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A CLASS OF MODIFIED ζ AND L-FUNCTIONS

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The purpose of this paper is the construction of a class of functions that have exactly the same complex zeros as the Riemann zeta function $\zeta(s)$, or as any Dirichlet function $L(s, \chi)$. The motivation for this construction is found in certain attempts to study the Riemann hypothesis.

The problem of the Riemann hypothesis has been approached, occasionally

(a) by attempts to study functions that share with $\zeta(s)(s=\sigma+it)$ certain analytic properties (e.g., representation by Dirichlet series, functional equation, Euler product, etc.), in order to see what restrictions these properties impose upon their zeros; or

(b) by the construction of functions, whose zeros are subject to certain restrictions (e.g., they do, or they don't satisfy a "Riemann hypothesis"), in the hope to detect similarities with, or differences from $\zeta(s)$, or $L(s, \chi)$.

To the first approach belong attempts to show that some of those analytic properties suffice to impose some kind of "Riemann hypothesis." These attempts were not too successful, in part because it turned out that functions like the the Epstein zeta function, with many of the properties of $\zeta(s)$ (functional equation, Dirichlet series-but no Euler product) may have zeros with $\sigma > 1$, and also with $0 < s < 1$ (see [4]).

To the second approach belongs, e.g., an attempt by Rademacher [5] to disprove the Riemann hypothesis, by studying the class of functions for which, assuming "the Riemann hypothesis," the sum $\sum_{\gamma} \gamma^{-1} \sin \gamma t = f(t)$ ($\gamma =$ imaginary part of the complex zero $1/2 + i\gamma$) exhibits the discontinuities known to occur when $\rho = 1/2 + i\gamma$ are zeros of $\zeta(s)$. It was shown, however, by Rubel and Straus (see [6] and [7]) that the known behaviour of $f(t)$ for $\zeta(s)$ is implied already by conditions much weaker than the Riemann hypothesis.

The results of the present paper seem to indicate that the last approach is unlikely to lead to interesting conclusions, but suggest a new and potentially useful approach. Indeed, we construct a class of functions of analytic character very different from that of $\zeta(s)$ and, nevertheless, with the same complex zeros. We also sketch the constrution of functions that share their complex zeros with Dirichlet's L-functions. In principle, the construction is valid for all Dirichlet series with an Euler product. These new functions have a Dirichlet series representation, also an Euler product, but, in general, they do not satisfy any functional equation and often cannot even be continued analytically beyond $\sigma > 0$.

Section 2 contains the principle of the construction and the statements of the main results. Sections 3 and 4 contain the proofs, Section 5 discusses analytic continuations and Section 6 mentions some possible applications.

2. Main results. Let p_n be the nth prime and select q_n so that

$$p_n \leqq q_n \leqq p_{n+1} . \tag{1}$$

With these q_n form the infinite product

$$\zeta^*(s) = \prod_{n=1}^{\infty} (1 - q_n^{-s})^{-1} . \tag{2}$$

The product converges absolutely for $\sigma > 1$ and uniformly $\sigma \geqq 1 + \varepsilon$ ($\varepsilon > 0$), so that $\zeta^*(s)$ is holomorphic for $\sigma > 1$.

THEOREM 1. *The function $\zeta^*(s)$ possesses the following properties:* (i) $\zeta^*(s) \neq 0$ *for* $\sigma > 1$; (ii) $\zeta^*(s)$ *can be continued as a meromorphic function in* $\sigma > 0$; (iii) *in* $\sigma > 0$, $\zeta^*(s)$ *has a single pole at* $s = 1$ *with residue* r, $1/2 \leqq r \leqq 1$; (iv) *in* $\sigma > 0$, $\zeta^*(s)$ *has the same zeros, with the same multiplicities as* $\zeta(s)$.

Theorem 1 remains valid if q_n is restricted only by $p_n \leqq q_n$ for all $n \geqq n_0$, with $\lim_{n\to\infty} p_n/q_n = 1$. In general, any restriction on q_n may be applied only to subscripts $n \geqq n_0$ (n_0 arbitrarily large). Many other generalizations of Theorem 1 are also possible.

