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COHERENT POLYNOMIAL RINGS OVER REGULAR RINGS OF FINITE INDEX

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It is shown that polynomial rings in finitely or infinitely many central indeterminates, over a regular ring of finite index, are right and left coherent.

In this paper all rings have unity and all ring homomorphisms preserve the unity.

DEFINITION 1. A ring R is:

(i) Regular, if it satisfies the sentence

$$(\forall r)(\exists s)[rsr = r] ;$$

(ii) Of index n, where $n \geqq 1$ is an integer, if for all $m \geqq n$, it satisfies the sentence

$$(\forall r)[r^m = 0 \longrightarrow r^n = 0] ;$$

(iii) Of finite index if it is of index n, for some integer $n \geqq 1$.

DEFINITION 2. A ring R is left coherent if:

(i) $U \cap V$ is a finitely generated left ideal in R, whenever U and V are finitely generated left ideals in R, and

(ii) For each $r \in R$, the left annihilator of r in R is finitely generated, as a left ideal in R.

Right coherence for R is similarly defined.

DEFINITION 3. Let f be an element of and I a finite subset of a polynomial ring $T[X_1, \cdots, X_q]$.
Then:

(i) $\deg(f)$ is the total degree of f,

(ii) $\deg(I) = \text{Sup}\{\deg(f) : f \in I\}$, and

(iii) $\langle I \rangle$ denotes the left ideal generated by I.

It is known (cf. [3, Theorem 2.2]) that a ring is left coherent iff each of its finitely generated left ideals is finitely presented. Thus, for certain homological applications, the left coherent rings seem to be a suitable generalization of the left Noetherian rings. In view of the Hilbert basis theorem (which states that $T[X]$ is left Noetherian if T is), this suggests the following conjecture: if R is a left coherent ring, then $R[X]$ is too. Soublin, in [11], disproved this conjecture, even when R is commutative. However he showed that it does hold when R is commutative and regular. (All regular rings are right and left coherent and all commutative regular rings have index 1.)

The main result of this paper is:

THEOREM 1. *Let R be a regular ring of finite index. Then the polynomial ring $R[\{X_\alpha\}]$ is left and right coherent, for any finite or infinite set $\{X_\alpha\}$ of central indeterminates.*

In [1] we established this result in the special case when R is also a commutative algebraic algebra over some field. To do this we effectively showed that the result held for any regular ring that can be embedded in a ring S such that, for each $q \geqq 1$, $S[X_1, \cdots, X_q]$ is left and right coherent. We then showed that, in this case, suitable S actually exist.

For the rest of this paper let R be an arbitrary regular ring of finite index, and let $q \geqq 1$ be any fixed integer. The following lemma yields Theorem 1:

LEMMA 1. *There exists a ring S containing R as a subring such that $S[X_1, \cdots, X_q]$ is left and right coherent.*

Our proof of Lemma 1 hinges upon Lemma 2. Our approach to Lemma 2 is model theoretic.

Basic concepts of model theory, such as a (well formed) formula, a free variable, and a bound variable are found in [9]. A sentence is a formula in which all variables are bound. Let L denote the first order predicate calculus for rings.

The major obstacle in applying model theory to our problem is that many useful statements cannot be expressed in L. For example, there is no sentence in L which is satisfied by, and only by, polynomial rings. Further, the statement "$f \in U$", where U is an ideal in some ring, cannot be expressed in L. To overcome these difficulties, we note that certain formulae ϕ, concerning polynomial rings in $X_1, \cdots, X_q$, can be translated as formulae Φ in L such that for any ring T, ϕ holds in $T[X_1, \cdots, X_q]$ iff Φ holds in T.

