

HEREDITARY CROSSED PRODUCT ORDERS

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In this paper one deals with crossed product orders Λ of the following form: Let \mathcal{R} be a Dedekind domain with quotient field \mathcal{F} and \mathcal{E} a semisimple, commutative, algebra of finite dimension over \mathcal{F} . Let \mathcal{G} be a finite subgroup of the group of automorphisms of \mathcal{E} whose fixed subalgebra is \mathcal{F} , and let Λ_0 be an \mathcal{R} -order in \mathcal{E} , which is \mathcal{G} -stable. Then, if $[\mathfrak{f}]$ is an element of the second cohomology group $H^2(\mathcal{G}, U(\Lambda_0))$, our order is $\Lambda = \Lambda(\mathfrak{f}, \Lambda_0, \mathcal{G})$. One is interested in the set of all maximal orders in $\mathcal{A} = \Lambda(\mathfrak{f}, \mathcal{E}, \mathcal{G})$ which contain Λ and also in all hereditary orders in \mathcal{A} which contain Λ . In particular, one is interested in knowing sufficient conditions for Λ itself to be hereditary. This last question is answered by Theorem 1, and the other, more general question, is successively reduced to the classical complete case (i.e., when \mathcal{R} is a local complete Dedekind domain and \mathcal{E} is a Galois field extension of \mathcal{F} with group \mathcal{G}), to the totally ramified case (i.e., when, furthermore, $\mathcal{E}|\mathcal{F}$ is totally ramified) and, finally, to the wildly ramified case.

1. Introduction. In general, we use in this paper the terminology of [3]. With \mathcal{R} we denote a Dedekind domain whose field of quotients is \mathcal{F} . \mathcal{A} will be a separable \mathcal{F} -algebra and \mathcal{E} a finite dimensional, semisimple, commutative subalgebra of \mathcal{A} . We denote with \mathcal{G} a finite subgroup of the group of automorphisms of \mathcal{E} .

Let \mathcal{B} be a commutative ring with identity and \mathcal{G} a finite group of automorphisms of \mathcal{B} . Then the group of units of $\mathcal{B}, U(\mathcal{B})$, is a \mathcal{G} -module. Let \mathfrak{f} be a factor system: $[\mathfrak{f}] \in H^2(\mathcal{G}, U(\mathcal{B}))$. Then we define the ring $\Lambda(\mathfrak{f}, \mathcal{B}, \mathcal{G})$ as a free \mathcal{B} -module with basis $\{t_\sigma\} (\sigma \in \mathcal{G})$, for which a multiplication rule is given by means of:

$$(bt_\sigma)(ct_\tau) = bc^{\sigma} \mathfrak{f}(\sigma, \tau) t_{\sigma\tau} \quad (b, c \in \mathcal{B}; \sigma, \tau \in \mathcal{G})$$

and extended by additivity.

DEFINITION 1. Given \mathcal{E} , a finite dimensional, commutative, semisimple \mathcal{F} -algebra, and a finite subgroup \mathcal{G} of the automorphisms of \mathcal{E} such that the fixed subalgebra of \mathcal{E} under the action of \mathcal{G} is \mathcal{F} : $\mathcal{E}^{\mathcal{G}} = \mathcal{F}$, we say that \mathcal{A} is a crossed product of \mathcal{E} over \mathcal{F} when \mathcal{A} is isomorphic to the \mathcal{F} -algebra $\Lambda(\mathfrak{f}, \mathcal{E}, \mathcal{G})$, for some factor system \mathfrak{f} corresponding to an element of $H^2(\mathcal{G}, U(\mathcal{E}))$.

DEFINITION 2. We say that the \mathcal{R} -order Λ is a crossed product of \mathcal{E} over Λ_0 when: Λ_0 is an \mathcal{R} -order in \mathcal{E} , when the fixed subalgebra

of \mathcal{E} under the action of \mathcal{G} is \mathcal{F} , when A_0 is stable under \mathcal{G} , and when A is isomorphic to $\Delta(\mathfrak{f}, A_0, \mathcal{E})$, for some factor system \mathfrak{f} corresponding to an element of $H^2(\mathcal{E}, U(A_0))$.

In the case of Definition 1 we usually allow ourselves to write $\mathcal{A} = \Delta(\mathfrak{f}, \mathcal{E}, \mathcal{E})$, and $A = \Delta(\mathfrak{f}, A_0, \mathcal{E})$ in the case of Definition 2. In both cases it is possible to assume, without loss of generality, that \mathfrak{f} is normalized, which we will do unless it is otherwise stated. If $\mathcal{A} = \Delta(\mathfrak{f}, \mathcal{E}, \mathcal{E})$, one can introduce crossed product orders in \mathcal{A} by merely taking an \mathcal{R} -order A_0 in \mathcal{E} which is \mathcal{G} -stable and such that $\mathfrak{f}(\mathcal{E}, \mathcal{E}) \subset U(A_0)$, and forming $\Delta(\mathfrak{f}, A_0, \mathcal{E})$ within \mathcal{A} . What we mean by this is the following: If $\mathcal{A} = \Delta(\mathfrak{f}, \mathcal{E}, \mathcal{E})$, there is a basis $\{t_\sigma\}$ ($\sigma \in \mathcal{E}$) of the \mathcal{E} -module \mathcal{A} such that $(\mathfrak{g}t_\sigma)(\mathfrak{h}t_\tau) = \mathfrak{g}\mathfrak{h}\mathfrak{f}(\sigma, \tau)t_{\sigma\tau}$ and, if 1 is the identity of \mathcal{E} , $t_1 = 1$, the identity of \mathcal{A} . Such a basis will be called a *natural \mathcal{E} -basis* for \mathcal{A} . In a similar way, we have a natural A_0 -basis for a crossed product order A . In these cases each $\sigma \in \mathcal{E}$ coincides with the restriction to \mathcal{E} (resp. A_0) of the conjugation by t_σ . Now, if \mathcal{A} is such a crossed product with such a natural \mathcal{E} -basis, given an \mathcal{R} -order A_0 in \mathcal{E} , which is \mathcal{G} -stable and such that $\mathfrak{f}(\mathcal{E}, \mathcal{E}) \subset U(A_0)$ it is enough to form $A = \sum_{\sigma \in \mathcal{E}} A_0 t_\sigma = \bigoplus_{\sigma \in \mathcal{E}} A_0 t_\sigma$ to have a crossed product order in \mathcal{A} which is isomorphic to $\Delta(\mathfrak{f}, A_0, \mathcal{E})$.

Also, given a crossed product order $A = \Delta(\mathfrak{f}, A_0, \mathcal{E})$, it is an order in a crossed product algebra of the form $\Delta(\mathfrak{f}, \mathcal{E}, \mathcal{E})$, which is obtained as the usual algebraic container: $\mathcal{A} = \mathcal{F} \otimes_{\mathcal{R}} A$.