THEOREM 2. *Let* $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$, *where* $\chi(n)$ *is a Dirichlet character modulo the natural integer* k. *For* $K \geqq k$, *select rational integers* q_n *that satisfy the two conditions*

$$p_n \leqq q_n \leqq p_n + K \quad \textit{and} \quad p_n \equiv q_n \pmod{k} . \tag{3}$$

For $\sigma > 1$ *define* $L^*(s, \chi)$ *by the absolutely convergent infinite product*

$$L^*(s, \chi) = \prod (1 - \chi(q)q^{-s})^{-1} , \tag{4}$$

extended over all $q = q_n$. *The function* $L^*(s, \chi)$ *can be continued into the whole half plane* $\sigma > 0$, *where it has exactly the same zeros (including multiplicities) as* $L(s, \chi)$. *If* $\chi(n)$ *is not the principal character, then* $L^*(s, \chi)$ *is holomorphic in* $\sigma > 0$.

The sketch of the proof of Theorem 2 (see §4) will suggest to the reader a number of more general possible versions of the theorem.

3. Proof of Theorem 1. Set

$$\varphi(s) = \prod_p \{(1 - p^{-s})(1 - q^{-s})^{-1}\}\ , \tag{5}$$

where p runs through the primes p_n and q through the corresponding real numbers q_n; here and in what follows the subscript n is suppressed whenever this does not lead to ambiguities. For $\sigma > 1$, the infinite product (5) converges absolutely (in fact, each factor converges separately) and

$$\zeta^*(s) = \varphi(s)\zeta(s)\ . \tag{6}$$

LEMMA 1. *The infinite product* (5) *converges absolutely in* $\sigma > 0$, *where* $\varphi(s) \neq 0$. *The convergence is uniform on compact sets* $\sigma_0 \leqq \sigma \leqq \sigma_1$, $|t| \leqq T$, *for any given constants with* $\sigma_1 > \sigma_0 > 0$ *and* $T > 0$.

Assuming Lemma 1, (6) may be used to define the analytic continuation of $\zeta^*(s)$ to the whole set into which $\varphi(s)$ can be continued, which contains the half plane $\sigma > 0$. From this all statements of Theorem 1 follow immediately, except the value of the residue r. For real s one obtains, by using the uniformity of the convergence, that

$$\begin{aligned} r &= \lim_{s\to 1+} \{\zeta^*(s)/\zeta(s)\} = \lim_{s\to 1+} \prod_p \{1(1 - p^{-s})(1 - q^{-s})^{-1}\} \\ &= \prod_p \{(1 - p^{-1})(1 - q^{-1})^{-1}\} = \lim_{P\to\infty} \prod_{p \leqq P} \{(1 - p^{-1})(1 - q^{-1})^{-1}\} \\ &= \lim_{N\to\infty} \prod_{n=1}^{N} \{(1 - p_n^{-1})(1 - q_n^{-1})^{-1}\}\ . \end{aligned}$$

By (1) and $p_1 = 2$ it follows that

$$\begin{aligned} \frac{1}{2} = 1 - \frac{1}{2} &= \lim_{N\to\infty} \prod_{n=1}^{N} \{(1 - p_n^{-1})(1 - p_{n+1}^{-1})^{-1}\} \leqq r \\ &\leqq \lim_{N\to\infty} \prod_{n=1}^{N} \{1 - p_n^{-1})(1 - p_n^{-1})^{-1}\} = 1\ , \end{aligned}$$

as claimed.

The proof of Theorem 1 has been reduced to that of Lemma 1. In its proof (and also in that of Lemma 3) we shall need several, presumably wellknown inequalities, which we collect, for convenience, in

LEMMA 2. *For integral, rational* m_0 *and real* α, x, y, ε, *and* u, *with* $0 < x < 1$, $0 < y < 1$, $0 < \varepsilon < 1/2$, $u > 0$, *the following inequalities hold*:

(i) $1 - \cos \alpha < \alpha^2/2$;
(ii) $|\log x| < (1 - x)/x$;
(iii) $1 < (1 - x)/(1 - x^y) < y^{-1}$, *with* $\lim_{x \to 1^-} \{(1 - x)/(1 - x^y)\} = y^{-1}$;
(iv) if $f(x) = \sum_{m=m_0+1}^{\infty} m^{-1}x^m$, *then* $0 < f(x) < x^{m_0+1}/(1 - x)(m_0 + 1)$;
(v) $1 - (1 - \varepsilon)^u < (1 + \varepsilon)(1 - e^{-\varepsilon u})$.

