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EXTREMAL PROPERTIES OF REAL BIAXIALLY SYMMETRIC POTENTIALS IN $E^{2(\alpha+\beta+2)}$

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The set $\mathscr{B}$ consists of all real biaxially symmetric potentials $U^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} a_n(x^2+y^2)^n P_n^{(\alpha,\beta)}(x^2-y^2/x^2+y^2)/P_n^{(\alpha,\beta)}(1)$, $\alpha > \beta > -1/2$ which are regular in the open unit sphere Σ about the origin in $E^{2(\alpha+\beta+2)}$. Three problems appear regarding $\mathscr{B}$ and subset $\mathscr{B}_*$ whose members have the first $m+1$ coefficients $a_0, \cdots, a_m$ specified. (1) For $U^{(\alpha,\beta)} \in \mathscr{B}$, determine $I(U^{(\alpha,\beta)}) = \inf\{U^{(\alpha,\beta)}(x, y) \mid (x, y) \in \Sigma\}$ as limit of a monotone sequence of constants $\{\lambda_{2n}(a_0, \cdots, a_n)\}_{n=0}^{\infty}$ which can be computed algebraically. (2) Find $U_0^{(\alpha,\beta)} \in \mathscr{B}_*$ and the constant $\lambda_{2m}(a_0, \cdots, a_m) = \sup\{I(U^{(\alpha,\beta)}) \mid U^{(\alpha,\beta)} \in \mathscr{B}_*\} = I(U_0^{(\alpha,\beta)})$. (3) Determine necessary and sufficient conditions from the Fourier coefficients so that $U^{(\alpha,\beta)} \in \mathscr{B}$ and $U^{(\alpha,\beta)}$ is nonnegative in Σ. We develop solutions using operators based on Koornwinder's Laplace type integral for Jacobi polynomials, along with applications of the methods of ascent and descent to the Caratheodory-Fejer and Caratheodory-Toeplitz problems which focus on the properties of harmonic functions in E^2.

1. Introduction. Real biaxially symmetric potentials (BASP) $U^{(\alpha,\beta)}$ which are regular in some domain Ω about the origin in $E^{2(\alpha+\beta+2)}$ may be expanded uniquely as a series

$$(1) \qquad U^{(\alpha,\beta)}(x, y) = a_0 + 2\sum_{n=1}^{\infty} a_n U_n^{(\alpha,\beta)}(x, y)\ , \quad \alpha, \beta > -1/2$$

in terms of the complete set of biaxially symmetric harmonic polynomials

$$(2) \qquad U_n^{(\alpha,\beta)}(x, y) = (x^2 + y^2)^n P_n^{(\alpha,\beta)}(x^2 - y^2/x^2 + y^2)/P_n^{(\alpha,\beta)}(1)\ ,$$

defined from the Jacobi polynomials [1, p. 9]. These functions are necessarily even, satisfying the Cauchy data

$$(3) \qquad U_x^{(\alpha,\beta)}(0, y) = U_y^{(\alpha,\beta)}(x, 0) = 0$$

along the singular lines $x = 0$, $y = 0$ in Ω.

Symmetry about one axis reduces $U_n^{(\alpha,\beta)}$ to zonal harmonics ($\alpha=\beta$), identifying $U^{(\alpha,\beta)}$ as a generalized axially symmetric potential (GASP) [1, p. 10; 5, p. 167] which corresponds to the real part of an analytic function of one complex variable when $\alpha = \beta = -1/2$. This simple correspondence provides characterizations of the fundamental properties of harmonic functions in E^2 from their Fourier coefficients in circular harmonics as they are determined by those of the associated analytic functions; serving as a point of reference in seeking the singularities, zeros, and extremal values for other parameters α, β.

R. P. Gilbert [5, 6] employed properties of the Jacobi polynomials to represent each (complex valued) BASP as the integral transform of a unique associated analytic function of one complex variable and conversely. Then reasoning as in the "Envelope Method" [4, 5], a generalization of the Hadamard argument in the Singularities Theorem [3, 5], he showed that the classical criterion of Hadamard and Mandelbrojt [3] for determining the location and structure of the singularities of harmonic functions in E^2 from their Fourier coefficients provides analogous information for BASP in $E^{2(\alpha+\beta+2)}$, $\alpha, \beta > -1/2$.