It is an easy exercise to prove that a crossed product algebra is centrally simple, and hence separable (cf. [2]). If \mathcal{A} is a crossed product, let \mathcal{E}_i ($i = 1, \dots, m$) be the simple components of \mathcal{E} , and $1 = e_1 + \dots + e_m$ the corresponding decomposition into primitive orthogonal idempotents. The requirement that $\mathcal{E}^{\mathcal{G}} = \mathcal{F}$ in Definition 1 implies that \mathcal{G} operates transitively on $\{e_1, \dots, e_m\}$. Each \mathcal{E}_i is a field, which is Galois over $\mathcal{F}e_i$ with Galois group $\mathcal{H}_i = \text{Stab}_{\mathcal{G}}(e_i)$, and each pair $\mathcal{E}_i, \mathcal{E}_j$ is a pair of fields conjugate with respect to \mathcal{F} . Sometimes, by abuse of language, we say that \mathcal{E}_i is Galois over \mathcal{F} with group \mathcal{H}_i . If n denotes the order of \mathcal{G} , the dimension of \mathcal{E} over \mathcal{F} and the rank of \mathcal{A} over \mathcal{E} are both equal to n , while the dimension of \mathcal{A} over \mathcal{F} is n^2 . It may be noted (which is also easily proven) that \mathcal{E} is a maximal commutative subalgebra of \mathcal{A} .

We will refer to the case in which \mathcal{E} is simple (i.e., when it is a field) by saying that we have a *classical crossed product algebra* or a *classical crossed product order*.

2. Hereditary crossed product orders.

THEOREM 1. *Let \mathcal{R} be a local complete Dedekind domain with field of quotients \mathcal{F} . If the extension $\mathcal{E}_1/\mathcal{F}$ is tamely ramified and if A_0 is the maximal \mathcal{R} -order Ω in \mathcal{E} , then $A = A(\mathfrak{f}, A_0, \mathcal{E})$ is a hereditary order and, moreover, a principal hereditary order (for the definition of it, see [6]).*

This result, which is essentially equivalent to a theorem of S. Williamson (cf. [5]), is, in a more general form, a corollary of Proposition 1, except for the part that affirms that the hereditary order is principal. We give a direct and easy proof, based on S. Williamson's result.

Proof. By calling \mathfrak{f}_1 the function $e_1 \cdot \mathfrak{f}$ and \mathcal{A}_1 the algebra $e_1 \mathcal{A} e_1$, we have that \mathcal{A}_1 is a crossed product $A(\mathfrak{f}_1, \mathcal{E}_1, \mathcal{H}_1)$ that is a classical crossed product. If we call A_1 the \mathcal{R} -order $e_1 A e_1$, we see that it is a classical crossed product order $A(\mathfrak{f}_1, A_{01}, \mathcal{H}_1)$ in \mathcal{A}_1 , where A_{01} is the \mathcal{R} -order $e_1 A_0$ in the field \mathcal{E}_1 .

According to S. Williamson's result, A_1 is principal hereditary if $\mathcal{E}_1/\mathcal{F}$ is tamely ramified. In this case, the radical of A_1 , \mathcal{N}_1 , is of the form: $\mathcal{N}_1 = \pi_1 A_1$, where π_1 is a prime element of A_{01} (and, hence, π_1 generates the radical of A_{01} , which is its unique maximal ideal). Let $\sigma_1 = 1, \sigma_2, \dots, \sigma_m$ be a system of representatives of the cosets $\sigma_i \mathcal{H}$ of \mathcal{G} -modulo $\mathcal{H} = \mathcal{H}_1$ such that $\sigma_1(e_i) = e_i$, and then define π to be the element of A_0 obtained by:

$$\pi = \sum_{i=1}^m \sigma_i(\pi_1).$$

Let \mathcal{N} be the radical of A . Theorem 1 will be proved if we succeed in showing that $\mathcal{N} = \pi A = A\pi$. The second equality is clear, in view of the definition of π . It follows readily from Nakayama's lemma that $\pi A \subset \mathcal{N}$. To prove the reverse inclusion, we form the quotient

$$\frac{A}{\pi A} = \bar{A},$$

which is in an obvious way of the form

$$A\left(\bar{\mathfrak{f}}, \frac{A_0}{\pi A_0}, \mathcal{E}\right)$$

(bars denoting objects modulo π), and show that $\text{rad}(\bar{A}) = 0$.

But here we notice at once that $(\bar{A})_1 = (\bar{A}_1)$, where the left-hand side means $\bar{e}_1 \bar{A}_1 \bar{e}_1$ and the right-hand side means $A_1/(\pi_1 A_1) = A_1/\mathcal{N}_1$.

Hence, \bar{A}_1 is semisimple. Now, if we express elements of \bar{A}_0 as the corresponding sums of their projections into the simple components, an arbitrary element $\bar{\delta}$ in $\text{rad}(\bar{A})$ is of the form

$$\bar{\delta} = \sum_{i,\sigma} \alpha_{i,\sigma} \bar{e}_i \bar{t}_\sigma$$

and a direct calculation shows that (for each i , j) $\bar{t}_{\sigma_i^{-1}e_i} \bar{\delta} \bar{e}_i \bar{t}_{\sigma_i}$ is equal to 0 because it is an element of $\text{rad}(\bar{A}) \cap \bar{A}_1 \subset \text{rad}(\bar{A}_1)$. It follows that all the $\alpha_{i,\sigma}$ are 0, so that $\bar{\delta} = 0$.

2. Reduction to the local case. In this section we assume \mathcal{R} is a local Dedekind domain and we keep the preceding notations and conventions. We assume further that A_0 is equal to the sum of its simple components (meaning that all the idempotents e_i are elements of A_0).

Let us be given a crossed product order $\Lambda = \Delta(f, A_0, \mathcal{G})$ in the crossed product algebra $\mathcal{A} = \Delta(f, \mathcal{E}, \mathcal{G})$. We consider a Λ -ideal, \mathcal{M} , i.e., a full \mathcal{R} -lattice in \mathcal{A} such that $\Lambda \mathcal{M} \subset \mathcal{M}$ and $\mathcal{M} \Lambda \subset \mathcal{M}$. In particular, \mathcal{M} may be any order containing Λ . We define, $\mathcal{M}_1 = e_i \mathcal{M} e_i$. It is immediate then that $\mathcal{M}_1 = \mathcal{M} \cap \mathcal{A}_1$.

LEMMA 1. We have:

$$\mathcal{M} = \Lambda \mathcal{M}_1 \Lambda = \sum_{\sigma,\tau} t_\sigma \mathcal{M}_1 t_\tau.$$

Proof. It is clear that \mathcal{M}_1 is a A_0 -module (left and right). Next, given an element x of \mathcal{M} , we have through a direct computation that

$$t_{\sigma_i^{-1}e_i} x e_i t_{\sigma_i} \in \mathcal{M}_1$$

so that elements:

$$x_{ii} = \frac{1}{f(\sigma_i, \sigma_i^{-1})^{\sigma_i^{-1}}} \cdot t_{\sigma_i^{-1}e_i} x e_i t_{\sigma_i} \cdot \frac{1}{f(\sigma_i, \sigma_i^{-1})^{\sigma_i^{-1}}}$$

are in \mathcal{M}_1 and are such that

$$x = \sum_{i,j} t_{\sigma_i} x_{ii} t_{\sigma_i}^{-1}.$$

This shows that $\mathcal{M} \subset \Lambda \mathcal{M}_1 \Lambda \subset \mathcal{M}$, which gives the desired result.

Note. For this result one does not need to assume that \mathcal{R} is local.