Proof of Lemma 2. (i) is classical. (ii) follows from

$$|\log x| = |\log(1 - (1 - x))| = \sum_{n=1}^{\infty} n^{-1}(1 - x)^n < (1 - x) \sum_{n=0}^{\infty} (1 - x)^n = (1 - x)/x .$$

For (iii), observe that $x < x^y < 1$ and the first inequality holds. Set $g(x, y) = 1 + xy - y - x^y$; the second inequality is equivalent to $g(x, y) > 0$. This follows from $g(1, y) = 0$ and $\{\partial g/\partial x\}_{x<1} = y(1 - x^{y-1}) < 0$. The limit for $x \to 1^-$ may be obtained by L'Hospital's rule. To prove (iv), consider $f'(x) = \sum_{m=m_0+1}^{\infty} x^{m+1} = x^{m_0}/(1 - x)$, so that $0 < f(x) < \int_0^x u^{m_0}(1 - u)^{-1}du < (1 - x)^{-1} \int_0^x u^{m_0}du = x^{m_0+1}/(1 - x)(m_0 + 1)$. For (v), set $h(u) = (1 + \varepsilon)(1 - e^{-\varepsilon u})$; then (v) is equivalent to $h(u) > 1$, for $u > 0$. We verify that $h(0) = 1$ and $h'(u) > 0$ for $u > 0$. The first is obvious and the second is equivalent to $(e^{-\varepsilon}/(1-\varepsilon))^u(1 + \varepsilon) > -\varepsilon^{-1} \log(1 - \varepsilon)$. As $e^{-\varepsilon} > 1 - \varepsilon$, we only need to have $1 + \varepsilon > \sum_{n=1}^{\infty} \varepsilon^{n-1}/n$, or $1/2 > \sum_{n=3}^{\infty} \varepsilon^{n-2}/n$, easily verified to hold, say, for $\varepsilon < 3/5$.

Proof of Lemma 1. We consider

$$\log \varphi(s) = \sum_p \{\log (1 - p^{-s}) - \log(1 - q^{-s})\}$$
$$= \sum_p \sum_{m=1}^{\infty} m^{-1}(q^{-ms} - p^{-ms}) = \sum_{m=1}^{\infty} \sum_p m^{-1}(q^{-ms} - p^{-ms}) ,$$

where the interchange of summations is easily justified for $\sigma > 1$. By the absolute and uniform convergence of the double series for $\sigma \geqq 1 + \varepsilon$, it is sufficient to study its convergence on compact sets C of the form $\sigma_0 \leqq \sigma \leqq 1$, $|t| \leqq T$, with given $\sigma_0 > 0$ and $T > 0$. For fixed $s = \sigma + it \in C$, define $m_0 = [\sigma^{-1}] + 1$, where $[x]$ stands for the greatest integer function. We split the sum over m into two parts, $\sum_{m=1}^{\infty} = \sum_{m=1}^{m_0} = \sum_{m=m_0+1}^{m_0} + \sum^1 + \sum^2$, say. The absolute and uniform convergence of $\sum^2$ is almost obvious. Indeed,

$$\left| \sum_{m=m_0+1}^{\infty} \sum_p m^{-1}(q^{-ms} - p^{-ms}) \right| \leqq \sum_{m=m_0+1}^{\infty} m^{-1} \sum_p (q^{-m\sigma} + p^{-m\sigma})$$
$$\leqq 2 \sum_{m=m_0+1}^{\infty} m^{-1} \sum_p p^{-m\sigma} = 2 \sum_p \sum_{m>m_0} (mp^{m\sigma})^{-1} .$$

By Lemma 2(iv) with $x = p^{-\sigma}$, it follows that the inner sum is majorized by

$$\frac{p^{-\sigma(m_0+1)}}{(1-p^{-\sigma})(m_0+1)} \leqq \frac{1}{(1-2^{-\sigma_0})(m_0+1)p^{\sigma m_0+\sigma}} < \frac{c_1}{p^{1+\sigma_0}},$$

with $c_1 = (1-2^{-\sigma_0})^{-1}$, because $\sigma m_0 + \sigma = \sigma([\sigma^{-1}]+2) > \sigma(\sigma^{-1}+1) = 1+\sigma \geqq 1+\sigma_0$. Here and in what follows, $c_1, c_2, \cdots$ stand for constants depending at most on σ_0 and T.