M. Marden [11] and P. McCoy [12, 13] applied convexity arguments and conformed mapping techniques to the Bergman $\mathscr{B}_3$ [2, 5] and Gilbert $\mathscr{A}_\mu$ [4, 5] integral representations of GASP, describing their value distribution as in the classical Cauchy [10, p. 123], Caratheordory-Toeplitz [16, p. 153] and Schur [17, p. 159] coefficient theorems for harmonic functions in E^2. T. Koornwinder's [1, 9] new Laplace type integral for Jacobi polynomials was used by P. McCoy and J. D'Archangelo [15] to extend properties developed by Marden for the zeros of axially symmetric harmonic polynomials to harmonic polynomials with biaxial symmetry.

Further applications of Koornwinder's integral by McCoy [16] produced operators mapping analytic functions of one complex variable onto (complex valued) BASP and conversely. These operators, valid for limited ranges of the parameters, permitted a partial extension of the Caratheodory-Toeplitz and Schur theorems. Moreover, a new aspect of the coefficient problem—that of the extremal properties of the real axially symmetric potentials of the Caratheodory-Fejer [8, p. 145ff] type—was introduced by operators related to $\mathscr{B}_3$ and $\mathscr{A}_\mu$.

This article provides a unified treatment of the above mentioned theorems and properties, extending them to real BASP without restriction beyond Koornwinder's on the parameters. Taken in union with Gilbert's theory of singularities, it completes the generalization of the classical coefficient theorems pertaining to real harmonic (or analytic) functions in E^2. These may be also viewed as a means of calculating the infimum (supremum) of solutions to the biaxially symmetric potential equation [5, 9] from the Fourier coefficients which taken with the methods of ascent and descent [4] indicates similiar possibilites for solutions to more general partial differential equation generated by operators whose properties are analogous to those found in

2. Basic formulas and definitions. Koornwinder's formula [9, p. 130] represents the biaxially symmetric harmonic polynomials as

$$U_n^{(\alpha,\beta)}(x, y) = \int_0^1\int_0^\pi \zeta^n d\mu_{\alpha,\beta}(t, s) , \quad \alpha > \beta > -1/2 \tag{4}$$

$$\zeta = x^2 - y^2t^2 + i2xyt \cos s \tag{5}$$

with nonnegative measure

$$\begin{aligned} d\mu_{\alpha,\beta}(t, s) &= \gamma_{\alpha,\beta}(1 - t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin s)^{2\alpha}dtds \\ \gamma_{\alpha,\beta} &= 2\Gamma(\alpha + 1)/\Gamma(1/2)\Gamma(\alpha - \beta)\Gamma(\beta + 1/2) \end{aligned} \tag{6}$$

normalized so that

$$\int_0^1\int_0^\pi d\mu_{\alpha,\beta}(t, s) = 1 . \tag{7}$$

The real harmonic polynomials

$$\begin{aligned} u_n(x, y) &\equiv \boldsymbol{u}_n(x^2 - y^2, 2xy) \\ v_n(x, y) &\equiv \boldsymbol{v}_n(x^2 - y^2, 2xy) \end{aligned} \tag{8}$$

are defined by

$$u_n(x, y) + iv_n(x, y) = (x + iy)^{2n} .$$

Expanding the vector ζ^n in terms of these as

$$\zeta^n = \boldsymbol{u}_n(x^2 - y^2t^2, 2xyt \cos s) + i\boldsymbol{v}_n(x^2 - y^2t^2, 2xyt \cos s) \tag{9}$$

and transforming according to Koornwinder's formula establishes that v_n, the harmonic conjugate of u_n, is in the null space of (4). This suggests the relation

$$U_n^{(\alpha,\beta)}(x, y) = \int_0^1\int_0^\pi \boldsymbol{u}_n(x^2 - y^2t^2, 2xyt \cos s)d\mu_{\alpha,\beta}(t, s) \tag{10}$$

associating the real (even) BASP (1) and the real (even) harmonic function

$$u(x, y) = a_0 + 2\sum_{n=1}^{\infty} a_n u_n(x, y) \tag{11}$$

viz.