PROPOSITION 1. Let \mathcal{I} be the set of all Λ -ideals in \mathcal{A} and \mathcal{I}_1

the set of all A_1 -ideals in \mathcal{A}_1 . We introduce the following mappings:

$$R: \mathcal{F} \longrightarrow \mathcal{F}_1; R(\mathcal{M}) = \mathcal{M}_1 .$$

$$S: \mathcal{F}_1 \longrightarrow \mathcal{F}; S(\mathcal{M}_1) = \Lambda \mathcal{M}_1 \Lambda .$$

Then R and S are inverse bijections and have the following properties:

- (i) $\mathcal{M} \subset \mathcal{M}' \Rightarrow R(\mathcal{M}) \subset R(\mathcal{M}')$.
- (ii) $\Gamma \in \mathcal{F}$ is an order if and only if $R(\Gamma)$ is an order.
- (iii) If $\Gamma \in \mathcal{F}$ is an order, then \mathcal{L} is a Γ -ideal if and only if $R(\mathcal{L})$ is an $R(\Gamma)$ -ideal. In particular, $\text{rad}(R(\Gamma)) = R(\text{rad}(\Gamma))$, $\text{rad}(S(\Gamma)) = S(\text{rad}(\Gamma))$.
- (iv) $\Gamma \in \mathcal{F}$ is a maximal (resp. hereditary) order if and only if $R(\Gamma)$ is a maximal (resp. hereditary) order.

Proof. (i) clear. (ii) If Γ is an order containing Λ , it is clear that Γ_1 is a finitely generated \mathcal{B} -module which is a ring containing A_1 and, hence generates \mathcal{A}_1 under \mathcal{F} . Conversely, if Γ_1 is an order in \mathcal{A}_1 containing Λ_1 , $S(\Gamma_1) \supset S(\Lambda_1) = \Lambda$, whence $S(\Gamma_1)$ is a finitely generated \mathcal{B} -module containing Λ . Its closure under multiplication follows from: $\Lambda \Gamma_1 \Lambda \Gamma_1 \Lambda = \Lambda \Gamma_1 e_1 \Lambda e_1 \Gamma_1 \Lambda = \Lambda \Gamma_1 \Lambda_1 \Gamma_1 \Lambda = \Lambda \Gamma_1 \Lambda$. (iii) If \mathcal{L} is a Γ -ideal, $\Gamma \mathcal{L} = \mathcal{L} \Gamma = \mathcal{L}$ implies: $\Gamma_1 \mathcal{L}_1 = e_1 \Gamma e_1 \mathcal{L} e_1 \subset e_1 \Gamma \mathcal{L} e_1 = \mathcal{L}_1$ and, similarly, $\mathcal{L}_1 \Gamma_1 \subset \mathcal{L}_1$. The converse is equally easy. (iv) Let Γ be a hereditary order containing Λ and let us call \mathcal{N} the radical of Γ . We also introduce:

$$\mathcal{N}^{-1} = \{x \in \mathcal{A} / x \mathcal{N} \subset \Lambda\}, \mathcal{N}_1 = \text{rad}(\Gamma_1)$$

and $\mathcal{N}_1^{-1} = \{x \in \mathcal{A}_1 / x \mathcal{N}_1^{-1} \subset \Gamma_1\}$. Then, there exist two finite families: x_i, y_i , such that $x_i \in \mathcal{N}, y_i \in \mathcal{N}^{-1}$ and $\sum y_i x_i = 1$. It follows that the elements $t_{\sigma_q}^{-1} e_q x_i e_1$ are in \mathcal{N}_1 , and the elements $e_1 y_i e_q t_{\sigma_q}$ in \mathcal{N}_1^{-1} . It follows easily that $\mathcal{N}_1^{-1} \mathcal{N}_1 = \Gamma_1$. This suffices to say that \mathcal{N}_1 is a projective Γ_1 -module, which is a necessary and sufficient condition for an order (in the local case) to be hereditary. Conversely, if Γ_1 is hereditary (that is, if $\mathcal{N}_1^{-1} \mathcal{N}_1 = \Gamma_1$) it follows: $\Lambda \mathcal{N}_1^{-1} \Lambda \cdot \Lambda \mathcal{N}_1 \Lambda = \Lambda \mathcal{N}_1^{-1} \Lambda_1 \mathcal{N}_1 \Lambda = \Lambda \mathcal{N}_1^{-1} \mathcal{N}_1 \Lambda = \Lambda \Gamma_1 \Lambda = \Gamma$, which is enough to guarantee that Γ is hereditary.

Note. The assumption that \mathcal{B} is local is used only in the proof of part (iv) in the preceding theorem.

COROLLARY. , Let $\Lambda_0 = \Omega$, the unique maximal order in the commutative algebra \mathcal{E} . Then the following propositions are equivalent:

- (i) Λ is hereditary for every factor system f ;
- (ii) if $f = 1$, Λ is hereditary;

(iii) \mathcal{E}_1 is a tamely ramified extension of \mathcal{F} .

If, furthermore, the residue class field of A_{01} is perfect, then the propositions above are equivalent to the following:

(ii)' There exists a factor system f such that A is hereditary.

Proof. (i) implies (ii), and (ii) implies that $A(1, \mathcal{E}, \mathcal{E})_1$ is hereditary which, by S. Williamson's results (cf. [5]) implies (iii). Again, by S. Williamson's results, (iii) implies that A_1 is hereditary, which implies (i). (ii)' implies that $A(f, \mathcal{E}, \mathcal{E})_1$ is hereditary for some f . Then, if A_{01} has a perfect residue class field, by Harada's results (cf. [1]), (iii) follows from (ii)'. Clearly, (i) implies (ii)'.

Remark on the possibility of an effective computation of the Schur index. Let \mathcal{R} be a local complete Dedekind domain and let us assume that $\mathcal{E}_1/\mathcal{F}$ is tamely ramified. We are going to show how, using only the information contained in the data for the definition of A as a crossed product order, it is possible to compute the Schur index of both \mathcal{A} and \mathcal{A}_1 (which, as it turns out, are equal to each other).

Let us work first under the simplifying assumption that we are in the classical case: $\mathcal{A} = \mathcal{A}_1, \mathcal{E} = \mathcal{E}_1$. The centrally simple algebra \mathcal{A} can be viewed as a matrix algebra of degree f , say, over a division algebra \mathcal{D} . Since A is a principal hereditary order, it is a block order corresponding to a matrix of the form:

$$\begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \cdot & & \cdot & & \\ \cdot & & \cdot & & \\ \cdot & & \cdot & & \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

whose blocks are all of the same size ($r \times r$, say) (cf. [3], Ch. IX, §2). In [5], S. Williamson has proved that if t is the number of blocks (and, hence $f = rt$) then t can be computed as the order of the conductor group of $[f], H_{[f]}$ which depends only on $[f]$.

It is well known that \mathcal{D} has a unique maximal order $\hat{\mathcal{Q}}$ with a unique maximal ideal $\hat{\mathcal{N}}$ and that the Schur index of \mathcal{A} (which is, by definition, the square root of the \mathcal{F} -dimension of \mathcal{D}), \mathfrak{s} , is equal to the dimension of $\hat{\mathcal{D}} = \hat{\mathcal{Q}}/\hat{\mathcal{N}}$ over $\hat{\mathcal{F}} = \mathcal{R}/\text{rad}(\mathcal{R})$. Let us introduce the notation f_π for the residue class degree of \mathcal{E} over \mathcal{F} .