Robinson observes (cf. [10, Chapter 5, §4]) that if r and n are fixed and we have bounds on the degrees of f and of each g_i, then the formula for polynomials (in $X_1, \cdots, X_q$) over a division ring

$$\text{(a)} \qquad (\exists h_1) \cdots (\exists h_r)[f = \sum_{i=1}^{r} h_i g_i \text{ and } \deg(h_j) \leqq n, \text{ when } 1 \leqq j \leqq r]$$

can be translated into a formula of L, in the above sense. The translation is a conjunction of certain formulae involving the coefficients of these polynomials. In this situation we shall always assume that $\deg(f) \leqq \max\{\deg(h_i g_i): 1 \leqq i \leqq r\}$. This bounds the number of variables required in the translation, in place of f. Further, there exists a function r such that for any $n \geqq 1$, division ring D, and finite subset $K \subseteqq D[X_1, \cdots, X_q]$ such that $\deg(K) \leqq n$, $\{f \in \langle K\rangle : \deg(f) \leqq n\}$ is a vector space over D of dimension $\leqq r(n)$. This is because $\{f \in D[X_1, \cdots, X_q]: \deg(f) \leqq n\}$ is a vector space generated by those products π of indeterminates satisfying $\deg(\pi) \leqq n$. Thus K may (and always will) be identified with a set $A = \{g_1, \cdots, g_{r(n)}\}$ (with repetitions if necessary) such that $\langle A\rangle = \langle K\rangle$.

For each m and $n \geqq 1$ define predicates $\in_{m,n}$ and $\subseteqq_{m,n}$, where f is a polynomial, $K = \{g_1, \cdots, g_{r(m)}\}$ satisfies $\deg(K) \leqq m$, and $K' = \{g'_1, \cdots, g'_{r(m+n)}\}$ satisfies $\deg(K') \leqq m + n$, by $f \in_{m,n} K$ iff (a) holds when $r = r(m)$; and $K' \subseteqq_{m,n} K$ iff $g'_i \in_{m,n} K$, for all $g'_i \in K$.

Using Robinson's observation, identify these predicates with their translations into L. Let $f \in_{m,n}(K, K')$ be the conjunction

$$[(f \in_{m,n} K) \wedge (f \in_{m,n} K')] ,$$

where $\deg(K') \leqq m$ too. Similarly define $G \subseteqq_{m,n} (K, K')$. Let $f \notin_{m,n} K$, $K' \not\subseteqq_{m,n} K$, and $G \not\subseteqq_{m,n} (K, K')$ be the negations of $f \in_{m,n} K$, $K' \not\subseteqq_{m,n} K$, and $G \not\subseteqq_{m,n} (K, K')$, respectively.

Although not themselves in the first order language L, the traditional $\in$ and $\subseteqq$ are related to these predicates as follows, where (m, n) and (m', n') take values in $\{(m, n): \deg(K) \leqq m$ and $\deg(K') \leqq m + n\}$ and in $\{(m', n'): \deg(K) \leqq m'$ and $\deg(f) \leqq m' + n'\}$, respectively:

$f \in \langle K\rangle$ iff $f \in_{m',n'} K$ for sufficiently large n',

$f \notin \langle K\rangle$ iff $f \notin_{m',n'} K$ for all n',

$\langle K'\rangle \subseteqq \langle K\rangle$ iff $\langle K'\rangle \subseteqq_{m,n} \langle K\rangle$ for sufficiently large n,

and $\langle K'\rangle \not\subseteqq \langle K\rangle$ iff $\langle K'\rangle \not\subseteqq_{m,n} \langle K\rangle$ for all n.

The next result is crucial in establishing Theorem 1.

LEMMA 2. *There exist (for each q) integral valued functions $M(-)$ and $N(-, -)$ such that for any division ring D and finite subsets I and J of $D[X_1, \cdots, X_q]$, having degrees $\leqq$ some m, there is a subset G such that:*

(i) $G \subseteqq_{m,M(m)}(I, J)$(*so that* $\deg(G) \leqq m + M(m)$);

(ii) *Whenever* $f \in_{m,n} (I, J)$, *then* $f \in_{m+M(m),N(m,n)} G$;

and thus

(iii) $\langle G\rangle = \langle I\rangle \cap \langle J\rangle$.

If the result were false, it would be obvious to any model theorist (cf. [4]) that ultraproducts could be used to construct a division ring E and finite subsets I and J of $E[X_1, \cdots, X_q]$ such that

$\langle I\rangle \cap \langle J\rangle$ is not finitely generated. This would contradict the Hilbert basis theorem.