$$\boldsymbol{u}(x^2 - y^2, 2xy) = a_0 + 2\sum_{n=1}^{\infty} a_n \boldsymbol{u}_n(x^2 - y^2, 2xy)$$

by the operator

$$U^{(\alpha,\beta)} = A_{\alpha,\beta}(u) \tag{12}$$

whose definition is

$$U^{(\alpha,\beta)}(x, y) = \int_0^1\int_0^\pi u(x^2 - y^2t^2, 2xyt\cos s)d\mu_{\alpha,\beta}(t, s) . \tag{13}$$

Evidently, if u is harmonic in the open unit disk D_ρ of radius ρ about the origin in E^2, $U^{(\alpha,\beta)}$ is a BASP in the open unit sphere Σ_ρ of radius ρ about the origin in $E^{2(\alpha+\beta+2)}$.

The inverse operator (related to $\mathscr{G}^{-1}_{\alpha,\beta}$ [see 6]) uses orthogonality of the Jacobi polynomials

$$\int_{-1}^{+1} P_n^{(\alpha,\beta)}(\tau)P_m^{(\alpha,\beta)}(\tau)(1-\tau)^\alpha(1+\tau)^\beta d\tau = h_n^{(\alpha,\beta)}\delta_{nm},\ \alpha, \beta > -1$$

to define the measure

$$\begin{aligned} d\nu_{\alpha,\beta}(\xi, \eta, \tau) &= S_{\alpha,\beta}(\xi, \eta, \tau)(1-\tau)^\alpha(1+\tau)^\beta d\tau , \\ S_{\alpha,\beta}(\xi, \eta, \tau) &= \sum_{n=0}^{\infty} u_n(\xi, \eta)P_n^{(\alpha,\beta)}(\tau)P_n^{(\alpha,\beta)}(1)/h_n^{(\alpha,\beta)} \end{aligned} \tag{14}$$

inverting the relation (10) as

$$u_n(x, y) = \int_{-1}^{+1} U_n^{(\alpha,\beta)}\left(r\sqrt{\frac{1-\tau}{2}}, r\sqrt{\frac{1+\tau}{2}}\right)d\nu_{\alpha,\beta}(xr^{-1}, yr^{-1}, \tau) ,$$

determining the inverse operator

$$u = A^{-1}_{\alpha,\beta}(U^{(\alpha,\beta)})$$

as

$$u(x, y) = \int_{-1}^{+1} U^{(\alpha,\beta)}\left(r\sqrt{\frac{1-\tau}{2}}, r\sqrt{\frac{1+\tau}{2}}\right)d\nu_{\alpha,\beta}(xr^{-1}, yr^{-1}, \tau) . \tag{15}$$

An absolutely and uniformly convergent dominant of $S_{\alpha,\beta}$ for (ξ, η, τ) on compact subsets of $[0, 1) \times [0, 1) \times [-1, +1]$ is the Poisson kernel [1, p. 11]. By construction of the operators, it follows directly that $A_{\alpha,\beta}$ and $A^{-1}_{\alpha,\beta}$ are one-one onto maps between the families

$$\mathscr{H}_\rho^{(\alpha,\beta)} = \{U^{(\alpha,\beta)} \mid \text{expansion (1) regular in } \Sigma_\rho\} ,\quad \alpha > \beta > -1/2$$

and

$$h_\rho = \{u \mid \text{expansion (11) regular in } D_\rho\}$$

which share the normalization

$$A_{\alpha,\beta}(1) = A^{-1}_{\alpha,\beta}(1) = 1 .$$

A principle interest is in the values of the functionals

$$\begin{aligned} I(U^{(\alpha,\beta)}) &= \inf_{\Sigma} U^{(\alpha,\beta)} ,\quad & U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)} \\ i(u) &= \inf_{D} u , & u \in h \end{aligned}$$