Then we can look at the quotient $\bar{A} = A/\mathcal{N}$ in two different ways,

namely as a direct sum of $n = (\mathcal{G}:1)$ copies of $\bar{\mathcal{E}} = \Omega/\text{rad}(\Omega)$ and as the direct sum of “residue matrix blocks” which arises when we compute the quotient starting from the general form of the matrices in A . A computation of the $\bar{\mathcal{F}}$ -dimension of \bar{A} from each point of view gives:

$$\text{tr}^2 \bar{\mathfrak{s}} = n \bar{f}_\pi$$

from which we get that:

$$\bar{\mathfrak{x}} = \bar{f}_\pi.$$

Then, we obtain the desired result in the form of the formula:

$$\bar{\mathfrak{s}} = \frac{n}{\bar{f}_\pi t}.$$

If we proceed, for the general case, in a similar way, we get that the Schur index of \mathcal{A} is equal to the Schur index of \mathcal{A}_1 .

Remark on the reduction to the complete case. Assume once again that \mathcal{R} is a local Dedekind domain. Questions regarding the hereditariness or maximality of orders in \mathcal{A} containing A can be reduced to the same questions concerning orders in $\hat{\mathcal{A}}$ containing the completion \hat{A} of A . However, the completion $\hat{\mathcal{A}}$ of a classical crossed product $\mathcal{A} = \Delta(\bar{f}, \mathcal{E}, \mathcal{G})$ is in general no longer a classical crossed product, but rather a crossed product in the sense of our Definition 1 (see Proposition 1.4 of [5]). More precisely, $\hat{\mathcal{A}} = \Delta(\bar{f}, \hat{\mathcal{E}}, \mathcal{G})$ where $\hat{\mathcal{E}} = \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_s$ and the \mathcal{E}_i denote the completions of \mathcal{E} at each of the prime ideals \mathcal{P}_i lying over the prime ideal of \mathcal{F} . The rest of the paper is devoted to reducing the study of A in \mathcal{A} to the local complete case, namely to the study of \hat{A}_1 in $\hat{\mathcal{A}}_1 = \Delta(\bar{f}_1, \mathcal{E}_1, \mathcal{G}_1)$ where \mathcal{G}_1 denotes the decomposition group of \mathcal{P}_1 .

3. Reduction to the totally ramified case. Throughout this section we keep our general notations but we assume that \mathcal{R} is a local complete Dedekind ring and that $\mathcal{E}|\mathcal{F}$ is a Galois extension with Galois group \mathcal{G} .

We fix a normal subgroup \mathcal{H} of \mathcal{G} and call \mathcal{L} the fixed field of \mathcal{H} so that $\mathcal{E}|\mathcal{L}$ is Galois with group \mathcal{H} and $\mathcal{L}|\mathcal{F}$ is Galois with group $\mathcal{G}|\mathcal{H}$.

We use the following notations:

π is a prime element of \mathcal{R} , so that $\text{rad}(\mathcal{R}) = \pi \mathcal{R}$.

$\bar{\mathcal{F}} = \mathcal{R}/\pi \mathcal{R}$.

$A_{0, \mathcal{R}} = A_0 \cap \mathcal{L} = \Omega =$ the maximal \mathcal{R} -order in \mathcal{L} .

$A_0^{\mathcal{R}} = \Omega \otimes_{\mathcal{R}} A_0$.

$$\mathcal{E}^\times = \mathcal{L} \otimes_{\mathcal{F}} \mathcal{E}.$$

$$\mathcal{A}^\times = \mathcal{L} \otimes_{\mathcal{F}} \mathcal{A}.$$

$$\Delta^\times = \Omega \otimes_{\mathcal{E}} \Delta.$$

$\sigma_1 = 1, \sigma_2, \dots, \sigma_m$ are representatives of the cosets of \mathcal{G} -modulo \mathcal{H} , or of the elements of \mathcal{G}/\mathcal{H} .

We assume that \mathcal{A} is identified to the subalgebra $1 \otimes \mathcal{A}$ of \mathcal{A}^\times .

LEMMA 2. *If $\mathcal{L}|\mathcal{F}$ is an unramified extension, then Δ_0^\times is equal to the sum of its simple components.*

Proof. The \mathcal{F} -algebra $\mathcal{L} \otimes_{\mathcal{F}} \mathcal{E}$ is semisimple and it contains in an obvious way the semisimple \mathcal{F} -algebra $\mathcal{L} \otimes_{\mathcal{F}} \mathcal{L}$ and both have the same primitive orthogonal idempotents. Let $(\eta_i)(i = 1, \dots, m)$ be an \mathcal{R} -basis of Ω and let Δ be the discriminant of this basis. The standard decomposition of a tensor product of \mathcal{F} -extensions: $\mathcal{H} \otimes_{\mathcal{F}} \mathcal{L}$, where $\mathcal{L}|\mathcal{F}$ is separable, through the chinese remainder theorem allows for a direct computation of the irreducible idempotents e_i . In our case, one obtains:

$$e_i = \frac{1}{\sqrt{\Delta}} \sum_{i=1}^m \begin{vmatrix} \eta_1^{\sigma_1} & \dots & \eta_{i-1}^{\sigma_{i-1}} & 0 & \eta_{i+1}^{\sigma_{i+1}} & \dots & \eta_m^{\sigma_1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \eta_1^{\sigma_{i-1}} & \dots & \eta_{i-1}^{\sigma_{i-1}} & 0 & \eta_{i+1}^{\sigma_{i-1}} & \dots & \eta_m^{\sigma_{i-1}} \\ \eta_1^{\sigma_i} & \dots & \eta_{i-1}^{\sigma_i} & 1 & \eta_{i+1}^{\sigma_i} & \dots & \eta_m^{\sigma_i} \\ \eta_1^{\sigma_{i+1}} & \dots & \eta_{i-1}^{\sigma_{i+1}} & 0 & \eta_{i+1}^{\sigma_{i+1}} & \dots & \eta_m^{\sigma_{i+1}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \eta_1^{\sigma_m} & \dots & \eta_{i-1}^{\sigma_m} & 0 & \eta_{i+1}^{\sigma_m} & \dots & \eta_m^{\sigma_m} \end{vmatrix} \otimes \eta_i.$$

Now, if $\mathcal{L}|\mathcal{F}$ is unramified, Δ is a unit and the lemma follows.

The group \mathcal{G} may be considered as a group of automorphisms of the semisimple commutative \mathcal{F} -algebra \mathcal{E}^\times , provided that we define the action of $\sigma \in \mathcal{G}$ in \mathcal{E}^\times by establishing that $(x \otimes y)^\sigma = x \otimes y^\sigma$ ($x \in \mathcal{L}, y \in \mathcal{E}, \sigma \in \mathcal{G}$). On the other hand, the factor system f may be viewed as a factor system $f: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{E}^\times$ (by identifying each $f(\sigma, \tau)$ with $1 \otimes f(\sigma, \tau)$). If we do this, we can state:

LEMMA 3. *\mathcal{A}^\times is a crossed product $\Delta(f, \mathcal{E}^\times, \mathcal{G})$.*

Proof. Straightforward.