It now follows that $|\sum^2| \leqq 2\sum_p p^{-(1+\sigma_0)}$ and the right-hand side converges and does not depend on s.

As for $\sum^1$, the outer sum contains the finitely many terms with $1 \leqq m \leqq m_0$, and $m_0 = [\sigma^{-1}]+1 \leqq \sigma^{-1}+1 < \sigma_0^{-1}+1$. Hence, it is sufficient to prove the absolute and uniform convergence of the inner sums $\sum_p |p^{-ms} - q^{-ms}|$ for $1 \leqq m \leqq m_0$. Let $q_n = p_n + r \leqq p_{n+1}$; then (see, e.g., [2] (14.3), page 131) $r = r(p) \leqq p^{3/5+\varepsilon}$ for every $\varepsilon > 0$, provided that $p \geqq P(\varepsilon)$. It follows that $0 \leqq r(p) \leqq p^{4/5}$, except, perhaps for finitely many primes, which have no influence upon the convergence of the series. If $x = x(p) = p/q$, then $1 - r/q = (q-r)/q = p/q = x(p) \leqq 1$, and, for $p \geqq P(1/5)$, $x(p) > 1 - q^{-1/5}$, so that $\lim_{p\to\infty} x(p) = 1^-$. In particular, for sufficiently large p, $x(p)^2 > 1/3$. It follows that $|p^{-ms} - q^{-ms}| = p^{-m\sigma}|1 - x^{ms}| = p^{-m\sigma}|1 - x^{m\sigma}e^{itm\log x}|$. If $p = q$, this difference vanishes; otherwise, $x < 1$, $r > 0$, and the last factor can be estimated for $3^{-1/2} \leqq x < 1$ and $m < m_0$, by using (i), (ii), and (iii) of Lemma 2, with $0 < y = m\sigma < 1$, as follows:

$$\begin{aligned}
|1 - x^{m\sigma}e^{itm\log x}|^2 &= (1 - x^{m\sigma}\cos(tm\log x))^2 + x^{2m\sigma}\sin^2(tm\log x) \\
&= (1-x^{m\sigma})^2 + 2x^{m\sigma}(1-\cos(tm\log x)) < (1-x^{m\sigma})^2 \\
&\quad + x^{m\sigma}(tm\log x)^2 \leqq (1-x^{m\sigma})^2 + x^{m\sigma-2}t^2m^2(1-x)^2 \\
&= (1-x^{m\sigma})^2\left\{1 + \left(\frac{1-x}{1-x^{m\sigma}}\right)^2 x^{m\sigma-2}t^2m^2\right\} \\
&< (1-x^{m\sigma})^2\left\{1 + \left(\frac{1-x}{1-x^{m\sigma}}\right)^2 x^{-2}t^2m^2\right\} < (1-x^{m\sigma})^2\{1 + (\sigma m)^{-2}x^{-2}t^2m^2\} \\
&= (1-x^{m\sigma})^2(1 + t^2/x^2\sigma^2) < (1-x^{m\sigma})^2(1 + 3t^2/\sigma^2) \\
&\leqq (1-x^{m\sigma})^2(1 + 3T^2/\sigma_0^2),
\end{aligned}$$

so that $|1 - x^{m\sigma}e^{itm\log x}| < c_2'(1-x^{m\sigma})$, with $c_2' = (1+3T^2/\sigma_0^2)^{1/2}$. This sequence of inequalities does not hold for the last term of the sum, with $m = m_0$, because for it $m\sigma = m_0\sigma = ([\sigma^{-1}]+1) > \sigma^{-1}\sigma = 1$. In that case, however, $(1-x)/(1-x^{m_0\sigma}) < 1$, so that

$$\begin{aligned}
(1-x^{m\sigma})^2&\left\{1 + \left(\frac{1-x}{1-x^{m_0\sigma}}\right)^2 x^{-2}t^2m^2\right\} \leqq (1-x^{m\sigma})^2(1 + x^{-2}t^2m_0^2) \\
&= (1-x^{m\sigma})^2(1 + x^{-2}t^2([\sigma^{-1}]+1)^2) \leqq (1-x^{m\sigma})^2(1 + x^{-2}t^2(\sigma_0^{-1}+1)^2) \\
&< (1-x^{m\sigma})^2(1 + 3T^2(\sigma_0^{-1}+1)^2)
\end{aligned}$$

and

$$|1 - x^{m\sigma}e^{itm\log x}| < c_2(1 - x^{m\sigma}) ,$$

with $c_2 = (1 + 3T^2(\sigma_0^{-1} + 1)^2)^{1/2}$. As $c_2 > c_2'$, the inequality $|1 - x^{m\sigma}e^{itm\log x}| < c_2(1 - x^{m\sigma})$ holds for all m with $1 \leqq m \leqq m_0$.