(subscripts $\rho = 1$ are dropped) as they are determined by the minimal eigenvalues $\lambda_{2k}(a_0, \cdots, a_k)$ of the Toeplitz matrices

$$(16) \qquad T_{2k}(a_0, \cdots, a_k) = \begin{pmatrix} a_0 & 0 & a_1 & 0 & a_2 & 0 & a_3 \cdots & a_k \\ 0 & a_0 & 0 & a_1 & 0 & a_2 & 0 \cdots & 0 \\ a_1 & 0 & a_0 & 0 & a_1 & 0 & a_2 \cdots & a_{k-1} \\ \vdots & & & & & & & \vdots \\ a_k & 0 & \cdots & & & & & a_0 \end{pmatrix}$$

found by applying theorem (a) [8, p. 147] to the function $F(z) = f(z^2)$. We now turn to

3. Extremal properties. The following is an extension of theorem (a) [8, p. 147] referred to in an equivalent form [7, p. 499ff] as the Caratheodory-Fejer theorem which is how we identify it.

THEOREM 1. *Let $U^{(\alpha,\beta)}(x, y) = a_0 + 2\sum_{n=1}^{\infty} a_n U_n^{(\alpha,\beta)}(x, y)$ be a real BASP regular in the sphere Σ and $\{\lambda_{2k}(a_0, \cdots, a_k)\}_{k=0}^{\infty}$ be the sequence of smallest eigenvalues associated with the Toeplitz matrices $\{T_{2k}(a_0, \cdots, a_k)\}_{k=0}^{\infty}$. Then*

$$(17) \qquad I(U^{(\alpha,\beta)}) = \lim_k \lambda_{2k}(a_0, \cdots, a_k) , \quad \alpha > \beta > -1/2 .$$

Proof. For the nonnegativity of the measure (6) and the normalization (7), it is immediate that

$$U^{(\alpha,\beta)}(x, y) = A_{\alpha,\beta}(u) \geqq i(u) , \quad \alpha > \beta > -1/2$$

and

$$(18) \qquad I(U^{(\alpha,\beta)}) \geqq i(u) = \lim_k \lambda_{2k}(a_0, \cdots, a_k) .$$

The smaller functional is evaluated by the Caratheodory-Fejer theorem [8, p. 147]. Anticipating the reverse inequality, we define the functionals

$$I_{\rho_0}(U^{(\alpha,\beta)}) = \inf_{\Sigma_{\rho_0}} U^{(\alpha,\beta)}$$
$$i_{\rho_0}(u) = \inf_{D_{\rho_0}} u$$

with

$$\rho_0 = \sup\{\rho \mid S_{\alpha,\beta}(\xi, \eta, \tau) > 0 , \; \xi^2 + \eta^2 < \rho^2 < 1 , \quad \tau \in [-1, 1]\} .$$

The number ρ_0 exists since $S_{\alpha,\beta}$ is continuous in a cylinder of small enough radius with center on the τ-axis, $\tau \in [-1, +1]$, and ρ_0 is positive as $S_{\alpha,\beta}(0, 0, \tau) = 1$ there.

Now, if $U_*^{(\alpha,\beta)}(x_1, y_1)$ is a BASP which is nonnegative for $x_1^2 + y_1^2 \leqq \rho_0^2$ and the $A_{\alpha,\beta}^{-1}$ associate is u_*, then

$$u_*(x_1, y_1) = A_{\alpha,\beta}^{-1}(U_*^{(\alpha,\beta)})$$
$$\geqq A_{\alpha,\beta}^{-1}(I_{\rho_0}(U_*^{(\alpha,\beta)})) = I_{\rho_0}(U_*^{(\alpha,\beta)})$$

so that

$$i_{\rho_0}(u_*) \geqq I_{\rho_0}(U_*^{(\alpha,\beta)}) . \tag{19}$$

The homothetic transformations $x_1 = \rho_0 x$, $y_1 = \rho_0 y$, and the homogeneity of the harmonic polynomials

$$u_n(x, y) = \rho_0^{-2n} u_n(x\rho_0, y\rho_0)$$

and

$$U_n^{(\alpha,\beta)}(x, y) = \rho_0^{-2n} U_n^{(\alpha,\beta)}(x\rho_0, y\rho_0)$$