We introduce the following sets:

\mathcal{O}^\times is the set of all orders Γ in \mathcal{A}^\times which contain Δ^\times and verify

the following condition:

for each element ξ in Γ there is a representation of the form

$$(*) \quad \xi = \sum_i m_i \otimes \xi_i$$

(with $m_i \in \Omega$, $\xi_i \in \mathcal{A}$) such that each $1 \otimes \xi_i$ is also an element of Γ .

\mathcal{O} is the set of all orders in \mathcal{A} which contain Λ .

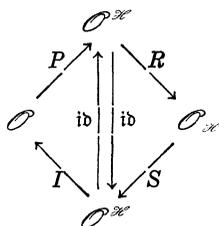
Next, we consider the following two functions:

$$\begin{aligned} P: \mathcal{O} &\longrightarrow \mathcal{O}^\times \\ P(\Gamma) &= \Omega \otimes_{\mathcal{A}} \Gamma =: \Gamma^\times \\ I: \mathcal{O}^\times &\longrightarrow \mathcal{O} \\ I(\Gamma) &= \Gamma \cap \mathcal{A}. \end{aligned}$$

Then we have:

PROPOSITION 2. *P and I are order preserving bijections, inverses of each other.*

If, furthermore, $\mathcal{L}|\mathcal{F}$ is an unramified extension, the following diagram is commutative:



where R and S are the mappings which were introduced in the Proposition 1, applied to the case in which \mathcal{A} is \mathcal{A}^\times and, thus, \mathcal{A}_1 is $(\mathcal{A}^\times)_1 =: \mathcal{A}_\times =: \Delta(\mathfrak{f}|\mathcal{H} \times \mathcal{H}, \mathcal{E}, \mathcal{H})$. It follows that \mathcal{O}^\times (the image of \mathcal{O}^\times under R) is the set of all \mathcal{R} -orders in \mathcal{A}_\times which contain

$$\Lambda_\times = \Delta(\mathfrak{f}|\mathcal{H} \times \mathcal{H}, \Lambda_0, \mathcal{H}).$$

Furthermore, in this latter situation both P and I carry hereditary orders into hereditary orders.

Proof. Most of the proof is direct. For example, to prove that $P \circ I = \text{id}$, we take $\Gamma \in \mathcal{O}^\times$ and form $\Omega \otimes_{\mathcal{A}} (\Gamma \cap \mathcal{A})$, which has to be shown is equal to Γ . It is obviously contained in Γ . But also, if $\gamma \in \Gamma$, according to (*), we can write it in the form: $\sum_i m_i \otimes \xi_i$,

with each $1 \otimes \xi_i$ belonging to Γ . But this actually means that each $1 \otimes \xi_i$ belongs to $\Gamma \cap \mathcal{A}$, whence $\gamma \in \Omega \otimes_{\mathcal{A}} (\Gamma \cap \mathcal{A})$. The fact that the other composite, $I \circ P$, is the identity of \mathcal{O} goes as follows: Let $\Gamma \in \mathcal{O}$; then $\Gamma \subset (\Lambda_{0, \mathcal{A}} \otimes \Gamma) \cap \mathcal{A}$. Now, if γ is in the latter order and if m_i is an \mathcal{B} -basis of Ω , then there are elements ξ_i in Γ such that $\gamma = \sum_i m_i \otimes \xi_i = 1 \otimes \xi$ (with ξ in \mathcal{A}). Then, if α_i are elements of \mathcal{L} such that $1 = \sum_i \alpha_i m_i$, it follows that $\alpha_i \xi = \xi_i \in \Gamma$, which implies that $\xi = \sum_i m_i \xi_i \in \Gamma$.

In the case that $\mathcal{L} | \mathcal{F}$ is unramified, according to Lemma 1, $\Lambda_0^{\mathcal{A}}$ is the sum of its simple components and we can apply Proposition 1 to our situation in which $\mathcal{A}^{\mathcal{A}}$ plays the role of \mathcal{A} , and $\Lambda^{\mathcal{A}}$ the role of Λ .

As for the statement on the behavior of P and I in this unramified case, we proceed as follows. Let $\mathcal{N}, \mathcal{N}^{\mathcal{A}}$, be the radicals of Γ and $P(\Gamma) = \Gamma^{\mathcal{A}}$ and let

$$\mathcal{N}^{-1} = \{ \xi \in \mathcal{A} | \xi \mathcal{N} \subset \Gamma \}, (\mathcal{N}^{\mathcal{A}})^{-1} = \{ \xi \in \mathcal{A}^{\mathcal{A}} | \xi \mathcal{N}^{\mathcal{A}} \subset \Gamma^{\mathcal{A}} \}.$$

We claim that $\mathcal{N}^{\mathcal{A}} = P(\mathcal{N}) = : \Omega \otimes_{\mathcal{A}} \mathcal{N}$. In fact, \mathcal{N} is π -nilpotent (meaning that for some integer m , $\mathcal{N}^m \subset \pi \Gamma$), so that $\Omega \otimes_{\mathcal{A}} \mathcal{N}$ is π -nilpotent too and hence $\Omega \otimes_{\mathcal{A}} \mathcal{N} \subset \mathcal{N}^{\mathcal{A}}$. On the other hand, the assumption that $\mathcal{L} | \mathcal{F}$ is unramified forces $\Omega | \pi \Omega$ to be a separable extension of \mathcal{F} and from this it follows readily that $\Gamma^{\mathcal{A}} / (\Omega_{0, \mathcal{A}} \otimes_{\mathcal{A}} \mathcal{N})$ is semisimple, and hence that $\mathcal{N}^{\mathcal{A}} \subset \Omega \otimes_{\mathcal{A}} \mathcal{N}$.

It is then easy to see that also $(\mathcal{N}^{\mathcal{A}})^{-1} = P(\mathcal{N}^{-1}) = : \Omega \otimes_{\mathcal{A}} \mathcal{N}^{-1}$. Now, if Γ is hereditary, we have $\mathcal{N}^{-1} \mathcal{N} = \Gamma$, from which it follows that $(\mathcal{N}^{\mathcal{A}})^{-1} \cdot \mathcal{N}^{\mathcal{A}} = \Gamma^{\mathcal{A}}$ which says that $\Gamma^{\mathcal{A}}$ is hereditary. Conversely, if $\Gamma^{\mathcal{A}}$ is hereditary, $(\mathcal{N}^{\mathcal{A}})^{-1} \cdot \mathcal{N}^{\mathcal{A}} = \Gamma^{\mathcal{A}}$ and then we have also that $\mathcal{N}^{-1} \mathcal{N} = \Gamma$, or that Γ is hereditary. This completes the proof.

Let us remark before proceeding any further that the algebra $\mathcal{A}_{\mathcal{A}} = \Delta(\mathfrak{f} | \mathcal{H} \times \mathcal{H}, \mathcal{E}, \mathcal{H})$ is a crossed product of the \mathcal{L} -algebra \mathcal{E} and either $\Lambda_{\mathcal{A}}$ or any other \mathcal{B} -order in $\mathcal{O}_{\mathcal{A}}$ is at the same time an order over the Dedekind ring $\Omega = \Lambda_{0, \mathcal{A}}$. Also, it is important to keep in mind that questions of maximality or heredity for elements of $\mathcal{O}_{\mathcal{A}}$ are the same independently that one considers them as \mathcal{B} -orders or as Ω -orders.