If the inner sums are extended only over the primes $p < p_N$, the error of each sum is majorized by

$$\begin{aligned}\sum_{p \geqq p_N} |p^{-ms} - q^{-ms}| &= \sum_{p \geqq p_N} p^{-m\sigma}|1 - x^{m\sigma}e^{itm\log x}| \leqq c_2 \sum_{p \geqq p_N} (1 - x^{m\sigma})p^{-m\sigma} \\ &= c_2 \sum_{n=N}^{\infty} (p_n^{-m\sigma} - q_n^{-m\sigma}) \leqq c_2 \sum_{n=N}^{\infty} (p_n^{-m\sigma} - p_{n+1}^{-m\sigma}) ,\end{aligned}$$

a telescoping series, with sum $c_2 p_N^{-m\sigma}$. It follows that the total error of $\sum^1$ is majorized by $c_2 \sum_{m=1}^{m_0} p_N^{-m\sigma_0} \leqq c_2 m_0 p_N^{-\sigma_0}$ and can be made arbitrarily small, by taking N sufficiently large. This estimate is independent of s and the uniform convergence of the double series $\sum^1$, of the absolute values $m^{-1}|q^{-ms} - p^{-ms}|$ is proved. From the convergence of the series for $\log \varphi(s)$ it also follows that $\varphi(s) \neq 0$ and this finishes the proof of Lemma 1, hence also that of Theorem 1.

4. Proof of Theorem 2.

Let

$$\varphi_\chi(s) = \prod_p \{(1 - \chi(p)p^{-s})(1 - \chi(q)q^{-s})^{-1}\} . \tag{7}$$

Then

$$L^*(s, \chi) = \varphi_\chi(s)L(s, \chi) \tag{8}$$

and Theorem 2 is an immediate corollary of

LEMMA 3. *For $\sigma > 0$, the infinite product* (7) *converges absolutely and $\varphi_\chi(s) \neq 0$. The convergence is uniform on compact sets $\sigma_0 \leqq \sigma \leqq \sigma_1$, $|t| \leqq T$, for any constants σ_0, σ_1, T, that satisfy $\sigma_1 > \sigma_0 > 0$, $T > 0$.*

Proof of Lemma 3. The absolute convergence of (7) for $\sigma > 1$ is obvious; hence, it is sufficient to consider the convergence on the stated compact sets. As in the proof of Lemma 1, we take the logarithm of $\varphi_\chi(s)$ and, by taking into account also (3) and the absolute convergence for $\sigma > 1$, we get

$$\begin{aligned}\log \varphi_\chi(s) &= \sum_p \sum_{m=1}^{\infty} m^{-1}\{\chi(q^m)q^{-ms} - \chi(p^m)p^{-ms}\} \\ &= \sum_{m=1}^{\infty} m^{-1} \sum_p \chi(p^m)(q^{-ms} - p^{-ms}) .\end{aligned} \tag{9}$$

Hence, $|\log\varphi_\chi(s)|$ is majorized by $\sum_{m=1}^\infty m^{-1}\sum_p |q^{-ms}-p^{-ms}|$. The uniform convergence of this series is proved exactly as in the proof of Lemma 1, except for minor details in the handling of $\sum^1$. Its outer sum, we recall, contains no more than $m_0 \leqq \sigma_0^{-1}+1$ terms; this bound does not depend on s, so that it is sufficient to prove the uniform convergence of the inner sums. The terms with $q=p$ vanish. In the others, set $x=p/q$ and observe that $0<x<1$. If $q=p+r$ and $\varepsilon=r/q$, then $r\leqq K$, $\varepsilon\leqq K/q$ and $1-K/p<1-K/q\leqq 1-\varepsilon=1-r/q=(q-r)/q=p/q=x<1$. Except for finitely many primes, $\varepsilon=r/q$ stays below any preassigned positive quantity, so that $\lim_{p\to\infty}x(p)=1^-$. It follows as before that

$$|1-x^{m\sigma}e^{itm\log x}|<c_2(1-x^{m\sigma})=c_2(1-(1-\varepsilon)^{m\sigma})\ .$$