produce harmonic functions

$$u_*(x_1, y_1) = a_0 + 2\sum_{n=1}^{\infty} a_n \rho_0^{-2n} u_n(x_1, y_1)$$

and

$$U_*^{(\alpha,\beta)}(x_1, y_1) = a_0 + 2\sum_{n=1}^{\infty} a_n \rho_0^{-2n} U_n^{(\alpha,\beta)}(x_1, y_1)$$

regular for $x_1^2 + y_1^2 < \rho_0^2$ corresponding to the regular functions (1) and (11) in $x^2 + y^2 < 1$. Evidently,

$$I_{\rho_0}(U_*^{(\alpha,\beta)}) = I(U^{(\alpha,\beta)})$$
$$i_{\rho_0}(u_*) = I(U^{(\alpha,\beta)})$$

and because of inequality (19),

$$i(u) \geqq I(U^{(\alpha,\beta)}) . \tag{20}$$

Thus,

$$i(u) = I(U^{(\alpha,\beta)}) \tag{21}$$

and because of (18) the theorem is proved.

We next define the set $\mathscr{H}_*^{(\alpha,\beta)} = \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)$ as the subset of $\mathscr{H}^{(\alpha,\beta)}$ whose members have their first $m+1$ coefficients $a_0, \cdots, a_m$ fixed and turn to the analogy of the second classical theorem [8, p. 151].

THEOREM 2. *Let $U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)$ be expanded as in* (1) *and $\lambda_{2m}(a_0, \cdots, a_m)$ be the smallest eigenvalue of the Toeplitz matrix $T_{2m}(a_0, \cdots, a_m)$. Then*

$$I(U^{(\alpha,\beta)}) \leqq \lambda_{2m} , \quad \alpha > \beta > -1/2$$

and

$$\sup \{I(U^{(\alpha,\beta)}) \,|\, U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)\} \\ = I(U_0^{(\alpha,\beta)}) = \lambda_{2m}$$

for unique $U_0^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)$ *expanded as*

$$\begin{aligned} U_0^{(\alpha,\beta)}(x, y) &= \lambda_{2m} + \sum_{k=1}^{j} \sigma_k W_k^{(\alpha,\beta)}(x, y) , \\ W_k^{(\alpha,\beta)}(x, y) &= A_{\alpha,\beta}(w_k) , \\ w_k(x, y) &= [1 - (x^2 + y^2)][g_k(x, y)]^{-1} , \\ g_k(x, y) &= 1 - 2(x^2 + y^2) \cos \{2 \operatorname{arc} \cos x/\sqrt{x^2 + y^2} - \phi_k\} \\ &\quad + (x^2 + y^2)^2 \end{aligned} \tag{22}$$

for unique $1 \leqq j \leqq m$, $\phi_k \in [0, 2\pi)$, $\sigma_k > 0$ *provided* $c_1^2 + \cdots + c_m^2 \neq 0$, *otherwise if and only if*

$$U_0^{(\alpha,\beta)}(x, y) = c_0 = \lambda_{2m} . \tag{23}$$

Proof. For the subfamily $\mathscr{H}_*^{(\alpha,\beta)}$ we associate the subfamily $\mathscr{h}_* = A_{\alpha,\beta}^{-1}\{(U^{(\alpha,\beta)} \,|\, U^{(\alpha,\beta)} \in \mathscr{H}_*^{(\alpha,\beta)})\}$ whose members u satisfy the requisite inequality [8, p. 151],

$$i(u) \leqq \lambda_{2m}(a_0, \cdots, a_m) . \tag{24}$$

The matrix $T_{2m}(a_0, \cdots, a_m)$ is identified from [8, p. 146] by $u(x, y) = \operatorname{Re} f((x + iy)^2)$. Because of the relation (21), we find

$$I(U^{(\alpha,\beta)}) \leqq \lambda_{2m}(a_0, \cdots, a_m) .$$

The $A_{\alpha,\beta}^{-1}$ associate of the extremal function $U_0^{(\alpha,\beta)}$ is

$$\begin{aligned} u_0(x, y) &= \lambda_{2m} + \sum_{k=1}^{j} \sigma_k w_k(x, y) , \\ w_k(x, y) &= 1/2[g_k(z^2) + \bar{g}_k(\bar{z}^2)] , \quad z = x + iy \\ g_k(z) &= (1 + \varepsilon_k z)/(1 - \varepsilon_k z) , \quad |\varepsilon_k| = 1 . \end{aligned}$$

Because of (12), u_0 transforms onto the required extremal function (22) as defined.