LEMMA 3. *If* $S = \Pi\{D_\alpha: \alpha \in A\}$ *is a product of division rings and* $T = S[X_1, \cdots, X_q]$, *then* T *is left and right coherent.*

Proof. By symmetry it suffices to show that T is left coherent.

An element $s \in S$ is a function such that $s(\alpha) \in D_\alpha$, for each $\alpha \in A$. For any $t = \Sigma s_i \pi_i \in T$, where each $s_i \in S$ and each π_i is a product of indeterminates, let $t(\alpha) = \Sigma s_i(\alpha)\pi_i$. For each subset U of T let $U_\alpha = \{u(\alpha): u \in U\}$. Clearly, if U is a left ideal in T, then U_α is a left ideal in T_α and $T_\alpha = D_\alpha[X_1, \cdots, X_q]$, for each $\alpha \in A$.

To see that the left annihilator of any $t = \Sigma s_i \pi_i \in T$ is finitely generated (in fact generated by an idempotent) choose $e \in S$ such that $e(\alpha) = 1$ if $t(\alpha) = 0$, and $e(\alpha) = 0$ otherwise. Clearly $T \cdot e$ is the left annihilator of s.

Now let I and J be finite subsets of T and choose m such that $\deg(I) \leqq m$ and $\deg(J) \leqq m$. For each $\alpha \in A$, $\deg(I_\alpha) \leqq m$ and $\deg(J_\alpha) \leqq m$ so that there exists a subset $G_{(\alpha)} = \{g_{\alpha,1}, \cdots, g_{\alpha,s}\} \subseteqq T_\alpha$ (where $s = r(M(m))$) such that conditions (i), (ii), and (iii) from Lemma 2 hold, when I, J, G, and D are replaced by I_α, J_α, $G_{(\alpha)}$, and D_α respectively. Define a finite subset $G = \{g_1, \cdots, g_s\} \subseteqq T$ by $g_i(\alpha) = g_{\alpha,i}$, for each $\alpha \in A$. We must now show that these g_i actually exist. For any i, $g_i \in T$ exists as defined iff $\{\deg(g_{\alpha,i}): \alpha \in A\}$ is bounded above. By Lemma 2 (i), $m + M(m)$ is such a bound.

We shall establish that $\langle I\rangle \cap \langle J\rangle = \langle G\rangle$. To see that $\langle I\rangle \cap \langle J\rangle \subseteqq \langle G\rangle$ let $f \in \langle I\rangle \cap \langle J\rangle$ and $n = \deg(f)$. Then, for each $\alpha \in A$, $f(\alpha) \in \langle I_\alpha\rangle \cap \langle J_\alpha\rangle = \langle G_\alpha\rangle \subseteqq \langle T_\alpha\rangle$. Lemma 2 (ii) yields elements $h_{\alpha,i} \in T_\alpha$ such that $f(\alpha) = \sum_{i=1}^{s} h_{\alpha,i} g_i(\alpha)$, and an upper bound, $N(m, n)$, to $\{\deg(h_{\alpha,i}): 1 \leqq i \leqq s \text{ and } \alpha \in A\}$. Thus there are elements $h_i \in T$ satisfying $h_i(\alpha) = h_{\alpha,i}$. Therefore $f = \sum_{i=1}^{s} h_i g_i \in \langle G\rangle$. The proof that $\langle G\rangle \subseteqq \langle I\rangle \cap \langle J\rangle$ is similar, and uses Lemma 2 (i).