By Lemma 2(v), this is less than $c_3(1-e^{-\varepsilon m\sigma})<c_3\varepsilon m\sigma\leqq c_3 m\sigma K/q\leqq c_4 m\sigma/p\leqq c_5/p$. Here $c_3\leqq(1+\varepsilon)c_2\leqq(3/2)c_2$, $c_4=Kc_3$ and c_5 may be taken equal to $2c_4$, because $m\sigma\leqq m_0\sigma\leqq(\sigma^{-1}+1)\sigma=1+\sigma\leqq 2$. It now follows that

$$\begin{aligned}|\chi(q^m)q^{-ms}-\chi(p^m)p^{-ms}| &\leqq p^{-m\sigma}|1-x^{m\sigma}e^{itm\log x}|<c_5p^{-1-m\sigma}\\ &\leqq c_5p^{-1-\sigma}<c_5p^{-1-\sigma_0}\ .\end{aligned}$$

By summing only over $p<p_N$, each sum has an error not in excess of $\sum_{p>p_N}c_5p^{-1-\sigma_0}<\eta$, with η arbitrarily small, provided that $p_N=p_N(\sigma_0,T,\eta)$ is chosen large enough. This choice does not depend on s and leads to a total error on $\sum^1$ that is majorized by $m_0\eta<(\sigma_0^{-1}+1)\eta$. The proof of the absolute and uniform convergence of (9) is complete, and with it the proofs of Lemma 3 and of Theorem 2.

5. Analytic continuation. We have shown that $\varphi(s)$ and, more generally, $\varphi_\chi(s)$ can be continued analytically into the whole half plane $\sigma>0$. By (6) and (8) this implies the analytic continuability to $\sigma>0$ for $\zeta^*(s)$ and $L^*(s,\chi)$. In some cases it is obvious that $\varphi(s)$, or even $\varphi_\chi(s)$ can be continued as meromorphic functions into the whole complex plane, e.g., if $q_n=p_{n+1}$. On the other hand, one may select the q_n's so that $\sigma=0$ becomes a natural boundary for $\varphi(s)$ (see [1] for a similar result). Indeed, let all $q_n\neq p_n, p_{n+1}$; then $\varphi(s)$ has poles at all the points $s=it$, with $t=t_{n,k}=2k\pi(\log q_n)^{-1}$, $k\in Z$. One may select the real numbers q_n so that the poles $it_{n,k}$ will be dense on the imaginary axis. Apparently, this will be the case, whenever at most a finite number of factors $1-q^{-s}$ in the denominator of $\varphi(s)$ are cancelled by identical factors in the numerator, but we have not pursued this matter further. Similar considerations hold for $\varphi_\chi(s)$.

When $\sigma=0$ is a natural boundary for $\varphi(s)$, then it also is one for $\zeta^*(s)$; hence, $\zeta(s)$ and $\zeta^*(s)$, while sharing all their complex zeros, have an entirely different analytic character and a similar statement holds for $L(s, \chi)$ and $L^*(s, \chi)$.

6. Possible applications. In 1948 Turán ([8]; see also [9]) showed that if the partial sums $\zeta_n(s) = \sum_{m=1}^{n} m^{-s}$ of the ζ-function do not vanish in the half planes $\sigma > 1 + n^{-1/2+\varepsilon}(\varepsilon > 0)$, then the Riemann hypothesis holds. Recently [3] H. Montgomery showed, however, that these partial sums do, in fact, vanish for sufficiently large n even in the half planes $1 + c(\log\log n)(\log n)^{-1}$, provided that $c < (4 - \pi)/\pi$, that bound being best possible. The question arises, whether it is possible to choose the q_n so that Turán's theorem remains valid, while Montgomery's construction may not apply.

Another possible approach is the following: The functions holomorphic in the half-plane $\sigma > 0$ form a ring H under ordinary addition and multiplication. Within H the functions with the same zeros in $\sigma > 0$ as $(s - 1)\zeta(s)$ form an ideal I, to which belong all functions $(s - 1)\zeta^*(s)$. The study of I and of H/I may throw some light on the problem of the Riemann hypothesis. We intend to return to this topic on a later date.

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