Let $\mathcal{O}'_{\mathcal{A}}$ be the family of all orders in $\mathcal{O}_{\mathcal{A}}$ which are invariant under \mathcal{E} , i.e., the set of all orders Γ in $\mathcal{O}_{\mathcal{A}}$ for which it is true that:

$$t_{\sigma} \Gamma t_{\sigma}^{-1} \subset \Gamma \quad (\forall \sigma \in \mathcal{E})$$

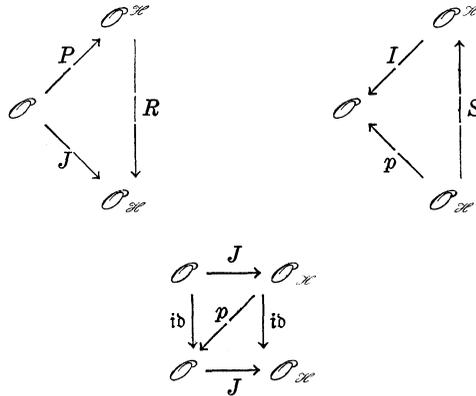
and let us introduce the following two functions:

$$\begin{aligned}
 J: \mathcal{O} &\longrightarrow \mathcal{O}'_{\mathcal{A}} \\
 J(\Gamma) &= \Gamma \cap \mathcal{A}_{\mathcal{A}} \\
 p: \mathcal{O}'_{\mathcal{A}} &\longrightarrow \mathcal{O} \\
 p(\Gamma) &= \Gamma\Lambda = \Lambda\Gamma\Lambda.
 \end{aligned}$$

Then we have:

PROPOSITION 3. *J_0p is the identity map of $\mathcal{O}'_{\mathcal{A}}$ and, for each Γ in \mathcal{O} , $p(J(\Gamma))$ is the minimum of the elements of \mathcal{O} whose under J is $J(\Gamma)$.*

If \mathcal{L}/\mathcal{F} is unramified, the following diagrams are commutative:



and therefore $\mathcal{O}'_{\mathcal{A}} = \mathcal{O}_{\mathcal{A}}$ and both bijections p and J preserve hereditary orders.

Proof. It is clear that J is a mapping from \mathcal{O} into $\mathcal{O}'_{\mathcal{A}}$, because $\mathcal{A}_{\mathcal{A}}$ is invariant under \mathcal{G} . It is also easy to see that, for Γ in $\mathcal{O}'_{\mathcal{A}}$, $\Gamma\Lambda = \sum_{\sigma} \Gamma t_{\sigma}$, from which one obtains readily that $p(\Gamma)$ is an element of \mathcal{O} . The statements about the composites $J \circ p$ and $p \circ J$ are proved straightforwardly.

Let us assume that \mathcal{L}/\mathcal{F} is unramified and let us take Γ in \mathcal{O} . Then $R(P(\Gamma))$ is the set of all elements of the form

$$\sum_{i, \sigma} m_i r_{i\sigma} t_{\sigma}$$

such that

$$m_i \in \Omega, \quad \sum_{\sigma} r_{i\sigma} t_{\sigma} \in \Gamma, \quad \sum_{\sigma \in \mathcal{A}} r_{i\sigma} t_{\sigma} = 0$$

for all i . But this means exactly that $R(P(\Gamma)) = J(\Gamma)$, which is the commutativity of the first diagram. On the other hand, if Γ is in $\mathcal{O}'_{\mathcal{A}}$, it contains $\Lambda_{\mathcal{A}}$, so that:

$$I(S(\Gamma)) = ((\Omega \otimes_{\mathcal{A}} \Lambda)\Gamma(\Omega \otimes_{\mathcal{A}} \Lambda)) \cap \mathcal{A} = (\Omega \otimes_{\mathcal{A}} (\Lambda\Gamma\Lambda)) \cap \mathcal{A}$$

and the order on the right contains $p(\Gamma) = \Lambda\Gamma\Lambda$. Conversely, an element ξ of $I(S(\Gamma))$ can be written in the form:

$$\begin{aligned} \xi &= \sum_i w_i \otimes v_i = 1 \otimes z \\ (w_i, \text{ an } \mathcal{B}\text{-basis of } \Omega; v_i \in \Lambda\Gamma\Lambda = p(\Gamma); z \in \mathcal{A}). \end{aligned}$$

If α_i are elements of \mathcal{L} such that $1 = \sum_i w_i \alpha_i$ we obtain at once that $v_i = \alpha_i z$ so that $z = \sum_i w_i v_i \in p(\Gamma)$. This proves the commutativity of the second diagram. The rest follows because of Proposition 2.

COROLLARY. *Let \mathcal{I} be the inertia group of $\mathcal{E}|\mathcal{F}$ and $\mathcal{E}_{\mathcal{I}} =: \mathcal{L}$ the inertia field, and let us assume that the order Λ_0 contains $\Omega = \Lambda_{0_{\mathcal{I}}} = \Omega_{\mathcal{I}}$ (the inertia ring of the maximal \mathcal{B} -order in \mathcal{E}). Then the lattice \mathcal{O} of orders in \mathcal{A} is isomorphic to the lattice of Ω -orders in $\mathcal{A}_{\mathcal{I}}$. The isomorphism preserves hereditary orders. Hence, the study of orders containing the crossed product $\Lambda = \Lambda(\mathfrak{f}, \Lambda_0, \mathcal{G})$ is reduced to the study of orders containing $\Lambda_{\mathcal{I}} = \Lambda(\mathfrak{f}, \Lambda_0, \mathcal{I})$, a classical crossed product order corresponding to a Galois extension $\mathcal{E}|\mathcal{L}$ which is totally ramified. In particular, if $\mathcal{E}|\mathcal{F}$ is an inertial extension and Λ_0 is the maximal \mathcal{B} -order in \mathcal{E} , then $\Lambda = \Lambda(\mathfrak{f}, \Lambda_0, \mathcal{G})$ is a maximal order in \mathcal{A} . [This last statement has been proved by Auslander and Goldman (see Th. A.12 and Prop. 7.1 of The Brauer group of a commutative ring).]*

4. **Reduction to the wildly ramified case.** In [2] we introduced in a first approach the idea of a noncommutative crossed product. For the time being, we write below an ad-hoc definition which suffices for the purposes of the present paper.

Let \mathcal{S} be a ring, \mathcal{B} an \mathcal{S} -algebra and $\bar{\mathcal{G}}$ a finite group. Given a mapping $\phi: \bar{\mathcal{G}} \rightarrow \text{Aut}(\mathcal{B})$ such that:

- (a) $\phi_1 = \text{id}_{\mathcal{B}}$,
- (b) $\phi_{\sigma} \phi_{\tau} \phi_{\sigma\tau}^{-1} \in \text{In Aut}(\mathcal{B})$,

we define the two mappings:

$$\begin{aligned} \alpha: \bar{\mathcal{G}} \times \bar{\mathcal{G}} &\longrightarrow \text{In Aut}(\mathcal{B}) \\ (\sigma, \tau) &\longrightarrow \alpha_{\sigma, \tau} \\ \mathfrak{f}: \bar{\mathcal{G}} \times \bar{\mathcal{G}} &\longrightarrow U(\mathcal{B}) \\ (\sigma, \tau) &\longrightarrow \mathfrak{f}(\sigma, \tau) \end{aligned}$$

by the conditions:

- (i) $\phi_{\sigma} \phi_{\sigma} = \alpha_{\sigma, \sigma} \phi_{\sigma\sigma}$
- (ii) $\alpha_{\sigma, \tau}(\xi) = \mathfrak{f}(\sigma, \tau) \xi \mathfrak{f}(\sigma, \tau)^{-1} \quad (\forall \sigma, \tau \in \bar{\mathcal{G}}, \forall \xi \in \mathcal{B})$.

