pi(A))$ be the norm-closed subalgebra of the Calkin algebra generated by $\pi(A)$ and the identity. Let $\operatorname{Lat} \pi(A)$ be the set $\{\pi(P): P \in \operatorname{Inv}_e(A)\}$. By [9, Theorem 1.8] we have

$$\operatorname{Alg}(\operatorname{Lat}\pi(A)) = \mathscr{A}(\pi(A)) .$$

Thus if A is essentially reductive, $\pi(A^*)$ is an element of $\mathscr{A}(\pi(A))$ and in particular commutes with $\pi(A)$.

It follows from the fact that $\pi(A^*)$ lies in $\mathscr{A}(\pi(A))$ and a theorem of Lavrentiev [7, Ch. II, 8.7] that the essential spectrum $\Lambda_e(A)$ of A must have no interior and must fail to separate the plane. In the next section we present an alternative approach to the proof of this fact, which uses only the essential nomality of A.

Essential reductivity and essential spectrum. If A and B are operators and if there exist a unitary U and a compact K such that $A = U^*BU + K$, we will write $A \sim B$. The next lemma follows from the fact that $\operatorname{Inv}_e(U^*BU + K) = \operatorname{Inv}_e(U^*BU)$.

LEMMA 2. *If $A \sim B$ and A is essentially reductive, then so is B.*

The following fact is central and is due to Brown, Douglas, and Fillmore [1, Corollary 2.3]:

THEOREM 3. *If A is essentially normal and N is normal with $\Lambda_e(N) \subseteqq \Lambda_e(A)$, then $A \oplus N$ is unitarily equivalent to a compact perturbation of A.*

THEOREM 4. *Suppose A is essentially normal, N is normal, and $\Lambda_e(N) \subseteqq \Lambda_e(A)$. If A is essentially reductive, then so is N.*

Proof. By Theorem 4.7, $N \oplus A \sim A$. Thus if A is essentially reductive, so is $N \oplus A$. Now suppose the subspace $\mathscr{M}$ is essentially invariant under N. It follows that the subspace $\mathscr{M} \oplus \mathscr{H}$ is essentially invariant under $N \oplus A$. But $N \oplus A$ is essentially reductive, so $\mathscr{M} \oplus \mathscr{H}$ essentially reduces $N \oplus A$, and therefore $\mathscr{M}$ is essentially reducing for N.

THEOREM 5. *Suppose A is essentially normal, N is normal, and $\Lambda(N) \subseteqq \Lambda_e(A)$. If A is essentially reductive, then N is reductive.*

Proof. We use the notation $N^{(\infty)}$ to represent the direct sum of countably many copies of the operator N (notice that $N^{(\infty)}$ still acts on a separable space). It is well known that the essential spectrum of a normal operator consists of all points in the spectrum of the operator except the isolated eigenvalues of finite multiplicity. Since $N^{(\infty)}$ cannot have any such eigenvalues, we have $\Lambda_e(N^{(\infty)}) = \Lambda(N^{(\infty)})$, and it is easy to confirm that the spectrum of $N^{(\infty)}$ is equal to the spectrum of N. Hence $N^{(\infty)}$ is a normal operator whose essential spectrum is contained in the essential spectrum of A.

Now suppose that N is not reductive, that is, there exists a projection P such that $NP - PNP = 0$, but $NP - PN$ is a nonzero operator. It is then obvious that $N^{(\infty)}P^{(\infty)} - P^{(\infty)}N^{(\infty)}P^{(\infty)} = 0$, so that the range of $P^{(\infty)}$ is certainly essentially invariant for $N^{(\infty)}$. On the other hand, $N^{(\infty)}P^{(\infty)} - P^{(\infty)}N^{(\infty)} = (NP - PN)^{(\infty)}$; even if the range of $NP - PN$ is one-dimensional, the range of $(NP - PN)^{(\infty)}$ contains an infinite-dimensional subspace, and the operator $(NP - PN)^{(\infty)}$ therefore cannot be compact. Hence if N is not reductive, $N^{(\infty)}$ is not essentially reductive, and the previous proposition establishes that A cannot be essentially reductive either. This completes the proof.

The upshot of the last theorem is that if there is any non-reductive normal operator at all whose spectrum is contained in the essential spectrum of the operator A, then A cannot be essentially reductive.

K. J. Harrison [8] has proved the following fact:

THEOREM 6. *If X is a compact set in the plane which either separates the plane or has interior, then there is a normal operator N which is not reductive and whose spectrum is contained in X.*

COROLLARY 1. *A is essentially reductive if and only if A is essentially normal and $\Lambda_e(A)$ neither separates the plane nor has interior.*

Proof. If $\Lambda_e(A)$ fails to satisfy the condition, Theorems 4 and 5 show that A is not essentially reductive.

Now suppose $\Lambda_e(A)$ has no interior and fails to separate the plane.

For the proof we forget about operators and deal entirely with elements of the Calkin algebra. Suppose that $a \in \mathscr{C}$ is a normal element whose spectrum $\Lambda(a)$ has no interior and does not separate the plane. Recall that the Gelfand map $c \rightarrow \hat{c}$ takes elements of a commutative C^*-algebra $\mathscr{A}$ into functions in $C(M)$, where M is the maximal ideal space of $\mathscr{A}$; the Gelfand-Naimark theorem [e.g., 5, p. 92] asserts that the map is an adjoint-preserving isometric isomorphism. We identify $\mathscr{A}$ with the subalgebra of $\mathscr{C}$ generated by a and a^*.

The range of $\hat{a}$ is exactly $\Lambda(a)$ and for such a set it is known that there exist polynomials p_n for which $p_n(z) \rightarrow \bar{z}$ uniformly for $z \in \Lambda(a)$. Thus $p_n(\hat{a}) \rightarrow \bar{\hat{a}}$ and $p_n(a) \rightarrow a^*$. It now follows easily that any projection in $\mathscr{C}$ that is invariant for a is invariant for a^*.

For normal operators, more can be said:

THEOREM 7. *If N is normal and essentially reductive, then N is reductive.*

Proof. By Corollary 1, $\Lambda_e(N)$ has no interior and does not separate the plane. On the other hand, $\Lambda(N)$ differs from $\Lambda_e(N)$ only by the addition of isolated points. It is an exercise in elementary topology to show that $\Lambda(N)$ also has no interior and fails to separate the plane. It now follows from [10, Theorem 7] that N is reductive.

The converse of this theorem fails, however, and provides us with an example of a reductive operator that is not essentially reductive. For suppose that A is a diagonal unitary operator whose diagonal entries are dense in the unit circle. Then $\Lambda_e(A)$ is the whole circle, so A is not essentially reductive; on the other hand, every diagonal unitary operator is reductive [10, Theorem 6].

We also cannot expect to get a statement like Theorem 7 where the word "normal" is replaced by "essentially normal"; for instance, every compact operator is trivially essentially normal and essentially reductive, but only the normal ones are reductive. However, our Theorem 5 is used in [8] to prove the following fact:

THEOREM 7 [Harrison]. *If A is essentially normal and essentially reductive, then there is a compact K such that $A + K$ is normal and reductive.*

Proof. $\Lambda_e(A)$ has no interior and does not separate the plane, so by [1, Corollary 11.3 and Theorem 11.1], A is a compact perturbation of a normal operator N such that $\Lambda(N) = \Lambda_e(N) = \Lambda_e(A)$. N is thus reductive by the same proof as in Corollary 1 (or by [10, Theorem 7]).

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