Proof of Lemma 1. Let Q be the complete left quotient ring of R. By [2, Theorem A] there is an isomorphism

$$Q \cong \bigoplus \sum_{i=1}^{a} (D_i)_{u(i)}$$

where a and the $u(i)$ are suitable integers, expressing Q as a direct sum of matrix rings over regular rings D_i, of index 1. For each i, let $S_i = \Pi\{D_i/M: M \text{ is a maximal ideal in } D_i\}$. Kaplansky has shown (cf. [8, Theorem 2.3]) that each D_i/M is a division ring and there is an embedding $D_i \subseteqq S_i$. Let $S = \bigoplus \sum_{i=1}^{a} (S_i)_{u(i)}$. Clearly $R \subseteqq S$ and

$$* \qquad S[X_1, \cdots, X_q] \cong \oplus \sum_{i=1}^{a} (S_i[X_1, \cdots, X_q])_{u(i)} .$$

By Lemma 3, each $S_i[X_1, \cdots, X_q]$ is left coherent. Thus [6, Corollary 2.2] establishes that each $(S_i[X_1, \cdots, X_q])_{u(i)}$ is left coherent whence, by * and ([7, Corollary 2.1], $S[X_1, \cdots, X_q]$ is too). Right coherence is established similarly.

REMARK 1. Let $\mathscr{C}$ be any class of rings closed under elementary equivalence. (I.e., if $D \in \mathscr{C}$ satisfies the same sentences from L as D', then $D' \in \mathscr{C}$.) The above methods may be used to show that if $D[X_1, \cdots, X_q]$ is (left) coherent for each $D \in \mathscr{C}$ and S is a product of rings from $\mathscr{C}$, then $S[X_1, \cdots, X_q]$ is (left) coherent too. However proving this is somewhat cumbersome since the number of elements in various $G \subseteqq D[X_1, \cdots, X_q]$ having degree $\leqq m$, for various $D \in \mathscr{C}$, need not be bounded. This complicates the definitions of $\in_{m,n}$, $\subseteqq_{m,n}$, and the statement and proof of Lemma 2. In addition, the $D[X_1, \cdots, X_q]$ need not be integral domains. Thus another lemma is required stating that there exist integral valued functions $A(-)$ and $B(-, -)$ such that for each $D \in \mathscr{C}$ and $f \in D[X_1, \cdots, X_q]$ having degree $\leqq m$, there exists a subset

$$\{g_1, \cdots, g_{A(m)}\} \subseteqq D[X_1, \cdots, X_q] \qquad \text{of degree} \quad \leqq A(m)$$

such that $G \cdot f = 0$ and, if $k \cdot f = 0$ and $\deg(k) \leqq n$, then there exist $h_1, \cdots, h_{A(m)}$ (each having degree $\leqq B(m, n)$) such that

$$k = \sum_{i=1}^{A(m)} h_i g_i .$$

REMARK 2. In particular, for any fixed n, if $\mathscr{C} = \{(D)_m : m \leqq n$ and D is a division ring$\}$, then it is closed under elementary equivalence since, by [8, Theorem 2.3], it consists of regular rings of index n satisfying

$$(\forall e)([e^2 = e \wedge (\forall r)(re = er)] \rightarrow [e = 0 \vee e = 1]) .$$

This provides an alternate proof of Lemma 1. Suppose that R is a regular ring of index n. Let $S = \Pi\{R/M : M$ is a maximal ideal in $R\}$. Since R is regular, it is standard that the natural map $R \rightarrow S$ is an embedding. By [8, Theorem 2.3] each $R/M \in \mathscr{C}$. As in the proof of Lemma 1, $T[X_1, \cdots, X_q]$ is left coherent, for each $T \in \mathscr{C}$. Thus, by Remark 1, $S[X_1, \cdots, X_q]$ is left coherent.

REMARK 3. Our approach to Theorem 1 uses the structure results for regular rings of finite index obtained in [2] and [8]. We do not know if Theorem 1 also holds for arbitrary regular rings.

REMARK 4. Eklof and Sabbagh have related coherence for rings to certain model theoretic concepts. They show (cf. [5, Theorem 3.16]) that a ring Λ is coherent iff each ultraproduct of $\aleph_0$-injective Λ-modules is $\aleph_0$-injective, and that (cf. [5, Theorems 4.1 and 4.8]) Λ is coherent iff the elementary theory of its modules has a model completion. (A ring Λ is $\aleph_0$-injective if, for each finitely generated ideal U and $f \in \mathrm{Hom}(U, \Lambda)$, there exists $g \in \mathrm{Hom}(\Lambda, \Lambda)$ such that $g|_U = f$.

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