Let $\{t_\sigma\}(\sigma \in \bar{\mathcal{G}})$ be any set in 1 - 1 correspondence with $\bar{\mathcal{G}}$ and let \mathcal{A} be the free \mathcal{B} -module with basis $\{t_\sigma\}$. Then the rules:

- (i) $t_\sigma b = \phi_\sigma(b)t_\sigma$,
- (ii) $t_\sigma t_\tau = \bar{f}(\sigma, \tau)t_{\sigma\tau}$ ($\sigma, \tau \in \bar{\mathcal{G}}; b \in \mathcal{B}$)

define a multiplication in \mathcal{A} which makes it into an \mathcal{S} -algebra. This algebra will be denoted by $\Delta(\phi, \mathcal{B}, \bar{\mathcal{G}})$.

DEFINITION 3. If \mathcal{F} is any field and \mathcal{B} is a finite dimensional \mathcal{F} -algebra, any algebra $\tilde{\mathcal{B}}$ which is isomorphic to one of the form $\Delta(\phi, \mathcal{B}, \bar{\mathcal{G}})$ discussed above, will be called a (noncommutative) crossed product of $\bar{\mathcal{G}}$ over \mathcal{B} . If Γ is an \mathcal{B} -order in \mathcal{B} which is invariant under each of the ϕ_σ 's any order $\tilde{\Gamma}$ which is isomorphic to $\Delta(\phi, \Gamma, \bar{\mathcal{G}})$ will be called a (noncommutative) crossed product of $\bar{\mathcal{G}}$ over Γ . (For more details see [2], §6.)

The proof of the following result is fairly direct, but rather long. It is written in details in [2].

LEMMA 4 (*Maschke's theorem*). Let \mathcal{F} be any field and \mathfrak{p} its characteristic and let $\tilde{\mathcal{B}} = \Delta(\phi, \mathcal{B}, \bar{\mathcal{G}})$ be a crossed product \mathcal{F} -algebra. Then, if \mathfrak{p} does not divide the order of $\bar{\mathcal{G}}$, we have:

$$\begin{aligned} \mathcal{B} \text{ is semisimple} &\implies \tilde{\mathcal{B}} \text{ is semisimple;} \\ \mathcal{B} \text{ is separable} &\implies \tilde{\mathcal{B}} \text{ is separable.} \end{aligned}$$

For the remaining of this section we go back to the notations of §3. The situation of Proposition 3 provides us with an example of noncommutative crossed product algebra and an example of noncommutative crossed product order, which is actually the main justification for the introduction of this idea in the present paper.

Let us make \mathcal{A}_\times play the role of \mathcal{B} , and \mathcal{G}/\mathcal{H} the role of $\bar{\mathcal{G}}$. Let us choose a set of representatives of \mathcal{G} in $\bar{\mathcal{G}}$, and let, in general, σ be the representative of the coset $\bar{\sigma} \in \bar{\mathcal{G}}$. Let ϕ be the mapping which sends each $\bar{\sigma} \in \bar{\mathcal{G}}$ to the automorphism of \mathcal{A}_\times which is conjugation by t_σ . Then \mathcal{A} is a crossed product $\Delta(\phi, \mathcal{A}_\times, \mathcal{G}/\mathcal{H})$. In a similar way, Λ is a crossed product order: $\Lambda = \Delta(\phi, \Lambda_\times, \mathcal{G}/\mathcal{H})$, and, more generally, for each $\Gamma \in \mathcal{O}'_\times$, $p(\Gamma) = \Lambda\Gamma\Lambda = \Delta(\phi, \Gamma, \mathcal{G}/\mathcal{H})$.

PROPOSITION 4. If the characteristic of \mathcal{F} does not divide the order of $\bar{\mathcal{G}} = \mathcal{G}/\mathcal{H}$, we have:

1. For each $\Gamma \in \mathcal{O}'_\times$, $\text{rad}(p(\Gamma)) = \text{rad}(\Gamma) \cdot p(\Gamma)$;
2. $\Gamma \in \mathcal{O}'_\times$ is hereditary if and only if $p(\Gamma)$ is hereditary;
3. $\forall \Gamma \in \mathcal{O}$, if $J(\Gamma)$ is hereditary, then Γ is hereditary.

Note. This proposition applies in the case that \mathcal{E}/\mathcal{F} is totally ramified when $\mathcal{H} = \mathcal{R}_1$, the first ramification group. In this case, $\mathcal{A}_{\mathcal{H}} = \mathcal{A}_{\mathcal{R}_1} = \Lambda(\bar{f} | \mathcal{R}_1 \times \mathcal{R}_1, \mathcal{E}, \mathcal{R}_1)$ is a \mathcal{V} -algebra, where \mathcal{V} is the ramification field of \mathcal{E}/\mathcal{F} . The extension \mathcal{E}/\mathcal{V} is wildly ramified. Thus, this proposition provides a reduction for our study to the wildly ramified case.

Proof. 1. Let \mathcal{N} be the radical of Γ and let

$$\mathcal{N}' = \Delta(\phi, \mathcal{N}, \mathcal{E} | \mathcal{H}).$$

Then \mathcal{N}' is a two-sided ideal of $p(\Gamma)$ and, by Nakayama's lemma, $\mathcal{N}' \subset \text{rad}(p(\Gamma))$. On the other hand,

$$p(\Gamma) / \mathcal{N}' = \Delta(\phi, \Gamma, \mathcal{E} | \mathcal{H}) / \Delta(\phi, \mathcal{N}, \mathcal{E} | \mathcal{H}) \cong \Delta(\bar{\phi}, \Gamma / \mathcal{N}, \mathcal{E} | \mathcal{H}),$$

where, for each $\bar{\sigma} \in G/H$, $\bar{\phi}_{\bar{\sigma}}$ is the passage to the quotient Γ / \mathcal{N} of the automorphism $\phi_{\bar{\sigma}}$ of Γ . Then, by Maschke's theorem, $p(\Gamma) / \mathcal{N}'$ is semisimple, whence $\text{rad}(p(\Gamma)) = \mathcal{N}' = \mathcal{N} \cdot p(\Gamma) =: p(\mathcal{N})$.

2. Let $\Gamma \in \mathcal{O}_{\mathcal{H}}$ be hereditary. Then there are elements $x_i \in \mathcal{N}$, $x'_i \in \mathcal{N}^{-1}$ such that $\sum x'_i x_i = 1$. It follows from 1 that $x_i \in p(\mathcal{N})$, $x'_i \in p(\mathcal{N})^{-1}$, so that $p(\Gamma)$ is hereditary.