The final result is the generalization of the Caratheodory-Toeplitz theorem [8, p. 152; 17, p. 157] which classified nonnegative harmonic functions in D from their coefficients.

THEOREM 3. *Necessary and sufficient conditions for the expansion* $U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}$ *and*

$$U^{(\alpha,\beta)}(x, y) \geqq 0 , \quad x^2 + y^2 < 1 , \quad \alpha > \beta > -1/2$$

specify that the determinants

$$\Delta_n(a_0, \cdots, a_n) = \det T_{2n}(a_0, \cdots, a_n)$$

generated from the coefficients of the expansion are either

$$\text{(i)} \qquad \Delta_n(a_0, \cdots, a_n) > 0\,, \quad n = 0, 1, \cdots$$

or in case

$$\text{(ii)} \qquad \begin{aligned} &\Delta_n(a_0, \cdots, a_n) > 0\,, \quad n = 0, \cdots, m\,, \\ &\Delta_n(a_0, \cdots, a_n) = 0\,, \quad n = m + 1, \cdots\,, \\ &\textit{where } U^{(\alpha,\beta)}(x, y) = U_0^{(\alpha,\beta)}(x, y) - \lambda_{2m}\,. \end{aligned}$$

Proof. When u, the associate of $U^{(\alpha,\beta)}$, is nonnegative and regular in D so must $U^{(\alpha,\beta)}$ be nonnegative and regular in Σ because the measure of the transform is nonnegative. This is indeed the case [see 8, 18] if (i) or (ii), establishing the sufficiency. Conversely,

$$u_*(x_1, y_1) = A_{\alpha,\beta}^{-1}(U_*^{(\alpha,\beta)}) \geqq 0$$

when

$$U_*^{(\alpha,\beta)}(x_1, y_1) \geqq 0\,, \quad x_1^2 + y_1^2 < \rho_0^2\,.$$

However,

$$\operatorname{sgn} u_*(x_1, y_1) = \operatorname{sgn} u(x, y)$$
$$\operatorname{sgn} U_*^{(\alpha,\beta)}(x_1, y_1) = \operatorname{sgn} U^{(\alpha,\beta)}(x, y)$$

so that $U^{(\alpha,\beta)}$ nonnegative and regular in Σ implies u nonnegative and regular in D which asserts (i) or (ii).

4. Generalizations. For $\beta > \alpha$, the symmetry relations found from [1, p. 8]

$$U_n^{(\alpha,\beta)}(x, y) = (-1)^n U_n^{(\beta,\alpha)}(-x, y)$$

may be employed with the proper interpretation of the biaxially symmetric potential equation [6, 9]. The axisymmetric case $\alpha = \beta$, may be interpreted with $\alpha \downarrow \beta$ in Koornwinder's formula which becomes Gegenbauer's integral for the Jacobi polynomials.

The classical theorems of Caratheodory-Fejer and Caratheodory-Toeplitz have analogous calculations for the supremum [see 8] and bounds on the maximum modulus (Caratheodory-Schur [see 17]) which generalize directly by the methods contained here in. Domains $\Omega \subset E^{2(\alpha+\beta+2)}$ about the origin which are not spheres are defined by their projections into $\omega \subset E^2$ as

$$\Omega = \{(x, y) \,|\, \zeta^2 \in \omega, 0 \leqq s \leqq \pi, -1 \leqq t \leqq +1\}\,,$$

ω being a simply connected domain about the origin. To consider extensions of theorems and 1 and 2, ω is mapped conformally onto D. The required connection coefficients between $U^{(\alpha,\beta)}$ a regular BASP in Ω and the "associated" BASP regular in Σ may be found as in [13, 14]. The methods of ascent and descent may be utilized to extend the above properties.

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