Now let $p(\Gamma)$ be hereditary and let $\{x_i\}$ be an \mathcal{B} -basis of $p(\mathcal{N})$ and x'_i be elements of $p(\mathcal{N})^{-1}$ such that $\sum x'_i x_i = 1$. We can write each x_i in the form $\eta_i t_{\bar{\sigma}}$, where $\{\eta_i\}$ is an \mathcal{B} -basis of \mathcal{N} . If x_i happens to be, in this notation, $\eta_i t_{\bar{\sigma}}$ we denote it $x_{i,\bar{\sigma}}$ and $x'_{i,\bar{\sigma}}$ the corresponding x'_i . Hence, we can write:

$$x'_{i,\bar{\sigma}} = \sum_{\bar{\tau} \in \mathcal{G} | \mathcal{H}} b_{i,\bar{\sigma},\bar{\tau}} t_{\bar{\tau}}. \quad (b_{i,\bar{\sigma},\bar{\tau}} \in \mathcal{A}_{\mathcal{H}}).$$

It follows easily that each $b_{j,\bar{\sigma},\bar{\tau}}$ belongs to \mathcal{N}^{-1} and since

$$\sum_{i,\bar{\sigma},\bar{\tau}} b_{i,\bar{\sigma},\bar{\tau}} t_{\bar{\tau}} \eta_i t_{\bar{\sigma}} = 1,$$

which implies

$$\sum_{i,\bar{\sigma}} b_{i,\bar{\sigma},\bar{\sigma}^{-1}} \cdot \eta_i^{\bar{\sigma}^{-1}} \cdot \bar{f}(\bar{\sigma}^{-1}, \bar{\sigma}) = 1,$$

we have that Γ is hereditary.

3. We have $p(J(\Gamma)) \subset \Gamma$. Hence, if $J(\Gamma)$ is hereditary, $p(J(\Gamma))$ is hereditary, by 2, and Γ is hereditary because it contains the latter.

DEFINITION 4. Let $\tilde{\Lambda} = \Lambda(\phi, \Lambda, \tilde{\mathcal{E}})$ be a noncommutative crossed product in a separable \mathcal{F} -algebra $\tilde{\mathcal{A}} = \Lambda(\phi, \mathcal{A}, \tilde{\mathcal{E}})$, where \mathcal{A} is a central simple \mathcal{F} -algebra. Let $\mathcal{Z}(\Lambda)$ be the center of Λ . Then:

- (1) The inertia group, \mathcal{I} , of $\tilde{\Lambda}$ over Λ is:

$$\mathcal{I} = \{\sigma \in \bar{\mathcal{G}} / \forall \mathfrak{x} \in \mathcal{X}(A), \phi_\sigma(\mathfrak{x}) \equiv \mathfrak{x} \pmod{\text{rad}(A)}\}$$

- (\mathcal{I} is a normal subgroup of $\bar{\mathcal{G}}$);
- (2) \tilde{A} is tamely ramified over A if
 - (i) $\mathcal{X}(A/\text{rad}(A))$ is a field, an algebraic, separable extension of $\bar{\mathcal{F}}$ and
 - (ii) The characteristic, p , of $\bar{\mathcal{F}}$ does not divide the order of \mathcal{I} .

THEOREM 2. *With the notations of Definition 4, if \tilde{A} is tamely ramified over A and if A is hereditary, then \tilde{A} is hereditary and $\text{rad}(\tilde{A}) = \text{rad}(A) \cdot \tilde{A}$. [Note: In fact, for the proof of this theorem we do not make use of the separability of $\mathcal{X}(A/\text{rad}(A))/\bar{\mathcal{F}}$.]*

Proof. It is enough to prove the latter statement (cf. the proof of Proposition 4). As in Proposition 4, we can write $\bar{\Gamma} = \tilde{A}/\text{rad}(A)\tilde{A}$ in the form $A(\bar{\phi}, A/\text{rad}(A), \bar{\mathcal{G}})$, and it would be enough to show that this algebra is semisimple. Let $\bar{\Gamma}_0 = A(\bar{\phi}, A/\text{rad}(A), \mathcal{I})$. Then $\bar{\Gamma}$ is in an obvious way a noncommutative crossed product of $\bar{\mathcal{G}}/\mathcal{I}$ over $\bar{\Gamma}_0$. By Maschke's theorem, $\bar{\Gamma}$ is semisimple. Also, it is clear that $\text{rad}(\bar{\Gamma}) \cap \bar{\Gamma}_0 \subset \text{rad}(\bar{\Gamma}_0) = 0$. By contradiction, we assume that $\text{rad}(\bar{\Gamma}) \neq 0$, and take $\delta \neq 0, \delta \in \text{rad}(\bar{\Gamma})$. We write δ as a linear combination:

$$\delta = \sum \delta_i t_{\sigma_i} \quad (\delta_i \in \bar{\Gamma}_0)$$

wite a minimal set of nonzero coefficients and we can assume, w.l.o.g., that one of the t_{σ_i} is $t_{\sigma_1} = 1$. Also, since $\delta \neq \mathcal{O}$, the number of coefficients is at least two. Let $\bar{\mathfrak{x}}$ be an element of $\mathcal{X}(A)$ such that, for some index i which we keep fixed thereafter,

$$\phi_{\sigma_i}(\bar{\mathfrak{x}}) \not\equiv \bar{\mathfrak{x}} \pmod{\text{rad}(A)},$$

so that if $\bar{\mathfrak{x}}$ is its class in $\bar{\Gamma}_0$, $\bar{\phi}_{\sigma_i}(\bar{\mathfrak{x}}) \neq \bar{\mathfrak{x}}$. We have:

$$\begin{aligned} \bar{\mathfrak{x}} \cdot \delta &= \sum \bar{\mathfrak{x}} \delta_i t_{\sigma_i} = \sum \delta_i \bar{\mathfrak{x}} t_{\sigma_i} \\ \delta \cdot \bar{\phi}_{\sigma_i}^{-1}(\bar{\mathfrak{x}}) &= \sum \delta_i \bar{\phi}_{\sigma_i} \bar{\phi}_{\sigma_i}^{-1}(\bar{\mathfrak{x}}) t_{\sigma_i}. \end{aligned}$$

By the assumption of minimality, it follows that $\bar{\mathfrak{x}}\delta - \delta\bar{\phi}_{\sigma_i}^{-1}(\bar{\mathfrak{x}}) = 0$, which implies $\delta_i \cdot (\bar{\mathfrak{x}} - \bar{\phi}_{\sigma_i}^{-1}(\bar{\mathfrak{x}})) = 0$, a contradiction.

COROLLARY *If $\Gamma \in \mathcal{O}'_{\mathcal{X}}$ is a maximal order and $p \nmid [\mathcal{I} : \mathcal{H}]$, where $\mathcal{I} | \mathcal{H}$ is the inertia group of $p(\Gamma)$ over Γ , then $p(\Gamma)$ is hereditary.*

Proof. We know that $\mathcal{A}_{\mathcal{X}}$ is simple and if Γ is maximal, $\Gamma/\text{rad}(\Gamma)$ is a simple algebra whose center is a field, so that the proof of Theorem 2 applies to this case.

Note. This paper has been written under the assumption that for Galois (field) extensions \mathcal{E}/\mathcal{F} the residue class field extension $\bar{\mathcal{E}}/\bar{\mathcal{F}}$ is separable (cf. Cor. to Prop. 3, Prop. 4). This assumption is convenient for our purposes, but is not really necessary.

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